Geodesic Excursions into Cusps in Finite-Volume Hyperbolic Manifolds

María V. Melián & Domingo Pestana

0. Introduction

Throughout, $\mathfrak{M}^{d+1}$ will be a fixed, complete, noncompact Riemannian manifold of constant negative sectional curvature and finite volume. Given a point p on $\mathfrak{M}$, we denote by $S(p)$ the unit ball of the tangent space of $\mathfrak{M}$ at p, and for every $v \in S(p)$ let $\gamma_v(t)$ be the geodesic emanating from p in the direction v. In this paper, we study the long time behaviour of $\gamma_v(t)$.

Sullivan proved in [S] that for almost every direction $v \in S(p)$, one has

$$\limsup_{t \to \infty} \frac{\operatorname{dist}(\gamma_v(t), p)}{\log t} = \frac{1}{d},$$

where dist is the distance in $\mathfrak{M}$. On the other hand, for just a countable number of directions $v \in S(p)$,

$$\limsup_{t \to \infty} \frac{\operatorname{dist}(\gamma_v(t), p)}{t} = 1.$$

We give a result interpolating between these two.

THEOREM 1. *For* $0 \le \alpha \le 1$,

$$\operatorname{Dim}\left\{v : \limsup_{t \to \infty} \frac{\operatorname{dist}(\gamma_v(t), p)}{t} \ge \alpha\right\} = d(1-\alpha).$$

Here and hereafter, Dim denotes Hausdorff dimension. Dimension refers here to the induced distance in $S(p)$. Also, we will use the notation M_α for α-dimensional content. We refer to [C] or [R] for definitions and background on these metrical notions.

Let $\mathbf{H}^{d+1}$ be the upper half plane of $\mathbf{R}^{d+1}$,

$$\mathbf{H}^{d+1} = \{(x_1, \dots, x_{d+1}) \in \mathbf{R}^{d+1} : x_{d+1} > 0\},$$

and let λ be the hyperbolic metric in $\mathbf{H}^{d+1}$,

$$d\lambda = \frac{|dx|}{x_{d+1}}.$$

Received January 3, 1991. Revision received October 21, 1992.
Research supported by a grant from CICYT, Ministerio de Educación y Ciencia, Spain.
Michigan Math. J. 40 (1993).

We will denote by Möb($\mathbf{H}^{d+1}$) the group of orientation-preserving Möbius transformations which map $\mathbf{H}^{d+1}$ on itself. It is well known that $\mathbf{H}^{d+1}$ is the unique (up to isometries and a constant conformal factor) simply connected complete Riemannian manifold of constant negative sectional curvature and $\mathfrak{M}^{d+1}=\mathbf{H}^{d+1}/\Gamma$, where Γ is a discrete subgroup of Möb($\mathbf{H}^{d+1}$) with parabolic elements (since $\mathfrak{M}^{d+1}$ is noncompact) and finite covolume; that is, the hyperbolic volume of a Dirichlet region D_a of Γ is finite. We recall that

$$D_a=\{x\in\mathbf{H}^{d+1}: \rho_{\mathbf{H}^{d+1}}(x,a)\le\rho_{\mathbf{H}^{d+1}}(\gamma(x),a) \text{ for all } \gamma\in\Gamma\},$$

where $a\in\mathbf{H}^{d+1}$ is a non-fixed point of Γ and $\rho_{\mathbf{H}^{d+1}}$ is the hyperbolic distance in $\mathbf{H}^{d+1}$.

We remark that for the cases $d=1,2$, if Γ is any discrete subgroup of Möb($\mathbf{H}^{d+1}$) then we can ensure that $\mathbf{H}^{d+1}/\Gamma$ is a Riemannian manifold. We refer to [A] and [B] for general background on Möbius Transformations.

