Propagation of Singularities in a Locally Integrable Structure

S. Berhanu

0. Introduction

In a recent paper [13] Trépreau proved several theorems about the propagation of singularities for CR functions defined on generic CR submanifolds of $\mathbf{C}^n$. This work extends some of his results to a general first-order system of PDEs for which the question of holomorphic extendability (more precisely, hypoanalyticity [1]) makes sense.

A few years ago, Hanges and Treves [5] proved that connected elliptic submanifolds of a hypoanalytic manifold Ω propagate hypoanalyticity of a solution. When Ω is a CR manifold, elliptic submanifolds coincide with complex submanifolds and "hypoanalyticity of a solution" means holomorphic extendability of a CR distribution. As a corollary of our result, we will derive a microlocal version of the Hanges–Treves theorem; that is, we will get a propagator of microlocal hypoanalyticity in the part of the cotangent space $T^*\Omega$ lying above an elliptic submanifold. This corollary will in turn imply the main result of [5].

Another corollary concerns the propagation of microlocal analyticity for solutions of a formally integrable system of real analytic vector fields. It is shown here that the propagation occurs along a Nagano leaf [8] generated by the Hamiltonians of the real and imaginary parts of the vector fields and contained in the characteristic set of the system. We mention that Hanges and Sjöstrand [4] proved such a propagation for solutions of a differential operator of principal type with real analytic coefficients. In the case of systems of analytic vector fields, we believe the approach here is simpler. This paper is also related to the work of Baouendi and Rothschild in [2; 3] and that of Tumanov in [14] on wedge extendability in CR manifolds. We mention that our Lemma 4.2 is similar to Lemma 3.1 of [7].

The paper is organized as follows. In Section 1 we will discuss the locally integrable structures we work in and state our main results. Section 2 contains some corollaries to these results. In Section 3 we recall microlocal hypoanalyticity and prove a lemma concerning the wavefront characterization of the FBI transform. In Section 4 we embed our hypoanalytic structure into a CR structure and show that this embedding preserves microlocal

Received March 8, 1991. Revision received August 6, 1992.
Michigan Math. J. 40 (1993).

hypoanalyticity of a solution in certain relevant directions. We also show that this embedding preserves orbits of families of appropriate vector fields both in the base spaces and in the cotangent spaces. Finally, the lemmas of Sections 3 and 4 allow us to use the theorems of Trépreau to prove our theorem.

At this point I would like to thank Professors F. Treves and N. Hanges for several stimulating discussions.

1. Definitions and Statement of Results

For the general theory of hypoanalytic structures the reader is referred to [1]. We recall here what we will need. Let Ω be an open subset of R^{m+n}. We suppose, given a C^∞ map, that

$$Z=(Z_1,\dots,Z_m):\Omega\to\mathbf{C}^m,$$

with the differentials $dZ_1,\dots,dZ_m$ linearly independent. These differentials generate an m-dimensional subbundle of the complex cotangent bundle $CT^*\Omega$ which we denote by T'. T' will be called the *structure bundle.*

The orthogonal of T', denoted by $\mathbf{L}$, for the duality between tangent and cotangent vectors is an n-dimensional locally integrable Lie algebra. If $H=(H_1,\dots,H_m)$ is a biholomorphism defined in a neighborhood of $Z(\Omega)$, then $\{H_1(Z),\dots,H_m(Z)\}$ defines the same hypoanalytic structure on Ω.

A distribution u defined in an open subset V of Ω is called a *solution in V* if, for any C^∞ section L of $\mathbf{L}$ on V, $Lu=0$ in the distribution sense in V.

Let X be a C^∞ submanifold of Ω of dimension m. Denote by π_X the natural map $T^*\Omega\,|_X\to T^*X$ and by $\pi_X^{\mathbf{C}}$ the analogous map of the complex cotangent bundles. X is called *maximally real* if $\mathbf{C}T^*X=\pi_X^{\mathbf{C}}(T')$. Since a maximally real manifold X is noncharacteristic for $\mathbf{L}$, the trace on X of any solution is well defined.

If L is a smooth section of $\mathbf{L}$, its symbol $\sigma(L)$ vanishes on the bundle T'. In fact, T' is the set of all common zeros of all the functions $\sigma(L)$, with L a smooth section of $\mathbf{L}$. Therefore, $\operatorname{Char}\mathbf{L}=T^*\Omega\cap T'$ is the characteristic set of $\mathbf{L}$. In general, $\operatorname{Char}\mathbf{L}$ is not a vector bundle.

A solution u is called *hypoanalytic at a point p* in Ω if there is a holomorphic function $\tilde{u}$ defined near $Z(p)$ in $\mathbf{C}^m$ such that $u=\tilde{u}\circ Z$ in a neighborhood of p. The concept of hypoanalyticity was microlocalized in [1]. In Theorem 1.1 of this section, WF_{ha} will denote the hypoanalytic wavefront set of a solution u.

We need to recall briefly from [11] some of the main classes of hypoanalytic structures. The structure defined by T' on Ω is said to be *elliptic* if $\operatorname{Char}\mathbf{L}=0$; it is said to define a *CR structure* on Ω if $\mathbf{C}T^*\Omega=T'+\bar{T}'$; it defines a *complex structure* if it is elliptic and defines a *CR* structure, in other words, if

$$\mathbf{C}T^*\Omega=T'\oplus\bar{T}',$$

where $\oplus$ denotes direct sum. Since the questions we consider are always local, we will work near a central point, say 0 is in Ω. Accordingly, after contracting Ω about 0 and making an affine substitution of the Z_j's and a real local change of coordinates as in [1], we can obtain

$$(1.1)\qquad \begin{aligned} &Z_j = x_j + \sqrt{-1}\, y_j = z_j \quad \text{for } j = 1, \ldots, r \quad \text{and} \\ &Z_k = x_k + \sqrt{-1}\, \Phi_k(x, y) \quad \text{for } k = r+1, \ldots, m, \end{aligned}$$

with Φ_k real-valued, $\Phi_k(0,0) = 0$, and $d\Phi_k(0,0) = 0$ for $k \geq r+1$. The integer $m-r$ is $\dim \operatorname{Char} \mathbf{L}$ at 0. We remark that T' is elliptic when $r = m$; T' is a *CR* structure when $r = n$; and T' is a complex structure when $r = m = n$.

Finally, we need to cite the concepts and results of Sussmann [10] on orbits of families of vector fields. Let D be a set of C^∞ vector fields on a C^∞ manifold M. If $V \in D$, let $\Phi_V(t)$ denote the flow of V. If $p \in M$, there is a maximal open interval $J(p)$ such that $\Phi_V(t)p$ is defined from $J(p)$ into M. When we write $\Phi_V(t)p$, it will be understood that $t \in J(p)$. Two points p_1 and p_2 are said to be *D-equivalent* if there are finitely many vector fields $V_1, \ldots, V_r$ in D such that

$$p_2 = \Phi_{V_1}(t_1) \cdots \Phi_{V_r}(t_r) p_1.$$

This defines an equivalence relation among the points of M. An equivalence class for this relation is called a *D-orbit*. In [10] Sussmann proved that the D-orbits are smooth submanifolds of M. We will apply this to $D = \operatorname{Re} \mathbf{L}$ and $D = H_{\operatorname{Re} \mathbf{L}}$, where the elements of $\operatorname{Re} \mathbf{L}$ are the real parts of C^∞ sections of $\mathbf{L}$ and $H_{\operatorname{Re} \mathbf{L}}$ is the family of Hamilton fields $H_{\sigma(X)}$, $X \in \operatorname{Re} \mathbf{L}$. The notation $\mathcal{O}(D, x)$ will denote the D-orbit containing x. If N is a submanifold of Ω, $T_N^*\Omega$ will denote the conormal of N in Ω.

THEOREM 1.1. *Suppose N is a submanifold of Ω that is a D-orbit of* $\operatorname{Re} \mathbf{L}$. *If u is a solution and $\gamma \in T_N^*\Omega$, then*

$$\gamma \in WF_{ha} u \Leftrightarrow \mathcal{O}(H_{\operatorname{Re} \mathbf{L}}, \gamma) \subseteq WF_{ha} u.$$

THEOREM 1.2. *Suppose* $\Gamma \subseteq \operatorname{Char} \mathbf{L}$ *and* $\Gamma = \mathcal{O}(\gamma, H_{\operatorname{Re} \mathbf{L}})$ *for some* $\gamma \in \Gamma$. *If u is a solution, then*

$$\gamma \in WF_{ha} u \Leftrightarrow \Gamma \subseteq WF_{ha} u.$$

2. Consequences of Theorem 1.1 and Theorem 1.2

In this section we will first deduce Corollary 2.1 from Theorem 1.1. This corollary will in turn yield Corollary 2.2, which was proved in [5] by Hanges and Treves.

Let M be a submanifold of Ω, and set $T_M' = \pi_M^{\mathbf{C}}(T')$.

