An Example Concerning the Bass Conjecture

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In this paper an example is given of a 2-generator group G of cohomological dimension 2 which contains a subgroup $A \cong \mathbf{Q}$ such that $A\setminus 1$ is contained in a single conjugacy class of G. This answers a question of Schafer which arose when studying the Bass conjecture.

Let R be a commutative ring with a 1 and let G be a group. If P is a finitely generated projective RG-module then there is a map $r_P\colon G \to R$, called the *rank* of P, which is constant on the conjugacy classes of G (i.e., $r_P(xy) = r_P(yx)$ for all $x, y \in G$). This map can be considered as an extension of the usual trace map from square matrices over RG to RG. The precise definition of r_P, which is not required for this paper, is given in [1] (see also [6, p. 95]). If F is a finitely generated free RG-module then $r_F(x) = 0$ for all $x \in G\setminus 1$, and if Q is a finitely generated projective RG-module then $r_{P\oplus Q} = r_P + r_Q$. Let $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{C}$ denote (resp.) the integer, rational, and complex numbers. One form of the strong Bass conjecture [1, (4.5)] is the following.

CONJECTURE 1. *Let P be a finitely generated projective RG-module, and let $1 \neq x \in G$. If R is a subring of $\mathbf{C}$ such that $R \cap \mathbf{Q} = \mathbf{Z}$, then $r_P(x) = 0$.*

Let $\mathbf{Q}^+$ denote the additive group of $\mathbf{Q}$. We define an element $x \in G$ to be an *exceptional element* if $x \neq 1$ and there exist subgroups C and H of G such that $x \in C \subseteq H$, $C \cong \mathbf{Q}^+$, H is finitely generated, and the elements of C lie in finitely many H-conjugacy classes. The following related result is contained in a personal communication from J. A. Moody to the author; it extends [6, Lemma 4.1] and [7, Cor. 1].

THEOREM 2. *Let P be a finitely generated projective RG-module, and let x be a nonexceptional element of $G\setminus 1$. If R is a subring of $\mathbf{C}$ such that $R \cap \mathbf{Q} = \mathbf{Z}$, then $r_P(x) = 0$.*

The following question was posed in [7, p. 108].

QUESTION 3. If G is a finitely generated group containing a subgroup $A \cong \mathbf{Q}^+$ such that A is contained in only finitely many conjugacy classes of G, must G be of infinite cohomological dimension?

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If the answer to Question 3 were "yes", then it would be clear from Theorem 2 that the strong Bass conjecture would be true for all groups G of finite cohomological dimension. However, we shall show that the answer to Question 3 is "no".

EXAMPLE 4. *There is a 2-generator group G of cohomological dimension 2 containing a subgroup $A \cong \mathbf{Q}^+$ such that $A \setminus 1$ is contained in a single conjugacy class of G.*

Proof. We shall use the notation $\langle s_1, s_2, s_3, \ldots \rangle$ for the subgroup generated by elements $s_1, s_2, s_3, \ldots$ of a group. Let $B = \langle b_1 \rangle = \mathbf{Z}$, and for $2 \le i \in \mathbf{Z}$ define $\sigma_i : \mathbf{Z} \to \mathbf{Z}$ to be multiplication by i and define $C_i = B *_{B, \sigma_i}$, an HNN extension with stable letter x_i (see [2, p. 30]). Thus $C_i = \langle b_1, x_i \rangle$ and $x_i b x_i^{-1} = b^i$ for all $b \in B$. Also define $\sigma_1 : \mathbf{Z} \to \mathbf{Z}$ to be multiplication by -1 and $C_1 = B *_{B, \sigma_1}$, an HNN extension with stable letter x_1. Thus $C_1 = \langle b_1, x_1 \rangle$ and $x_1 b x_1^{-1} = b^{-1}$ for all $b \in B$. Let $H = C_1 *_B C_2 *_B C_3 * \cdots$, an infinite free product with amalgamation (this construction was pointed out to me by Peter Kropholler).

By Theorem 2 of [3], each C_i has cohomological dimension 2. Now H is the fundamental group of a graph of groups in which the vertex groups have cohomological dimension 2 and the edge groups have cohomological dimension 1. Therefore another application of Theorem 2 of [3] shows that H has cohomological dimension 2.

For $n \ge 2$, define $b_n = x_2^{-1} x_3^{-1} \cdots x_n^{-1} b_1 x_n \cdots x_3 x_2$. Then $b_n^n = b_{n-1}$ for all $n \ge 2$, so if $A = \langle b_1, b_2, b_3, \ldots \rangle$ then $A \cong \mathbf{Q}^+$ and the elements of $A \setminus 1$ lie in a single conjugacy class.

Let F be the nonabelian free group on two generators, and let $K = H * F$. We now employ the standard technique of embedding a countable group in a 2-generator group; one good reference for this is Theorem 36 in [4, p. 43], which shows that there is an HNN extension $G = K *_E$ where E is a free subgroup of K such that G is 2-generated. Now G has cohomological dimension 2 by Theorem 2 of [3], so we have the required example. □

REMARK. Although one cannot verify the strong Bass conjecture for the group G of Example 4 by using the techniques of [6; 7], one can verify it by the techniques of cyclic homology. In fact, the following is an immediate consequence of [5].

THEOREM 5. *Let G be the group of Example 4 and let P be a finitely generated projective $\mathbf{C}G$-module. If $1 \ne x \in G$ then $r_P(x) = 0$.*

All that needs to be done is to note that the last line of [5, Thm. 2.5] can be modified to "$H_i(G/\langle x \rangle; \mathbf{C}) = 0$ for $i \ge 2$", and then [5, Thm. 3.3] for the class of groups (d) applies if P is a finitely generated projective $\mathbf{C}G$-module.

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