Here is a brief description of the geometry at infinity of $\mathfrak{M}^2=\mathbf{H}^2/\Gamma$. It can be shown that $\mathfrak{M}^2=X_0\cup_{i=1}^k Y_i$, where X_0 is compact and Y_i is isometric to $S^1\times[a,+\infty)$ with the metric $dr^2+e^{-2r}\,d\theta^2$ [P]. The Y_i's are usually called *cusps*. Notice that the infimum of the lengths of curves in nontrivial free homotopy classes on each cusp is zero.

Moreover, given a fixed cusp $\mathcal{E}$ there exists a conjugacy class of maximal cyclic parabolic subgroups of Γ, usually also called a cusp, which contains a subgroup of Γ generated by a parabolic element γ with fixed point ξ in the limit set of Γ. Besides, there exists a Möbius transformation A such that $A(\infty)=\xi$ and $A^{-1}\circ\gamma\circ A$ is the translation $z\mapsto z+1$. Also, there exists a half-plane

$$U_c=\{z\in\mathbf{C}: \operatorname{Im} z>c\},$$

verifying that the image of $A(U_c)$ under $\pi:\mathbf{H}^2\to\mathbf{H}^2/\Gamma$, the canonical projection, is homeomorphic to $\mathcal{E}$ [K, p. 52].

By a theorem of H. Shimizu [K, p. 60] we have that the set

$$\bigcup\{g(U_c): g\in A^{-1}\circ\Gamma\circ A\setminus\{\text{identity}\}\}$$

consists of a pairwise disjoint and countable union of balls in $\mathbf{H}^2$ with diameter at most c. These balls are tangent to $\mathbf{R}$ in certain base-points a_i which are the parabolic fixed points fixed by the elements belonging to the conjugacy class in $A^{-1}\circ\Gamma\circ A$ of the translation $z\mapsto z+1$. Also, notice that

$$a_i=A^{-1}\circ\gamma_i\circ A(\infty) \quad \text{with } \gamma_i\in\Gamma\setminus\Gamma_\xi,$$

where $\Gamma_\xi=\{\gamma\in\Gamma:\gamma(\xi)=\xi\}$.

This description holds in higher dimensions. We have that a cusp $\mathcal{E}$ in $\mathbf{H}^{d+1}/\Gamma$ is isometric to $(S^1)^d\times[a,+\infty)$, and there exists a conjugacy class of infinite maximal parabolic subgroups of Γ associated to the cusp. Since Γ has finite covolume, each parabolic subgroup in the cusp is an abelian group with rank d. Besides, there exists a conjugate group $\bar{\Gamma}$ of Γ such that the inverse image of $\mathcal{E}$ by the canonical projection consists of a semispace above a hyperplane parallel to $\mathbf{R}^d$, at height c, and a pairwise disjoint and countable union of $(d+1)$-balls in $\mathbf{H}^{d+1}$ resting on $\mathbf{R}^d$ with base-points

$$a_i = \bar{\gamma}_i(\infty) \quad \text{where } \bar{\gamma}_i \in \bar{\Gamma} \backslash \bar{\Gamma}_\infty$$

and radii $R(a_i) \le c/2$.

Henceforth we will refer to these $(d+1)$-balls as the *horoballs* corresponding to the cusp $\mathcal{E}$. The boundary of a horoball is called a *horosphere.*

Following [S], we will study the excursions of geodesics into the cusps of $\mathbf{H}^{d+1}/\Gamma$ by translating this problem to $\mathbf{H}^{d+1}$ and considering there the corresponding geodesics and the set of horoballs associated to each cusp. Thus, the proof of Theorem 1 is reduced to the following theorem.

THEOREM 2. *Let $\{\mathcal{E}_l\}_{l=1}^n$ be the set of all cusps of $\mathfrak{M}$. Then, for $0<\tau<1$, the Hausdorff dimension of the set of $\xi \in \mathbf{R}^d$ such that $\|\xi - a_i\| < C(\xi)(R(a_i))^{1/\tau}$ for infinitely many a_i is τd. Here each a_i is a base-point of a horosphere corresponding to some cusp $\mathcal{E} \in \{\mathcal{E}_l\}_{l=1}^n$ and $R(a_i)$ is the radius of the horosphere.*

In fact, we can also prove the following improvement.