DEFINITION 2.1. M is called a *hypoanalytic submanifold* if it is equipped with a hypoanalytic structure whose structure bundle equals T_M' and which has the following property:

> Given any hypoanalytic function f in an open set $V \subseteq \Omega$, the restriction of f to $M \cap V$ is hypoanalytic.

If M is a hypoanalytic submanifold of Ω we call it elliptic, CR, complex if these apply to the hypoanalytic structure given to M.

COROLLARY 2.1. *Suppose N is an elliptic submanifold of Ω. If u is a solution in Ω and $\gamma \in \operatorname{Char} \mathbf{L}|_N$, then*

$$\gamma \in WF_{ha}u \Leftrightarrow \mathcal{O}(H_{TN}, \gamma) \cap \operatorname{Char} \mathbf{L}|_N \subseteq WF_{ha}u.$$

The preceding corollary is a microlocal version of the following corollary, which was the main result in [5].

COROLLARY 2.2 (Hanges-Treves [5]). *If a solution u is hypoanalytic at a point of a connected elliptic submanifold M of Ω, then u is hypoanalytic at every point of M.*

Proof of Corollary 2.1. As in [5], we consider three cases.

Case 1: Assume T' is a CR structure; that is, assume $r = n$ in (1.1). Then the ellipticity of N implies that it has a complex structure induced by T'. Assume further that $\dim_{\mathbf{C}} N = n$. Then over N, $\mathbf{L} \oplus \bar{\mathbf{L}} = CTN$. Hence Theorem 1.1 applies to N. Let $L_1, \ldots, L_n$ generate $\mathbf{L}$ (near 0). Let $X_j = \operatorname{Re} L_j$ and $Y_j = \operatorname{Im} L_j$ for each j. $\operatorname{Char} \mathbf{L}$ is now a manifold. If $X \in \{X_1, Y_1, \ldots, X_n, Y_n\}$, then $X \in TN$ and therefore $H_{\sigma(X)} \in T(\operatorname{Char} \mathbf{L}|_N)$. This fact, together with the equation

$$[H_{\sigma(X)}, H_{\sigma(Y)}] = H_{\sigma[X,Y]},$$

tell us that the family $H_{\operatorname{Re}\mathbf{L}}$ foliates $\operatorname{Char} \mathbf{L}|_N = T_N^*\Omega$ into orbits each of which has dimension $2n$, and has a basis for its tangent space given by the restrictions of

$$\{H_{\sigma(X_1)}, H_{\sigma(Y_1)}, \ldots, H_{\sigma(X_n)}, H_{\sigma(Y_n)}\}.$$

By Theorem 1.1, each of these orbits propagates the singularities of a solution. If $\gamma \in \operatorname{Char} \mathbf{L}|_N$, we clearly have $\mathcal{O}(H_{\operatorname{Re}\mathbf{L}}, \gamma) = \mathcal{O}(H_{TN}, \gamma)$.

To deduce Corollary 2.2 in this case, we look at an integral curve for $H_{\sigma(X_j)}$ or $H_{\sigma(Y_k)}$ in an orbit. We use local coordinates $(x', x'') \in \mathbf{R}^{2n} \times \mathbf{R}^{m-n}$ on Ω near 0, which we assume is in N, such that N is given by $x'' = 0$. In these coordinates, if

$$X = \sum_{j=1}^{n+m} a_j(x) \frac{\partial}{\partial x_j}$$

is tangent to N, then $a_j(x', 0) = 0$ for $j > 2n$. Therefore

$$H_{\sigma(X)} = X - \sum_{t=2n+1}^{m+n} \left(\sum_{j=2n+1}^{m+n} \frac{\partial a_j}{\partial x_t}(x', 0)\xi_j \right) \frac{\partial}{\partial \xi_t} \quad \text{over } \operatorname{Char} \mathbf{L}|_N = T_N^*\Omega,$$

where $\xi = (\xi_1, \ldots, \xi_{m+n})$ is the fiber variable in the cotangent space to Ω. An integral curve $\mathcal{C}(t)$ for $H_{\sigma(X)}$ through $\mathcal{C}(0) = (x'(0), 0; 0, \xi''(0)) \in \operatorname{Char} \mathbf{L}|_N$ has the form

$$\mathcal{C}(t) = (x'(t), 0; 0, \xi''(t)).$$

$\xi''(t)$ depends linearly on $\xi''(0)$ as follows:

$$\xi''(t) = \exp\left(\int_0^t A(s)\,ds\right)\xi''(0) = B(t)\,\xi''(0),$$

where $B(t)$ is a matrix depending only on the base projection of the curve $\mathcal{C}$. Since $B(t)$ is invertible, for any fixed t the mapping $\xi''(0) \to B(t)\,\xi''(0)$ from the fiber $\operatorname{Char}\mathbf{L}(x'(0), 0)$ to the fiber $\operatorname{Char}\mathbf{L}(x'(t), 0)$ is a bijection.

The latter implies Corollary 2.2, since a solution u is hypoanalytic at a point p if and only if its hypoanalytic wavefront set does not intersect $\operatorname{Char}\mathbf{L}$ over p.

Case 2: Ω a *CR* structure as in Case 1, but $\dim_{\mathbf{C}} N = m' < n$. Without loss of generality, we may assume that the restrictions of $Z_1, \dots, Z_{m'}$ generate the structure bundle on N. Since N is a hypoanalytic submanifold, for each $k = m'+1, \dots, m$ there is a holomorphic function h_k such that $Z_k = h_k(Z_1, \dots, Z_{m'})$ on N. We will use the new chart

$$Z_1, \dots, Z_{m'}, Z_{m'+1} - h_{m'+1}(Z_1, \dots, Z_{m'}), \dots, Z_m - h_m(Z_1, \dots, Z_{m'}).$$

We also make a real change of coordinates in Ω $(x, y) \to (\tilde{x}, \tilde{y})$, where

$$\tilde{x}_k = \begin{cases} x_k & \text{for } k = 1, \dots, m' \\ \operatorname{Re}(Z_k(x,y) - h_k(Z_1(x,y), \dots, Z_m(x,y))) & \text{for } k > m' \end{cases}$$

and

$$\tilde{y}_j = \begin{cases} y_j & \text{for } j = 1, \dots, m' \\ \operatorname{Im}(Z_j(x,y) - h_j(Z_1(x,y), \dots, Z_m(x,y))) & \text{for } m'+1 \le j \le n. \end{cases}$$

After dropping the tildes, we have coordinates x, y and a hypoanalytic chart $Z_1, \dots, Z_m$ such that

$$N = \{(x,y) : x_{m'+1} = \cdots = x_m = 0,\ y_{m'+1} = \cdots = y_n = 0\};$$
$$Z_j = x_j + \sqrt{-1}\,y_j \quad \text{for } j = 1, \dots, n;$$
$$Z_k = x_k + \sqrt{-1}\,\Phi_k(x,y) \quad \text{for } k = n+1, \dots, m,$$

where $\Phi_k|_N = 0$ and $d\Phi_k(0,0) = 0$.

In these coordinates, a basis $L_1, \dots, L_n$ for $\mathbf{L}$ can be chosen so that

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=n+1}^{m} a_k^j \frac{\partial}{\partial x_k} \quad \text{when } 1 \le j \le n$$

and $L_j \in CTN$ when $1 \le j \le m'$. In other words,

(2.1) $\qquad a_k^j(x', y', 0) = 0 \quad \text{for } j = 1, \dots, m' \text{ and } k = n+1, \dots, m,$

Let $\Omega' = \{(x,y) \in \Omega : y_{m'+1} = \cdots = y_n = 0\}$. Equip Ω' with the bundle $\tilde{\mathbf{L}}$ generated by

$$\tilde{\mathbf{L}}_j = L_j|_{\Omega'} \quad \text{for } 1 \le j \le m'.$$

Then $(\Omega', \tilde{\mathbf{L}})$ is a *CR* structure containing N as an elliptic submanifold and

(2.2) $$\tilde{\mathbf{L}}|_N = CTN.$$

Therefore, by Case 1, $\operatorname{Char}\tilde{\mathbf{L}}|_N = T_N^*\Omega'$ is a union of orbits along each of which the singularities of solutions in the structure $(\Omega', \tilde{\mathbf{L}})$ propagate. From (2.1), we get

$$\operatorname{Char}\mathbf{L}|_N \subseteq T_N^*\Omega' = \operatorname{Char}\tilde{\mathbf{L}}|_N.$$

If $\gamma \in \operatorname{Char}\mathbf{L}|_N$, (2.2) gives

$$\mathcal{O}(H_{TN}, \gamma) = \mathcal{O}(H_{\operatorname{Re}\tilde{\mathbf{L}}}, \gamma).$$

If u is a solution for $(\Omega, \mathbf{L})$ then $u' = u|_{\Omega'}$ is a solution for $(\Omega', \tilde{\mathbf{L}})$, and hence the singularities of u' propagate along $\mathcal{O}(H_{\operatorname{Re}\tilde{\mathbf{L}}}, \gamma)$.

Through each point $(x_0, y_0) \in \Omega'$, the maximally real manifold

$$\{(x, y) \in \Omega : y = y_0\} \subseteq \Omega'.$$

Since microlocal hypoanalyticity of a solution in Ω is determined by its trace on maximally real submanifolds, it follows that the singularities of u propagate along $\mathcal{O}(H_{\operatorname{Re}\tilde{\mathbf{L}}}, \gamma)$ and hence along $\mathcal{O}(H_{TN}, \gamma)$. Moreover,

$$\operatorname{Char}\mathbf{L}|_N = \bigcup_{\gamma \in \operatorname{Char}\mathbf{L}|_N} \mathcal{O}(H_{\operatorname{Re}\tilde{\mathbf{L}}}, \gamma).$$

The latter yields both Corollary 2.1 and Corollary 2.2 in this case.

Case 3: Ω is not a *CR* structure $(n > r)$ and N is elliptic, $\dim N = 2m' + s$. We may assume that the restrictions of $Z_1, \dots, Z_{m'}$ generate the structure bundle on N. As in Case 2, we can get coordinates x, y and a hypoanalytic chart such that

$$Z_j = x_j + \sqrt{-1}\, y_j \quad \text{for } 1 \le j \le r \quad \text{and}$$
$$Z_k = x_k + \sqrt{-1}\, \Phi_k(x, y) \quad \text{for } r+1 \le k \le m;$$

and on N,

$$x_k = 0,\ k = m'+1, \dots, m; \qquad y_l = 0,\ l = m'+1, \dots, r. \tag{2.3}$$

Moreover, $\Phi_k|_N = 0$ for $k = r+1, \dots, m$.

Unlike Case 2, (2.3) may not be all the defining functions for N. However, by applying the implicit function theorem as in [5] we may assume that N is given by

$$x_k = 0,\ k = m'+1, \dots, m; \qquad y_l = 0,\ l = m'+1, \dots, n-s. \tag{2.4}$$

We adopt here the following notation:

$$x' = (x_1, \dots, x_{m'}),\ y' = (y_1, \dots, y_{m'}),\ z' = x' + \sqrt{-1}\, y';$$
$$x'' = (x_{m'+1}, \dots, x_m),\ y'' = (y_{m'+1}, \dots, y_{n-s});$$
$$y^* = (y_{n-s+1}, \dots, y_n).$$

We contract Ω about 0 and assume that

$$\Omega = \Delta' \times V'' \times W'' \times W^*,$$

where

$$\Delta' = \{z \in C^{m'} : |z_j| < \delta, j = 1, \dots, m'\}$$

and V'', W'', W^* are open balls centered at the origin in the spaces of x'', y'', and y^*, respectively. Then

$$(2.5)\qquad \begin{gathered} N = \Delta' \times \{0\} \times \{0\} \times W^* \quad \text{and} \\ \Phi_k(x', y', 0, 0, y^*) = 0, \quad d\Phi_k(0) = 0 \ \forall k. \end{gathered}$$

In these coordinates $\mathbf{L}$ is spanned by a basis of the form

$$(2.6)\qquad \begin{aligned} L_j &= \frac{\partial}{\partial \bar{z}_j} + \sum_{k=r+1}^{m} a_j^k(x,y) \frac{\partial}{\partial x_k} \quad \text{for } 1 \le j \le r; \\ L_j &= \frac{\partial}{\partial y_j} + \sum_{k=r+1}^{m} a_j^k(x,y) \frac{\partial}{\partial x_k} \quad \text{for } r+1 \le j \le n. \end{aligned}$$

Let $\Omega' = \Delta' \times V'' \times \{0\} \times W^*$. The vector fields $L_1, \dots, L_{m'}$ together with $L_{n-s+1}, \dots, L_n$ are all tangent to Ω'. Let $\mathbf{L}'$ denote the bundle on Ω' generated by these vector fields. The restrictions of $Z_1, \dots, Z_m$ to Ω' generate the orthogonal of $\mathbf{L}'$ in $T^*\Omega'$. We now claim that $N = \Delta' \times \{0\} \times \{0\} \times W^*$ is an orbit of $\operatorname{Re}\mathbf{L}'$. To see this, it suffices to show that in (2.6)

$$a_j^k(x,y) = 0 \text{ on } N \text{ for } 1 \le j \le m' \text{ and } r+1 \le k \le m$$

and

$$a_j^k(x,y) = 0 \text{ on } N \text{ for } n-s+1 \le j \le n \text{ and } r+1 \le k \le m.$$

Fix $j \in \{1, \dots, m'\}$. The equations

$$L_j(x_k + \sqrt{-1}\,\phi_k(x,y)) = 0 \quad \text{for } r+1 \le k \le m$$

lead to the system

$$(2.7)\qquad \sqrt{-1}\,\frac{\partial \phi_k}{\partial \bar{z}_j} + a_j^k + \sqrt{-1} \sum_{t=r+1}^{m} a_j^t \frac{\partial \phi_k}{\partial x_t} = 0 \quad \text{for } r+1 \le k \le m.$$

By (2.5), since $\phi_k|_N = 0$ and $\partial/\partial \bar{z}_j$ is tangent to N, the functions $\partial \phi_k / \partial \bar{z}_j$ vanish on N. Therefore, on N, (2.7) becomes

$$(2.8)\qquad a_j^k + \sqrt{-1} \sum_{t=r+1}^{m} a_j^t \frac{\partial \phi_k}{\partial x_t} = 0 \quad \text{for } r+1 \le k \le m.$$

Since $d\phi_k(0) = 0 \ \forall k$, it follows that in a neighborhood of 0, and hence without loss of generality on all of N, we have

$$(2.9)\qquad a_j^k = 0 \text{ for } r+1 \le k \le m \text{ and } 1 \le j \le m'.$$

Similar reasoning gives

$$(2.10)\qquad a_j^k = 0 \text{ on } N \text{ for } r+1 \le k \le m \text{ and } n-s+1 \le j \le n.$$

From (2.9) and (2.10), we conclude that

$$(2.11)\qquad \operatorname{Re}\mathbf{L}'|_N = TN \text{ and hence } N = \mathcal{O}(\operatorname{Re}\mathbf{L}', 0).$$

The latter permits us to apply Theorem 1.1 to obtain: if $h \in \mathcal{D}'(\Omega')$, $\mathbf{L}'h = 0$ and $\gamma \in T_N^*\Omega'$ then

$$\gamma \in WF_{ha} h \Leftrightarrow \mathcal{O}(H_{\mathrm{Re}\,\mathbf{L}'}, \gamma) \subseteq WF_{ha} h. \tag{2.12}$$

We now note that the canonical map

$$\pi : T^*\Omega\,|_{\Omega'} \to T^*\Omega'$$

is an injection of $\mathrm{Char}\,\mathbf{L}\,|_N$ into $\mathrm{Char}\,\mathbf{L}'|_N$, and that if $\mathbf{L}u = 0$ in Ω then $\mathbf{L}'u' = 0$ in Ω', where $u' = u\,|_{\Omega'}$. Moreover, Ω' contains a maximally real submanifold through each point of N. It follows that if $\gamma \in \mathrm{Char}\,\mathbf{L}\,|_N$,

$$\gamma \in WF_{ha} u \Leftrightarrow \pi(\gamma) \in WF_{ha} u'. \tag{2.13}$$

From the latter, (2.11), and (2.12) we conclude that

$$\gamma \in WF_{ha} u \Leftrightarrow \mathcal{O}(H_{TN}, \gamma) \cap \mathrm{Char}\,\mathbf{L}\,|_N \subseteq WF_{ha} u.$$

To get Corollary 2.2, suppose u is hypoanalytic at $p \in N$. Then u' is also hypoanalytic there, and hence by (2.11) and (2.12) u' is hypoanalytic at every point of N. But then by (2.13) u is hypoanalytic at every point of N. □

COROLLARY 2.3 (Hanges–Sjöstrand [4]). *Suppose that the structure* $(\Omega, \mathbf{L})$ *is real analytic, and assume that*

$$\mathcal{O}(H_{\mathrm{Re}\,\mathbf{L}}, \gamma) \subseteq \mathrm{Char}\,\mathbf{L}.$$

If u *is a solution, then*

$$\gamma \in WF_a u \Leftrightarrow \mathcal{O}(H_{\mathrm{Re}\,\mathbf{L}}, \gamma) \subseteq WF_a u.$$

Here $WF_a u$ denotes the analytic wavefront set of u as defined in [9].

Proof. This follows from Theorem 1.2. Indeed, for solutions of a real analytic structure, the notion of microlocal analyticity coincides with that of microlocal hypoanalyticity, as demonstrated in [1] and [6]. □

3. A Lemma on Microlocal Hypoanalyticity

Let (Ω, Z) be as in Section 1 with the Z_j given by (1.1). We first briefly recall Sato's version of microlocal hypoanalyticity (see [1] for details).