THEOREM 3. *Let $\{\mathcal{E}_l\}_{l=1}^n$ be the set of all cusps of $\mathfrak{M}$. Then, for $0<\tau<1$, the Hausdorff dimension of the set of $\xi \in \mathbf{R}^d$ such that*

$$\|\xi - a_{l,i}\| < C(\xi)(R(a_{l,i}))^{1/\tau}$$

for infinitely many i and for all $l \in \mathcal{L}$, where $\mathcal{L}$ is a subset of $\{1, 2, \ldots, n\}$, is τd.

Here each $a_{l,i}$ and $R(a_{l,i})$ are respectively the base-points and the radii of the horospheres corresponding to the cusp $\mathcal{E}_l$.

In particular, when $\Gamma = SL(2, \mathbf{Z})$ we have that the base-points a_i run over all nonzero rationals p/q, with g.c.d.$(p,q) = 1$ and $R(p/q) = 1/q^2$. So, one obtains the following classical theorem on metrical diophantine approximation [Be; J; Ka].

COROLLARY 1 (Jarník–Besicovitch theorem). *For $\lambda \ge 1$, the Hausdorff dimension of the set of the points $\xi \in \mathbf{R}$ such that*

$$\left|\xi - \frac{p}{q}\right| < \frac{C(\xi)}{|q|^{2\lambda}}$$

for infinitely many relatively prime integers p, q is $1/\lambda$.

If $\Gamma = SL(2, \mathbf{Z}[i])$ or, more generally, if $\Gamma = SL(2, \mathfrak{R})$ where $\mathfrak{R}$ is the ring of integers of $\mathbf{Q}(\sqrt{-n})$ and n is a positive integer which is not a perfect square (see e.g. [PD, p. 77]), we obtain, as in [S], that the base-points a_i run over all the nonzero fractions p/q with p, q relatively prime integers in $\mathfrak{R}$, and

$$R\left(\frac{p}{q}\right) = \frac{1}{|q|^2}.$$

Hence, we obtain the next corollary.

COROLLARY 2. *For $\lambda \geq 1$, the Hausdorff dimension of the set of the points $\xi \in \mathbf{C}$ such that*

$$\left|\xi - \frac{p}{q}\right| < \frac{C(\xi)}{|q|^{2\lambda}}$$

for infinitely many p, q relatively prime integers in $\mathfrak{R}$ is $2/\lambda$.

The outline of this paper is as follows: In Section 1, we give the proofs of some lemmas on orbit distribution needed in the proof of theorems. In Section 2 we use the concept of regular system of Baker–Schmidt in order to prove some approximation results. In Section 3 we prove the theorems.

NOTATION. We will use $\|\cdot\|$, m, and Vol to denote Euclidean norm, Lebesgue measure, and hyperbolic volume, respectively. The notation $|z|$ will denote the absolute value of the complex number z. Ω_d will mean the Lebesgue measure of the unit ball of $\mathbf{R}^d$, and ∂A will be the boundary of the set A. We will denote by $B(a,r)$ the Euclidean open ball of center a and radius r; $\bar{B}(a,r)$ will be the corresponding closed ball. By $\#A$ we will denote the cardinality of the set A.

As usual, $C(a,b,\dots)$ will denote a variable constant whose value depends only on the arguments shown. Thus its value may vary from line to line and even in the same line.

We take this opportunity to thank our advisor, José L. Fernández, for suggesting the problem and for his help and encouragement during the preparation of this work. Also, we thank the referee for pointing out to us a serious mistake in the original version of this paper.