We assume $\Omega = U \times W$, where U is an open ball about 0 in x-space in $\mathbf{R}^m$ and W is one about 0 in y-space $\mathbf{R}^n$. Microlocal hypoanalyticity is defined for distributions in the maximally real manifold U.

In what follows Γ is a nonempty, acute and open cone in $R_m \setminus \{0\}$. For A an open subset of U, we shall use the notation

$$N_\delta(A, \Gamma) = \{Z(x) + \sqrt{-1}\,Z_x(x)v : x \in A,\ v \in \Gamma,\ |v| < \delta\}.$$

In this section, $Z(x) = Z(x, 0)$.

DEFINITION 3.1. We denote by $B_\delta(A,\Gamma)$ the space of holomorphic functions f in $N_\delta(A,\Gamma)$ satisfying the condition: To every compact subset K of $N_\delta(A,\Gamma)$ there exists an integer $k \geq 0$ and a constant $c > 0$ such that

$$|f(z)| \leq c(\operatorname{dist}[z, Z(A)])^{-k} \quad \text{for all } z \text{ in } K.$$

In [1] it is shown that if A is small enough and $f \in B_\delta(A,\Gamma)$ then for every $\psi \in C_c^\infty(A)$,

$$\lim_{t \to +0} \int_A f(Z(x) + \sqrt{-1}\, Z_x(x) t v)\, \psi(x)\, dZ(x)$$

exists and is independent of $v \in \Gamma$. The notation bf will be used for this limit distribution.

DEFINITION 3.2. Let $u \in \mathfrak{D}'(U)$ and $(x, \xi) \in U \times (R_m \backslash \{0\})$. u is said to be hypoanalytic at (x, ξ) if there is an open neighborhood $A \subseteq U$ of x, $\delta > 0$, and a finite collection of nonempty acute open cones Γ_k in $R_m \backslash \{0\}$ $(k = 1, \ldots, r)$ such that the following hold:

(a) for every k and every $v \in \Gamma_k$, $\xi \cdot v < 0$;
(b) for each k there is an $f_k \in B_\delta(A, \Gamma_k)$ such that $u = bf_1 + \cdots + bf_r$ in A.

We remark that the preceding definition of microlocal hypoanalyticity does not depend on the chart (U, Z).

DEFINITION 3.3. Let $u \in \mathfrak{D}'(U)$. The hypoanalytic wavefront set of the distribution u is denoted by $WF_{ha}u$ and is defined by

$$WF_{ha}u = \{(x, \xi) \in U \times (R_m \backslash \{0\}) : u \text{ is not hypoanalytic at } (x, \xi)\}.$$

We next recall the Fourier transform criterion of hypoanalyticity. First, we contract the neighborhood U about 0 sufficiently so that the mapping

$$Z = (Z_1, \ldots, Z_m) : U \to C^m$$

is a diffeomorphism onto $Z(U)$.

In (1.1) the Z_j were chosen so that the Jacobian $Z_x(0)$ is the identity matrix. We now select the Z_j so that all the derivatives of the ϕ_j up to order 2 vanish at zero. Indeed, it suffices to replace the Z_j of (1.1) by

$$Z_j - \frac{\sqrt{-1}}{2} \sum_k \sum_l \frac{\partial^2 \phi_j}{\partial x_k \partial x_l}(0) Z_k Z_l.$$

(In the notation of (1.1), here $\phi_j = y_j$ when $1 \leq j \leq r$.) We will use Z_x^* to denote the transpose of the inverse of the matrix Z_x.

In what follows, $\mathbf{C}_m$ and R_m will denote respectively the duals of complex m space $\mathbf{C}^m$ and real space R^m.

For ζ in $\mathbf{C}^m$ (or $\mathbf{C}^m$), ζ^2 will denote the sum $\sum_{j=1}^m \zeta_j^2$. For ζ in $\mathbf{C}_m$ with $|\operatorname{Im} \zeta| < |\operatorname{Re} \zeta|$, the notation $\langle \zeta \rangle$ will be used for the holomorphic branch of the square root of ζ^2 that agrees with $|\xi|$ when ξ is in R_m.

Since the first and second derivatives of all the ϕ_j are zero at the origin, after contracting U if necessary we can find a number K $(0<K<1)$ such that for all $x, y\in U$ and for all $\xi\in R_m$,

$$
\begin{aligned}
&|\operatorname{Im} Z_x^*(x)\xi|<K|\operatorname{Re} Z_x^*(x)\xi| \\
&\text{and} \\
&\operatorname{Re}\{\sqrt{-1}\,Z_x^*(x)\xi\cdot(Z(x)-Z(y))-\langle Z_x^*(x)\xi\rangle(Z(x)-Z(y))^2\} \\
&\qquad\qquad\leq -K|\xi||Z(x)-Z(y)|^2.
\end{aligned} \tag{3.1}
$$

Let $u\in\mathcal{E}'(U)$. The integral

$$F(u,z,\zeta)=\int_U \exp(\sqrt{-1}\,\zeta(z-Z(y))-\langle\zeta\rangle(z-Z(y))^2)u(y)\,dZ(y)$$

is said to be the Fourier–Bros–Iagolnitzer (in short, FBI) transform of u (see [9] and [1]). Here $z\in C^m$ and $\zeta\in C_m$ with $|\operatorname{Im}\zeta|<|\operatorname{Re}\zeta|$.

In [1] the authors showed the equivalence between exponential decay in the FBI transform of u and microlocal hypoanalyticity as defined in this section. They established the following theorem.

THEOREM 3.1 [1]. *The following two properties are equivalent:*

(i) *u is hypoanalytic at $(0,\xi^0)$ for $\xi^0\neq 0$.*

(ii) *There are open neighborhoods V of 0 in C^m, a conic open neighborhood $\mathcal{C}_0$ of ξ_0 in C_m, and constants $c, r>0$ such that $|F(u,z,\zeta)|\leq c\exp(-r|\zeta|)$ for all z in V and for all ζ in $\mathcal{C}_0$.*

We emphasize here that Theorem 3.1 is a statement about the central point 0, and indeed in [1] the vanishing of the derivatives of the ϕ_j at 0 was used in the proof. For the proof of Theorem 1.1 we will need the following lemma. We assume that the neighborhood U has been contracted so that (3.1) holds.

LEMMA 3.1. *There is a neighborhood U' of 0, $U'\subseteq U$, such that for any $u\in\mathcal{E}'(U')$, the following properties are equivalent:*

(i) *u is hypoanalytic at $(x_0,\xi_0)\in U'\times R_m$ for $\xi^0\neq 0$.*

(ii) *There is an open neighborhood V of $Z(x_0)$ in C^m, a conic open neighborhood $\mathcal{C}_0$ of $Z_x^*(x_0)\xi^0$ in C_m, and constants $c, r>0$ such that*

$$|F(u,z,\zeta)|\leq c\exp(-r|\zeta|)$$

for all z in V and for all ζ in $\mathcal{C}_0$.

Proof. Suppose (i) holds. According to the definition, it suffices to prove the result when u is the boundary value of a holomorphic function f of tempered growth defined in a set of the form

$$\{Z(x)+\sqrt{-1}\,Z_x(x)v\colon x+\sqrt{-1}\,v\in(W+\sqrt{-1}\,\Gamma),\ |v|<\delta_0\},$$

where $W \subseteq U$ is an open neighborhood of x_0, δ_0 is a positive number, and Γ is an acute open cone in $R_m \setminus \{0\}$ such that for every $v \in \Gamma$, $\xi^0 \cdot v < 0$. Thus, for $\phi \in C_c^\infty(W)$,

$$\langle u, \phi \rangle = \lim_{t \to +0} \int f(Z(x) + t\sqrt{-1}\, Z_x(x)v)\phi(x)\, dZ(x) \quad \text{for } v \in \Gamma.$$

After contracting Γ if necessary, we may assume that there is a number $c_0 > 0$ such that $\xi^0 \cdot v \le -c_0 |v| |\xi^0|$ whenever v is in Γ. We shall need the following lemma.

LEMMA 3.1′. *Suppose $u \in \mathcal{E}'(U)$ vanishes in an open neighborhood of $x_0 \in U$. Then there is an open neighborhood V of $Z(x_0)$ in C^m, a conic neighborhood $\mathcal{C}$ of $\{Z_x^*(x_0)\xi : \xi \in R_m \setminus \{0\}\}$ in C_m, and constants c, r such that $|F(u, z, \zeta)| \le ce^{-r|\zeta|}$ for all z in V and for all ζ in $\mathcal{C}$.*

Proof. For $z \in C^m$, $\zeta \in C_m$, and $|\mathrm{Im}\,\zeta| < |\mathrm{Re}\,\zeta|$, we consider the FBI

$$F(u, z, \zeta) = \int \exp(\sqrt{-1}\,\zeta \cdot (z - Z(y)) - \langle \zeta \rangle (z - Z(y))^2) u(y)\, dZ(y).$$

Let

$$Q(z, \zeta, y) = \mathrm{Re}\left\{ \sqrt{-1} \frac{\zeta}{|\zeta|} \cdot (z - Z(y)) - \frac{\langle \zeta \rangle}{|\zeta|} (z - Z(y))^2 \right\}.$$

We first freeze z to $Z(x_0)$ and ζ to $Z_x^*(x_0) \cdot \xi^0$ for some $\xi^0 \in R_m$, $|\xi^0| = 1$:

$$Q(Z(x_0), Z_x^*(x_0)\xi^0, y)$$
$$= \mathrm{Re}\left\{ \sqrt{-1} \frac{Z_x^*(x_0)\xi^0}{|Z_x^*(x_0)\xi^0|} (Z(x_0) - Z(y)) - \frac{\langle Z_x^*(x_0)\xi^0 \rangle}{|Z_x^*(x_0)\xi^0|} (Z(x_0) - Z(y))^2 \right\}.$$

Condition (3.1) tells us that

$$Q(Z(x_0), Z_x^*(x_0)\xi^0, y) \le -K|Z(x_0) - Z(y)|^2.$$

Suppose d is a positive number such that $|y - x_0| \ge d$ whenever $y \in \mathrm{supp}\, u$. Then, in the support of u, $Q(Z(x_0), Z_x^*(x_0)\xi^0, y) \le -Kd^2$. By continuity, there are open neighborhoods $\tilde{V}$ of $Z(x_0)$ in C^m and $\tilde{\mathcal{C}}$ of $Z_x^*(x_0)\xi^0$ in C_m such that

$$Q(z, \zeta, y) \le -\frac{Kd^2}{2} \quad \text{for all } z \text{ in } \tilde{V},\ \zeta \text{ in } \tilde{\mathcal{C}}.$$

By compactness of the unit sphere in R_m, we may assume that the open set $\tilde{\mathcal{C}}$ contains the set $\{Z_x^*(x_0)\xi : \xi \in R_m, |\xi| = 1\}$. Moreover, the homogeneity of Q implies that there is a conic neighborhood $\mathcal{C}$ of $\{Z_x^*(x_0)\xi : \xi \in R_m \setminus \{0\}\}$ in C_m such that

$$\mathrm{Re}\{\sqrt{-1}\,\zeta \cdot (z - Z(y)) - \langle \zeta \rangle (z - Z(y))^2\} \le -\frac{Kd^2}{2}|\zeta|$$

whenever z is in $\tilde{V}$ and ζ is in $\mathcal{C}$. This gives us the required decay of $F(u, z, \zeta)$. □

Proof of Lemma 3.1. Let $g \in C_c^\infty(W)$ with $g \equiv 1$ near x_0. Since $(1-g)u$ vanishes near x_0, by Lemma 3.1′ we know that $F((1-g)u, z, \zeta)$ decays exponentially in the sets of interest. Therefore it suffices to show a similar decay for $F(gu, z, \zeta)$. Let $\chi \in C_c^\infty(W)$ with $\chi \equiv 1$ near x_0 and $\operatorname{supp}\chi \subseteq \{x : g(x) \equiv 1\}$. Fix $v \in \Gamma$ with $|v| = 1$. When s is a suitably small positive number, we can deform the contour of integration in $F(gu, z, \zeta)$ under the mapping

$$Z(y) \to \tilde{Z}(y) = Z(y) + \sqrt{-1}\, s Z_y(y)\chi(y)v.$$

Thus

$$\begin{aligned} &F(gu, z, \zeta) \\ &\quad = \int_U \exp(\sqrt{-1}\,\zeta\cdot(z-\tilde{Z}(y)) - \langle\zeta\rangle(z-\tilde{Z}(y))^2) f(\tilde{Z}(y))\cdot g(y)\, d\tilde{Z}(y). \end{aligned}$$

We focus on the quantity

$$Q(z, \zeta, y, s) = \operatorname{Re}\left\{\sqrt{-1}\frac{\zeta}{|\zeta|}\cdot(z-\tilde{Z}(y)) - \frac{\langle\zeta\rangle}{|\zeta|}\cdot(z-\tilde{Z}(y))^2\right\}$$

and write it as $Q = Q_1 + Q_2$, where

$$Q_1(z, \zeta, y) = \operatorname{Re}\left\{\sqrt{-1}\frac{\zeta}{|\zeta|}\cdot(z-Z(y)) - \frac{\langle\zeta\rangle}{|\zeta|}\cdot(z-Z(y))^2\right\}$$

and

$$\begin{aligned} Q_2(z, \zeta, y, s) = \operatorname{Re}\Bigg\{ &\frac{\zeta}{|\zeta|}\cdot(sZ_y(y)\chi(y)v) \\ &+ \frac{\langle\zeta\rangle}{|\zeta|}[2\sqrt{-1}\,s(z-Z(y))\cdot(\chi(y)Z_y(y)v) \\ &\qquad + s^2|\chi(y)Z_y(y)v|^2]\Bigg\}. \end{aligned}$$

We first consider these quantities when $z = Z(x_0)$, $\zeta = Z_x^*(x_0)\cdot\xi^0$, and y varies in the support of g. From (3.1) we have

$$Q_1(Z(x_0), Z_x^*(x_0)\xi^0, y, s) \le -K|Z(x_0) - Z(y)|^2.$$

To estimate $Q_2(Z(x_0), Z_x^*(x_0)\xi^0, y, s)$, we note that for s sufficiently small, say $0 < s \le s_0$,

$$\begin{aligned} Q_2(Z(x_0), Z^*(x_0)\xi^0, x_0, s) &= \operatorname{Re}\left\{\frac{s(\xi^0\cdot v)}{|Z_x^*(x_0)\xi^0|} + \frac{\langle Z_x^*(x_0)\xi^0\rangle}{|Z_x^*(x_0)\xi^0|}s^2|Z_x(x_0)v|^2\right\} \\ &\le -sc_0/4. \end{aligned}$$

Therefore, by continuity we can find a number $d > 0$ satisfying

$$|y - x_0| \le d \Rightarrow Q_2(Z(x_0), Z_x^*(x_0)\xi^0, y, s) \le -sc_0/4.$$

We may assume that $\chi(y) = 1$ whenever $|y - x_0| \le d$. On the other hand, for each y,

$$Q_2(Z(x_0), Z_x^*(x_0)\xi^0, y, s) \le 4s\chi(y)(|Z(x_0) - Z(y)| + s).$$

Hence, when $|y-x_0|\le d$,

$$Q(Z(x_0), Z_x^*(x_0)\xi^0, y, s) \le -K|Z(x_0)-Z(y)|^2 - sc_0/2,$$

while if $|y-x_0|\ge d$ then $|Z(y)-Z(x_0)|\ge d$, so that

$$Q(Z(x_0), Z_x^*(x_0)\xi^0, y, s) \le -Kd|Z(x_0)-Z(y)| + 4s\chi(y)(|Z(x_0)-Z(y)|+s).$$

Therefore, by choosing s small in comparison with d, we get a positive number δ such that

$$Q(Z(x_0), Z_x^*(x_0)\xi^0, y, s) \le -\delta \quad \text{when } y \in \operatorname{supp} g.$$

By continuity, there are open neighborhoods $\tilde{V}$ of $Z(x_0)$ in C^m and $\tilde{\mathcal{C}}$ of $Z_x^*(x_0)\xi^0$ in C_m such that $Q(z,\zeta,y,s)\le -\delta/2$ for all $z\in\tilde{V}$, $\zeta\in\tilde{\mathcal{C}}$ and for all $y\in\operatorname{supp} g$. Now Q is positive homogeneous of degree 0 in ζ. Therefore, there is an open conic neighborhood $\mathcal{C}$ of $Z_x^*(x_0)\xi^0$ in C_m such that

$$\operatorname{Re}\{\sqrt{-1}\,\zeta\cdot(z-\tilde{Z}(y)) - \langle\zeta\rangle(z-\tilde{Z}(y))^2\} \le -\frac{\delta}{2}|\zeta|$$

whenever z is in $\tilde{V}$ and ζ in $\mathcal{C}$. From this we get the required decay of $F(u,z,\zeta)$.

That (ii) implies (i) can be seen by a slight modification of the arguments used in [1] and [9] to prove Theorem 3.1. We will therefore only give a short outline. The main idea is to use the inversion of the FBI transform. For M a compact neighborhood of 0 in U, define the set T_M by

$$T_M = \{(z,\zeta): z = Z(x),\ \zeta = Z_x^*(x)\xi \text{ for some } (x,\xi)\in M\times R_m\backslash\{0\}\}.$$

$\Delta(z,\zeta)$ will denote the Jacobian $\det(\partial\theta/\partial\zeta)$, where $\theta=\zeta+\sqrt{-1}\langle\zeta\rangle z$. For $\delta>0$ and $h\in\mathcal{E}'(U)$, define the holomorphic function

$$\begin{aligned} h_M^\delta(z) = (4\pi^3)^{-m/2}\int_{T_M}\!\!\int &\exp(\sqrt{-1}(z-w)\cdot\zeta - \langle\zeta\rangle(z-w)^2 - \delta\langle\zeta\rangle^2) \\ &\cdot F(h,w,\zeta)\langle\zeta\rangle^{m/2}\Delta(z-w,\zeta)\,dw\,d\zeta \end{aligned} \tag{3.2}$$

We will need the following lemma from [12].