1. Distribution of Orbits

In this section we collect some known results on distribution of orbits. The first one is an asymptotic result due to Nicholls [N1; N2, p. 204] concerning the distribution of orbits under a discrete group $\tilde{\Gamma}$ of hyperbolic isometries of B^d—the unit ball of $\mathbf{R}^d$ with the Euclidean metric—with finite hyperbolic covolume. This result is an improvement of a theorem of Tsuji [T, p. 518].

Given $\xi \in \partial B^d$ and α an angle satisfying $0 < \alpha < \pi/2$, consider the set $\Omega(\xi,\alpha)$ defined as

$$\Omega(\xi,\alpha) = \{\eta \in B^d : |\langle \eta, \xi \rangle| \geq \|\eta\| \cos\alpha\}.$$

Thus, $\Omega(\xi,\alpha)$ is the portion in B^d of the solid cone of axis $O\xi$ and aperture angle α.

For $\eta \in B^d$ we define $N(s,\eta,\xi,\alpha)$ as the number of elements $\gamma \in \tilde{\Gamma}$ such that

$$\gamma(\eta) \in \Omega(\xi,\alpha) \cap \{x : \rho_{B^d}(0,x) \leq s\},$$

where ρ_{B^d} denotes the hyperbolic distance in B^d associated to the metric

$$d\lambda = \frac{2|dx|}{1-|x|^2}.$$

LEMMA 1.1 [N1].

$$\lim_{s\to\infty} \frac{N(s,\eta,\xi,\alpha)}{\mathrm{Vol}\{x\colon \rho(x,0)<s\}} = C(\Gamma)\alpha^{d-1},$$

and the convergence is uniform in ξ.

In the next lemma we make precise an idea of Sullivan.

LEMMA 1.2. *Let H be any horoball and Γ be a discrete subgroup of* $\text{Möb}(\mathbf{H}^{d+1})$. *Consider the following sum with* $p_0, q_0 \in \mathbf{H}^{d+1}$:

$$\mathcal{S} = \sum_{\substack{\gamma\in\Gamma \\ \gamma(q_0)\in\partial H}} e^{-\delta\rho(p_0,\gamma(q_0))},$$

where $\rho = \rho_{\mathbf{H}^{d+1}}$. *If* $p_0 \notin H$ *then there exists a constant* $C_1 = C_1(q_0,\Gamma)$ *such that, for* $\delta > d/2$,

$$\mathcal{S} \le C_1 e^{-\delta\rho(p_0,\partial H)}.$$

As a matter of fact, C_1 *depends only on*

$$\omega = \min\{\rho(q_0,\eta(q_0)),\ \eta\in\Gamma\backslash\{\text{identity}\}\}$$

REMARK. If $p_0 \in H$ then there exists a constant $C_2 = C_2(\omega)$ such that, for $\delta > d/2$,

$$\mathcal{S} \le C_2 e^{-(\delta-d)\rho(p_0,\partial H)}.$$

Proof. We may assume by conjugation that ∂H is the hyperplane of equation $x_{d+1}=1$, $p_0=\lambda e_{d+1}$, where $e_{d+1}=(0,0,\dots,0,1)$ and $\lambda\le 1$.

There exists $a=a(\omega)>0$ such that if $P,Q\in\partial H$ and $\rho(P,Q)\ge\omega$ then $\|P-Q\|\ge a$. On $\Omega_k=\{P\in\partial H\colon \|P-e_{d+1}\|\in[k-1,k)\}$ there are at most $C(\omega)\cdot k^{d-1}$ points of $\Gamma(q_0)$ $(k=1,2,\dots)$, and if $P\in\Omega_k$ then

$$\begin{aligned}\rho(p_0,P) &\ge \rho(p_0,(k-1,0,\dots,0,1))\\ &= \rho_{\mathbf{H}^2}(i\lambda,(k-1)+i) \ge \log\frac{(k-1)^2+(\lambda+1)^2}{4\lambda}.\end{aligned}$$

Therefore, if $P\in\Omega_k$,

$$e^{-\delta\rho(P,p_0)} \le C\frac{\lambda^\delta}{((k-1)^2+(\lambda+1)^2)^\delta} \le C\left(\frac{\lambda}{k^2}\right)^\delta.$$