LEMMA 3.1″. *Let M be as above. There exist two open sets U_0 and U_1 containing 0, with $U_0\subseteq U_1\subseteq U$, such that if $h\in\mathcal{E}'(U_1)$ then $h_M^\delta\circ Z\to h+f\circ Z$ in $\mathcal{D}'(U_0)$, where f is holomorphic in an open set in C^m containing $Z(U_0)$.*

Fix M, U_0, U_1 as in the lemma, and let $U'=M\cap U_0$. Let x_0 be in U' and suppose that (ii) holds for x_0 and $u\in\mathcal{E}'(U')$. Define $\Gamma_0=(V\times\mathcal{C}_0)\cap T_M$, where V and $\mathcal{C}_0$ are the neighborhoods of $Z(x_0)$ and $Z_x^*(x_0)\xi^0$ (respectively) satisfying (ii).

Let $\Gamma_1,\dots,\Gamma_k$ be conically compact sets such that

(i) $T_M=\bigcup_{j=0}^k\Gamma_j$;
(ii) $\operatorname{measure}(\Gamma_i\cap\Gamma_j)=0$ when $i\ne j$; and
(iii) for each $j=1,\dots,k$ there is a convex open subset $\Gamma_j'\subseteq U_0\times\mathbf{R}^m$ whose base contains x_0 and which satisfies the following property:

$$v\cdot\xi^0<0, \qquad v\cdot\xi\geq d|v||\xi| \quad (d>0) \tag{3.3}$$

if $(x,v)\in\Gamma_j'$ and $(Z(x),Z_x^*(x)\xi)\in\Gamma_j$ for some x.

Let u_j^δ denote the integral (3.2) in which the integration is carried out over Γ_j. For $u_0^\delta(z)$ we use (3.1) and (ii) to estimate the integrand as follows: If $z=Z(x_0)$ and $(w,\zeta)=(Z(x),Z_x^*(x)\xi)\in\Gamma_0$, then

$$\begin{aligned}&|\exp(\sqrt{-1}(z-w)\cdot\zeta-\langle\zeta\rangle(z-w)^2)F(u,w,\zeta)\langle\zeta\rangle^{m/2}\Delta(z-w,\zeta)|\\&\qquad\leq\mathrm{const}|\zeta|^{m/2}\exp(-K|\xi||Z(x_0)-Z(x)|^2-r|Z_x^*(x)\xi|).\end{aligned}$$

Recall also that $Z_x^*(x)$ is very close to the identity matrix. Hence when $\delta\to 0$, u_0^δ converges uniformly to a holomorphic function in a neighborhood of $z=Z(x_0)$. Fix $j\geq 1$. We will show that $u_j^\delta(z)$ converges uniformly on the set $\{z: z=Z(x)+\sqrt{-1}Z_x(x)v, (x,v)\in\Gamma_j'\}$ when v is small.

Let $(x,v)\in\Gamma_j'$. If $z=Z(x)+\sqrt{-1}Z_x(x)v$ and $(w,\zeta)=(Z(x'),Z_x^*(x)\xi)\in\Gamma_j$, then using (3.1) and (3.3) one gets the estimate (see [12])

$$\begin{aligned}&|\exp(\sqrt{-1}(z-w)\cdot\zeta-\langle\zeta\rangle(z-w)^2)F(u,w,\zeta)\langle\zeta\rangle^{m/2}\Delta(z-w,\zeta)|\\&\qquad\leq\mathrm{const}(1+|\zeta|)^{N-m-1}\exp(-\tilde{c}(|v|+|z-w|^2)|\zeta|).\end{aligned}$$

Here N is the order of the distribution u.

This estimate implies that as $\delta\to 0$, the function $|v|^N u_j^\delta(Z(x)+\sqrt{-1}Z_x(x)v)$ converges uniformly. This proves that the limit of $u_j^\delta(Z(x))$ exists in a neighborhood of x_0 and is the boundary value of a tempered holomorphic function in a set of the form

$$\{Z(x)+\sqrt{-1}Z_x(x)v: (x,v)\in\Gamma, |v|<\delta\}.$$

It follows that $(x_0,\xi^0)\notin WF_{ha}u$. □

4. Proof of Theorems 1.1 and 1.2

In this section (Ω,Z) is as in Section 1 with $Z=(Z_1,\ldots,Z_m)$ satisfying (1.1). To exploit the theorems of Trépreau, we will begin by first associating a *CR* structure to $\Omega'=\Omega\times T$, where T is a neighborhood of 0 in $\mathbf{R}^{n-r}$. We will use the variable $t=(t_{r+1},\ldots,t_n)$ for points in T. The structure bundle in Ω' is generated by

$$Z_j=x_j+\sqrt{-1}y_j \quad \text{for } 1\leq j\leq r;$$

$$Z_k=x_k+\sqrt{-1}\phi_k(x,y) \quad \text{for } r+1\leq k\leq m;$$

$$W_k=t_{r+k}+\sqrt{-1}y_{r-k} \quad \text{for } 1\leq k\leq n-r.$$

We will use the notation

$$\tilde{Z}(x,y,t)=(Z(x,y),W(t,y)).$$

We recall that the bundle $\mathbf{L}$ had as a basis:

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=r+1}^{m} b_{jk} \frac{\partial}{\partial x_k}, \quad 1 \le j \le r;$$

$$L_j = \frac{\sqrt{-1}}{2} \frac{\partial}{\partial y_j} + \sum_{k=r+1}^{m} b'_{jk} \frac{\partial}{\partial x_k}, \quad r+1 \le j \le n.$$

The structure associated to Ω' has a bundle $\mathcal{V}$ with basis:

$$V_j = L_j, \quad 1 \le j \le r;$$

$$V_k = \frac{1}{2}\left(\frac{\partial}{\partial t_k} + \sqrt{-1}\frac{\partial}{\partial y_k}\right) + \sum_{j=r+1}^{m} b'_{kj} \frac{\partial}{\partial x_j}, \quad r+1 \le k \le n.$$

We use $(x', x'', y; \xi', \xi'', \eta)$ for points in $T^*\Omega$, where $x' = (x_1, \dots, x_l)$, $x'' = (x_{l+1}, \dots, x_m)$, $\xi' = (\xi_1, \dots, \xi_l)$, and $\xi'' = (\xi_{l+1}, \dots, \xi_m)$. Lemma 4.2 will show that if N is an orbit of $\operatorname{Re} \mathbf{L}$ through 0, coordinates (x', x'', y) can be found for Ω in which N is defined by $x'' = 0$.

Points in $T^*(\Omega \times T)$ will be denoted by $(x', x'', y, t; \xi', \xi'', \eta, \tau)$, where x', x'', ξ', ξ'' are as above and $\tau = (\tau_{r+1}, \dots, \tau_n)$ is dual to t. With this notation, we can write

$$T_N^*\Omega = \{(x', 0, y; 0, \xi'', 0) : (x', 0, y) \in N\} \quad \text{and}$$

$$T_{N\times T}^*(\Omega \times T) = \{(x', 0, y, t; 0, \xi'', 0, 0) : (x', 0, y, t) \in N \times T\}.$$

In the following lemma,

$$\sigma = (x_1^0, \dots, x_l^0, 0, y^0; 0, \xi_{l+1}^0, \dots, \xi_m^0, 0) \quad \text{and}$$

$$\tilde{\sigma} = (x_1^0, \dots, x_l^0, 0, y^0, 0; 0, \xi_{l+1}^0, \dots, \xi_m^0, 0, 0).$$

LEMMA 4.1. *Let u be a solution in $(\Omega, \mathbf{L})$ and let $\tilde{u}(x, y, t) = u(x, y)$. Then*

$$\tilde{\sigma} \notin WF_{ha}\tilde{u} \text{ in } (\Omega \times T, \mathcal{V}) \Leftrightarrow \sigma \notin WF_{ha} u \text{ in } (\Omega, \mathbf{L}).$$

Proof. Lemma 3.1 enables us to use the FBI transform. This may require the contraction of Ω about the origin. However, we note that we need only prove the propagation of Theorem 1.1 in some neighborhood independent of the solution u.

Since $\tilde{u}$ is a solution for $(\Omega \times T, \mathcal{V})$, we may use the maximally real manifold $\tilde{X} = \{(x, y^0, t)\}$ which contains the base projection of $\tilde{\sigma}$. For u we may use $X = \{(x, y^0)\}$. Let $(z, w) \in C^m \times C^{n-r}$ and $(\zeta, \tau) \in C_m \times C_{n-r}$ denote variable points.

The FBI transform of $\tilde{u}(x, y^0, t) = u(x, y^0)$ in $\tilde{X}$ can be factored as

$$F(\tilde{u}, z, w, \zeta, \tau) = I_1(z, \zeta, \tau) \cdot I_2(w, \zeta, \tau),$$

where

$$I_1(z, \zeta, \tau) = \int \exp(\sqrt{-1}\zeta \cdot (z - Z(x, y_0)) - \langle \zeta, \tau \rangle (z - Z(x, y_0))^2) u(x, y_0)\, dZ(x, y_0)$$

and

$$I_2(w,\zeta,\tau) = \int_T \exp(\sqrt{-1}\tau\cdot(w-W(t,y_0))-\langle\zeta,\tau\rangle(w-W(t,y_0))^2)\,dt_{r+1}\dots dt_n.$$

If $\sigma\notin WF_{ha}u$, Lemma 3.1 tells us that there exists a neighborhood V of $Z(x_1^0,\dots,x_l^0,0,y^0)$ in C^m and a conic open neighborhood $\mathcal{C}$ of $\zeta^0=Z_x^*(x_1^0,\dots,x_l^0,0,y^0)(0,\xi_{l+1}^0,\dots,\xi_m^0)$ in C_m, together with positive constants c_1 and c_2, such that $|I_1(z,\zeta,0)|\le c_1\exp(-c_2|\zeta|)$ for $z\in V$ and $\zeta\in\mathcal{C}$. The factor I_2 satisfies an estimate of the form $|I_2(w,\zeta,\tau)|\le d_1e^{d_2|\tau|}$ when $w=W(0,y_0)$ and (ζ,τ) satisfies $|\mathrm{Re}(\zeta,\tau)|>|\mathrm{Im}(\zeta,\tau)|$. Indeed, for such (ζ,τ), $\mathrm{Re}\langle\zeta,\tau\rangle\ge 0$.

Therefore, for each $\epsilon>0$, there is a neighborhood V_ϵ of $W(0,y_0)$ such that $|I_2(w,\zeta,\tau)|\le d_1e^{d_2|\tau|+\epsilon|\zeta|}$ whenever $w\in V_\epsilon$, $|\mathrm{Re}(\zeta,\tau)|>|\mathrm{Im}(\zeta,\tau)|$.

We now recall that

$$\tilde{Z}(x_1^0,\dots,x_l^0,0y^0,0)=(Z(x_1^0,\dots,x_l^0,0,y^0),W(0,y^0))$$

and

$$Z_{x,t}^*(x_1^0,\dots,x_l^0,0,y^0,0)(0,\xi_{l+1}^0,\dots,\xi_m^0,0) = (Z_x^*(x_1^0,\dots,x_l^0,0,y^0)(0,\xi_{l+1}^0,\dots,\xi_m^0),0).$$

The proof of Lemma 3.1 shows that I_1 satisfies an estimate $|I_1(z,\zeta,\tau)|\le c_1\exp(-c_2|\zeta|)$ for z in V a neighborhood of 0 in C^m and (ζ,τ) in a conic neighborhood of $Z_x^*(x_1^0,\dots,x_l^0,0,y^0)(0,\xi_{l+1}^0,\dots,\zeta_m^0)$ in $C_m\times C_{n-r}$, where $|\tau|<\delta|\zeta|$ for some $\delta>0$.

These estimates on I_1 and I_2 imply that there are constants $r_1,r_2>0$ such that

$$|F(\tilde{u},z,w,\zeta,\tau)|\le r_1\exp(-r_2|(\zeta,\tau)|) \tag{4.1}$$

for (z,w) near $\tilde{Z}(x_1^0,\dots,x_l^0,0,y^0,0)$ and (ζ,τ) in a conic neighborhood of $(\zeta^0,0)$ in $C_m\times C_{n-r}$. It follows that $\tilde{\sigma}\notin WF_{ha}\tilde{u}$.

Suppose now that $\tilde{\sigma}\notin WF_{ha}\tilde{u}$. Then Lemma 3.1 tells us that

$$F(\tilde{u},z,w,\zeta,\tau)=I_1(z,\zeta,\tau)I_2(w,\zeta,\tau)$$

decays exponentially as in (4.1). In particular, $F(\tilde{u},z,W(0,y_0),\zeta,0)$ decays exponentially for z near $Z(x_1^0,\dots,x_l^0,0,y^0)$ and ζ in a conic neighborhood of ζ^0. Since the t component of $\tilde{\sigma}$ is 0, we may contract T around 0 as much as we wish in the integral I_2. It follows that $I_1(z,\zeta,0)$ decays exponentially for z near $Z(x_1^0,\dots,x_l^0,0,y^0)$ and ζ in a conic neighborhood of ζ^0. Hence $\sigma\notin WF_{ha}u$. □

LEMMA 4.2. *If N is an orbit through 0 for* $\mathrm{Re}\,\mathbf{L}$ *in Ω, then $N\times T$ is an orbit for* $\mathrm{Re}\,\mathcal{V}$ *in $\Omega\times T$.*

Proof. We begin by first finding coordinates in Ω that flatten N and leave the "form" of $\mathbf{L}$ unchanged. Since $\mathbf{L}|_N\subseteq CTN$ and the fiber dimension of $\mathbf{L}\ge n$,

$\dim N = n + l$ for some $0 \le l \le m$. Let N be defined by $h_1 = \cdots = h_{m-l} = 0$. From $L_j h_k = 0$ on N we get

$$\frac{\partial h_k}{\partial x_j}(0) = 0 \quad \forall k = 1, \ldots, m-l \text{ and } j = 1, \ldots, r;$$

$$\frac{\partial h_k}{\partial y_j}(0) = 0 \quad \forall k = 1, \ldots, m-l \text{ and } j = 1, \ldots, n.$$

Since also the differentials of $h_1, \ldots, h_{m-l}$ are independent near 0, we have

$$\operatorname{rank}\left(\frac{\partial h_j}{\partial x_k}(0)\right)_{\substack{1 \le j \le m-l \\ r+1 \le k \le m}} = m - l.$$

Therefore, after a possible permutation of the variables $x_{r+1}, \ldots, x_m$ and using the implicit function theorem, we get functions $g_1, \ldots, g_{m-l}$ of $x_1, \ldots, x_l, y_1, \ldots, y_n$ such that

$$h_j(x, y) = 0 \Leftrightarrow x_{l+j} = g_j(x_1, \ldots, x_l, y_1, \ldots, y_n), \quad 1 \le j \le m-l.$$

Change coordinates in Ω by the mapping $(x, y) \mapsto (\tilde{x}, \tilde{y})$, so that

$$\tilde{x}_j = \begin{cases} x_j, & 1 \le j \le l, \\ x_j - g_{j-l}(x, y), & l+1 \le j \le m; \end{cases}$$

$$\tilde{y}_k = y_k \quad \forall k = 1, \ldots, n.$$

After dropping the tildes, we have found coordinates $(x, y) \in \mathbf{R}^m \times \mathbf{R}^n$ which flatten

$$N = \{(x, y) \in \Omega : x_j = 0,\ j = l+1, \ldots, m\},$$

and the vector fields L_j take the form

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=r+1}^{l} a_{jk} \frac{\partial}{\partial x_k} + \sum_{k=l+1}^{m} b_{jk} \frac{\partial}{\partial x_k} \quad \text{for } 1 \le j \le r;$$

$$L_j = \frac{\sqrt{-1}}{2} \frac{\partial}{\partial y_j} + \sum_{k=r+1}^{l} \tilde{a}_{jk} \frac{\partial}{\partial x_k} + \sum_{k=l+1}^{m} \tilde{b}_{jk} \frac{\partial}{\partial x_k} \quad \text{for } r+1 \le j \le n,$$

with $b_{jk}(x_1, \ldots, x_l, 0, y) = \tilde{b}_{jk}(x_1, \ldots, x_l, 0, y) = 0$.

In these coordinates, $\mathcal{V}$ is still spanned by

$$V_j = L_j \text{ for } 1 \le j \le r \quad \text{and}$$

$$V_k = \frac{1}{2} \frac{\partial}{\partial t_k} + L_k \text{ for } r+1 \le k \le n,$$

where the L_j and L_k are as above.

Recall from Sussmann [10] that given $m_0 \in N$ there exist $S = (S_1, \ldots, S_q) \in \mathbf{R}^q$ for some q, $\xi = (X^1, \ldots, X^q)$ (each $X^j \in \operatorname{Re} \mathbf{L}$), $m \in N$, and $\delta > 0$ such that $\rho_{\xi, m}(B_\delta(T)) = \Lambda$ is a neighborhood of m_0 in the orbit N. Here, for $s \in \mathbf{R}^q$, $\rho_{\xi, m}(s) = X^q_{s_q} \ldots X^1_{s_1} m$, $\rho_{\xi, m}(S_1, \ldots, S_q) = X^q_{S_q} \ldots X^{\prime}_{S_1} m = m_0$, and $B_\delta(S) = \{s \in \mathbf{R}^q : |s - S| < \delta\}$. Hence, if $m' \in \Lambda$, $\exists s' = (s_1', \ldots, s_q')$ such that

$$m' = X^q_{s'_q} \dots X^1_{s'_1} m = X^q_{s'_q} \dots X^1_{s'_1} X^1_{-s_1} \dots X^q_{-s_q} m_0.$$

Let

$$A = \{\operatorname{Re} L_1, \dots, \operatorname{Re} L_r, \operatorname{Im} L_1, \dots, \operatorname{Im} L_n\}$$
$$= \{\operatorname{Re} V_1, \dots, \operatorname{Re} V_r, \operatorname{Im} V_1, \dots, \operatorname{Im} V_n\}$$

and

$$B = \{\operatorname{Re} V_{r+1}, \dots, \operatorname{Re} V_n\}.$$

Recall that $\operatorname{Re} V_{r+1} = \frac{1}{2}(\partial/\partial t_{r+j}) + \operatorname{Re} L_{r+j}$ for $j = 1, \dots, n-r$.