Hence

$$\mathcal{S} = \sum_{k=1}^{\infty}\sum_{\substack{\gamma\in\Gamma\\ \gamma(q_0)\in\Omega_k}} e^{-\delta\rho(p_0,\gamma(q_0))} \le C(\omega)\lambda^\delta\sum_{k=1}^{\infty}\frac{1}{k^{2\delta-d+1}} = C_1(\omega)e^{-\delta\rho(p_0,\partial H)},$$

since $\log(1/\lambda)=\rho(p_0,\partial H)$. □

Next, using these two lemmas, we obtain a local version of an estimate of Sullivan [S, p. 227].

LEMMA 1.3. *There exists* $\mu\in(0,1)$ *such that the number* $\nu_n(\mathcal{E},\bar{\mathcal{B}})$ *of horoballs corresponding to a cusp* $\mathcal{E}$ *of* $\mathbf{H}^{d+1}/\Gamma$ *with base-points in a closed ball* $\bar{\mathcal{B}}$ *of* $\mathbf{R}^d$ *and radii* $R\in(\mu^{n+1},\mu^n]$ *satisfies, for all* $n\ge n_0(\Gamma,\mathcal{E},\bar{\mathcal{B}})$,

$$C_1\left(\frac{1}{\mu^n}\right)^d m(\bar{\mathcal{B}}) \le \nu_n(\mathcal{E},\bar{\mathcal{B}}) \le C_2\left(\frac{1}{\mu^n}\right)^d m(\bar{\mathcal{B}})$$

with constants $C_1 = C_1(\Gamma,\mathcal{E})$ *and* $C_2 = C_2(\Gamma,\mathcal{E})$.

Proof. We may assume without loss of generality that $\bar{\mathcal{B}}$ is contained in the unit ball of $\mathbf{R}^d$ and that $m(\bar{\mathcal{B}})$ is small. Let T be a Möbius transformation such that $T(\mathbf{H}^{d+1}) = B^{d+1}$ and let $\{H_i\}_{i=1}^{\infty}$ be the collection of horoballs in $\mathbf{H}^{d+1}$ corresponding to $\mathcal{E}$ with base-points in $\bar{\mathcal{B}}$ and radii $R_i \le 1$, say. Then $\{T(H_i)\}_{i=1}^{\infty}$ is a new collection of horoballs in B^{d+1}. For all i, the radii R_i and R_i' of H_i and $T(H_i)$ respectively satisfy

$$C_1(\bar{\mathcal{B}})R_i \le R_i' \le C_2(\bar{\mathcal{B}})R_i.$$

So, by conjugation, we can work in B^{d+1}. Also we can assume that the image of the origin, by the canonical projection, does not belong to $\mathcal{E}$ and therefore $R_i' < 1/2$. To simplify notation we still denote by $\bar{\mathcal{B}}$ a closed ball in ∂B^{d+1}, by $\{H_i\}_{i=1}^{\infty}$ the collection of horospheres in B^{d+1} corresponding to $\mathcal{E}$, and by R_i the radius of H_i. In this proof ρ means $\rho_{\mathbf{H}^{d+1}}$.

Take one of these horoballs, H_0, say, and let q be a point in ∂H_0. Let $\xi \in \partial B^{d+1}$ be the center of $\bar{\mathcal{B}}$ and α be the aperture of the cone with vertex at the origin whose intersection with ∂B^{d+1} is equal to $\bar{\mathcal{B}}$. Given $a, b \in \mathbf{R}$ with $a < b$, we will use the following notation:

$$L(a,b) = \{x \in B^{d+1} : \log(e^a - 1) \le \rho(0,x) < \log(e^b - 1)\}$$

$$N(a) = N(a,q,\xi,\alpha)$$

$$\mathcal{N}(a,b) = \#\{H_i : e^{-b} \le R_i < e^{-a}\}$$

We recall that $N(a,q,\xi,\alpha)$ is the number of elements $\gamma \in \Gamma$ such that $\rho(0,\gamma(q)) \le a$ and $\gamma(q)$ belongs to the portion in B^{d+1} of the solid cone of axis $O\xi$ and aperture angle α. $\#A$ means the cardinality of the set A.