Suppose $(x', 0, y) \in N$ and $t \in T$. If $X \in A$ then

$$X_s(x', 0, y, t) = (X_s(x', 0, y), t).$$

On the other hand, if $Y \in B$, say $Y = \operatorname{Re} V_{r+1}$ for definiteness, then

$$Y_s(x', 0, y, t) = \left((\operatorname{Re} L_{r+1})_s(x', 0, y), t_{r+1} + \frac{s_{r+1}}{2}, t_{r+2}, \dots, t_n\right).$$

Moreover, since $\operatorname{Re} L_{r+1}(0) = 0$, we have

$$Y_s(0, 0, t) = \left(0, 0, t_{r+1} + \frac{s_{r+1}}{2}, t_{r+2}, \dots, t_n\right).$$

We apply the preceding conclusions to $m_0 = 0$ and contract N about 0 so that $N = \Lambda$. Given $(x', 0, y) \in N$ and $t = (t_{r+1}, \dots, t_n) \in T$, let

$$X^1_{s_1} \dots X^k_{s_k}(0, 0) = (x', 0, y).$$

Then $\tilde{X}^1_{s_1} \dots \tilde{X}^k_{s_k}(0, 0, 0) = (x', 0, y; \tau_{r+1}, \dots, \tau_n)$, where $\sum |\tau_j| < \delta/2$. Here, when $X = \operatorname{Re} L_j$, $\tilde{X} = \operatorname{Re} V_j$. It follows that

$$\tilde{X}^1_{s_1} \dots \tilde{X}^k_{s_k}(\operatorname{Re} V_{r+1})_{2t_{r+1} - 2\tau_{r+1}} \dots (\operatorname{Re} V_n)_{2t_n - 2\tau_n}(0, 0, 0) = (x', 0, y, t).$$

Hence $\Lambda \times T$ is an orbit for $\operatorname{Re} \mathcal{V}$. □

Finally, we state the two results of Trépreau in [13] that we will use.

THEOREM 4.1 [13, Thm. 10]. *Suppose M is a generic CR manifold in C^n and N is a CR submanifold of M, with CR dimension of N = CR dimension of M. If u is a CR function on M and $x^* \in T^*_N M \setminus \{0\}$, then*

$$x^* \in WF_{ha} u \Leftrightarrow \mathcal{O}(H_X, x^*) \subseteq WF_{ha} u,$$

where H_X is the family of Hamilton fields of the real parts of the CR vecior fields tangential to M.

THEOREM 4.2 [13, Thm. 7]. *Suppose M is a generic CR manifold in C^n and $\mathcal{L} \subseteq T^*_M C^n \setminus \{0\}$ is a minimal CR manifold in $T^* C^n$. If u is a CR function on M, we have*

$$\mathcal{L} \subseteq WF_{ha} u \Leftrightarrow \mathcal{L} \cap WF_{ha} u \neq \emptyset.$$

In Theorem 4.2, $T_M^* C^n$ is defined as follows: Let $T^* C^n$ denote the bundle of (1,0) forms $\theta = \sum_{j=1}^n \zeta_j dz_j$. Then

$$T_M^* C^n = \{(z, \theta) \in T^* C^n : z \in M, \operatorname{Im} \theta |_{T_z M} = 0\}.$$

Proof of Theorem 1.1. Let $N = \mathcal{O}(\operatorname{Re} \mathbf{L}, 0)$. We continue to use $(z, w) = (z_1, \dots, z_m, w_1, \dots, w_{n-r})$ for a variable point in $C^m \times C^{n-r}$. Let M' denote the image of

$$\tilde{Z} = \tilde{Z}(x, y, t) : \Omega \times T \to C^m \times C^{n-r}$$

in complex space $C^m \times C^{n-r}$. M' is a generic CR manifold with a CR vector bundle $= \tilde{Z}_* \mathcal{V}$. $N' = \tilde{Z}(N \times T)$ is a submanifold of M' and by Lemma 4.2, since $\tilde{Z}_* \mathcal{V}$ is tangential to N', we know that N' is a CR manifold with $CR \dim M' = CR \dim N'$. Suppose $\mathbf{L} u = 0$ in Ω. If u is defined on M' by $\tilde{u}(\tilde{Z}(x, y, t)) = u(x, y)$, then $\tilde{u}$ is a CR distribution on M'. Since the Hamilton fields $H_{\operatorname{Re} \mathcal{V}}$ are related to $H_{\operatorname{Re} \mathbf{L}}$ in the same way as $\operatorname{Re} \mathcal{V}$ is related to $\operatorname{Re} \mathbf{L}$, the arguments of Lemma 4.2 lead to the following conclusion:

$$\mathcal{O}(H_{\operatorname{Re} \mathcal{V}}, \tilde{\sigma}) = \{(x', 0, y, t; 0, \xi'', 0, 0) : (x', 0, y; 0, \xi'', 0) \in \mathcal{O}(H_{\operatorname{Re} \mathbf{L}}, \sigma)\},$$

where $\sigma \in T_N^* \Omega$ and $\tilde{\sigma} \in T_{N \times T}^* (\Omega \times T)$ as in Lemma 4.1. Theorem 1.1 now follows from Lemma 4.1 and Theorem 4.1 by transferring the problem to M', N' and $\tilde{u}$. □

Proof of Theorem 1.2. As in the proof of Theorem 1.1, we transfer the problem to M' and use the lemmas of this paper. Let Γ' be the subset of $T^*(\Omega \times T)$ defined by

$$\Gamma' = \{x, y, 0; \xi, \eta, 0) : (x, y, \xi, \eta) \in \Gamma\}.$$

Since $\Gamma \subset \operatorname{Char} \mathbf{L}$, we have $\Gamma' \subset \operatorname{Char} \mathcal{V}$. Let Γ'' = the transfer to M' by the map $\tilde{Z}$. The condition $\Gamma' \subseteq \operatorname{Char} \mathcal{V}$ implies that $\Gamma'' \subseteq T_M^* C^n \setminus \{0\}$. Theorem 1.2 now follows from Lemma 4.1 and Theorem 4.2. □

References

[1] M. S. Baouendi, C. H. Chang, and F. Treves, *Microlocal hypo-analyticity and extension of CR functions,* J. Differential Geom. 18 (1983), 331–391.

[2] M. S. Baouendi and L. P. Rothschild, *Normal forms for generic manifolds and holomorphic extension of CR functions,* J. Differential Geom. 25 (1987), 431–467.

[3] ———, *Cauchy–Riemann functions on manifolds of higher codimension in complex space,* Invent. Math. 101 (1990), 45–56.

[4] N. Hanges and J. Sjöstrand, *Propagation of analyticity for a class of non-microcharacteristic operators,* Ann. of Math. (2) 116 (1982), 559–577.

[5] N. Hanges and F. Treves, *Propagation of holomorphic extendability of CR functions,* Math. Ann. 263 (1983), 157–177.

[6] ———, *On the analyticity of solutions of first-order nonlinear pde,* Preprint, 1990.

[7] M. Marson, *Minimality and the extension of functions from generic manifolds,* Preprint, 1989.
[8] T. Nagano, *Linear differential systems with singularities and an application to transitive Lie algebras,* J. Math. Soc. Japan 18 (1966), 398–404.
[9] J. Sjöstrand, *Singularités analytiques micorlocales,* Astérisque 95 (1982), 1–166.
[10] H. Sussmann, *Orbits of families of vector fields and integrability of distributions,* Trans. Amer. Math. Soc. 180 (1973), 171–188.
[11] F. Treves, *Approximation and representation of functions and distributions annihilated by a system of complex vector fields,* Ecole Polytechnique, Centre de Mathématiques, Palaiseau, 1981.
[12] ———, *Hypo-analytic structures, Microlocal theory* (in preparation).
[13] J. M. Trépreau, *Sur la propagation des singularités dans les variétés CR,* Preprint, 1990.
[14] A. E. Tumanov, *Extending CR functions on manifolds of finite type to a wedge,* Mat. Sb. (N.S.) 136 (1988), 128–139 (Russian).

Department of Mathematics
Temple University
Philadelphia, PA 19122