Notice that the orbit of q consists of points equally spaced on each of the horospheres ∂H_i, and therefore there exists a constant $k_0 = k_0(\Gamma,\mathcal{E})$ such that if H_i is a horoball of radius $R_i \ge e^{-b}$ then $L(b, b+k_0)$ contains at least a point $\gamma(q) \in \partial H_i$. So, for T, K real positive numbers

$$\mathcal{N}(T, T+K) \le N(\log(e^{T+K+k_0} - 1))$$

and for $T \ge T_0$, using that

$$\lim_{T\to\infty} \frac{\log(e^T - 1)}{T} = 1 \quad \text{and} \quad \lim_{T\to\infty} \frac{\mathrm{Vol}\{x : \rho(0,x) < T\}}{e^{dT}} = C(d), \tag{1.1}$$

we have by Lemma 1.1 that

$$\mathcal{N}(T, T+K) \le C(\Gamma,K)\alpha^d e^{d(T+K)}. \tag{1.2}$$

Next we will obtain an opposite inequality for some large enough K,

$$C'(\Gamma,K)\alpha^d e^{dT} \le \mathcal{N}(T, T+K), \tag{1.3}$$

and since the constants in (1.2) and (1.3) are independent of T we can conclude that, for $n=0,1,2,\ldots,$

$$C'(\Gamma,K)\alpha^d e^{d(T+nK)} \le \mathfrak{N}(T+nK, T+(n+1)K) \le C(\Gamma,K)\alpha^d e^{d(T+(n+1)K)}.$$

Let n_o be a positive integer such that $n_0 K \ge T_0$. Now, let T be such that $T=n_0K$. Then for $n \ge n_0$,

$$C'(\Gamma,K)\alpha^d e^{dnK} \le \mathfrak{N}(nK,(n+1)K) \le C(\Gamma,K)\alpha^d e^{d(n+1)K};$$

choosing $\mu = e^{-K}$ and $\nu_n(\Gamma,\mathcal{E}) = \mathfrak{N}(nK,(n+1)K)$, the lemma follows.

Now, we prove (1.3). Consider the following sum:

$$S(T,K) = \sum_{\substack{\gamma\in\Gamma \\ \gamma(q)\in\partial H_i\cap L(T,T+K) \\ e^{-(T+K)}\le R_i < e^{-T}}} e^{-\delta\rho(0,\gamma(q))},$$

where δ is a real number such that $d/2<\delta<d$. Notice that

$$S(T,K) = \sum_{\substack{\gamma\in\Gamma \\ \gamma(q)\in\partial H_i \\ e^{-(T+K)}\le R_i < e^{-T}}} e^{-\delta\rho(0,\gamma(q))}$$

and, by Lemma 1.2,

$$(1.4)\qquad S(T,K) \le A\mathfrak{N}(T,T+K)e^{-\delta T}.$$

So, in order to prove (1.3), it is enough to obtain a lower bound for $S(T,K)$. If we consider the sums

$$S_1(T,K) = \sum_{\substack{\gamma\in\Gamma \\ \gamma(q)\in\partial H_i\cap L(T,T+K) \\ R_i\ge e^{-T}}} e^{-\delta\rho(0,\gamma(q))}$$

and

$$S_2(T,K) = \sum_{\substack{\gamma\in\Gamma \\ \gamma(q)\in\partial H_i\cap L(T,T+K)}} e^{-\delta\rho(0,\gamma(q))}$$

then, since $\partial H_i \cap L(T,T+K) \ne \emptyset$ only if $R_i \ge e^{-(T+K)}$, we have that

$$(1.5)\qquad S_2(T,K) - S_1(T,K) = S(T,K).$$

On the other hand,

$$S_1(T,K) \le \sum_{j=2}^{[T+1]} \sum_{\substack{\gamma\in\Gamma \\ \gamma(q)\in\partial H_i\cap L(T,T+K) \\ e^{-j}\le R_i<e^{-(j-1)}}} e^{-\delta\rho(0,\gamma(q))} + \sum_{\substack{\gamma\in\Gamma \\ \gamma(q)\in\partial H_i\cap L(T,T+K) \\ e^{-1}\le R_i<1/2}} e^{-\delta\rho(0,\gamma(q))}$$

$$\le \sum_{j=2}^{[T+1]} (e^{j-1}-1)^{-\delta} N(\log(e^j-1)) + N(\log(e-1))$$

$$\le 2^\delta \sum_{j=1}^{[T+1]} e^{-\delta(j-1)} N(\log(e^j-1))$$

and

$$S_2(T,K) \geq e^{-\delta(T+K)}(N(\log(e^{T+K}-1)) - N(\log(e^T-1))),$$

where $[x]$ denotes the integer part of the real number x.

Using Lemma 1.1 and (1.1), we obtain

$$N(\log(e^j-1)) \leq C\alpha^d e^{d(j-1)} \quad \text{for all } j.$$

Therefore,

$$S_1(T,K) \leq C(\Gamma)\alpha^d \sum_{j=1}^{[T+1]} e^{(d-\delta)(j-1)} = C(\Gamma)\alpha^d e^{(d-\delta)T} \tag{1.6}$$

and for T large enough, again using Lemma 1.1 and (1.1),

$$S_2(T,K) \geq C(\Gamma,K)\alpha^d e^{(d-\delta)(T+K)} \quad \text{with } C(\Gamma,K) = C(\Gamma)(1-e^{-dK}). \tag{1.7}$$

Thus, by (1.5), (1.6), and (1.7),

$$S(T,K) \geq \alpha^d e^{(d-\delta)T}(C(\Gamma,K)e^{(d-\delta)K} - C(\Gamma)).$$

Finally, since we can choose K large enough so that

$$C(\Gamma,K)e^{(d-\delta)K} - C(\Gamma) > C > 0,$$

we obtain

$$S(T,K) \geq C\alpha^d e^{(d-\delta)T}, \tag{1.8}$$

and (1.3) is now a consequence of (1.4) and (1.8). □

2. Well-Distributed Systems of Balls

Baker and Schmidt introduced in [BS] the concept of regular system of intervals in order to get some results on diophantine approximation of algebraic numbers. We will extend their definition to systems of balls in $\mathbf{R}^d$ to obtain results of the same kind in any dimension.

DEFINITION. Let $\mathcal{W}$ be a countable collection of Euclidean balls $B_i = B(a_i, R_i)$ in $\mathbf{R}^d$. We will say that $\mathcal{W}$ is a *well-distributed system of balls* with constant Θ if, for every ball $\mathcal{B}$ in $\mathbf{R}^d$, there exists a positive number $K(\mathcal{B})$ such that for every K with $K \geq K(\mathcal{B})$ we have a subcollection $\mathcal{W}(K,\mathcal{B}) \subseteq \mathcal{W}$ satisfying:

(W1) $a_i \in \mathcal{B}$ and $R_i \geq 1/K$ for all $B_i \in (K,\mathcal{B})$;
(W2) For all $B_i, B_j \in \mathcal{W}(K,\mathcal{B})$ with $i \neq j$, $\|a_i - a_j\| > \min\{R_i, R_j\}$;
(W3) $\#\mathcal{W}(K,\mathcal{B}) \geq \Theta K^d m(\mathcal{B})$.

A simple example of a well-distributed system in $\mathbf{R}$ is the collection $\mathcal{W}$ of intervals with center a nonzero rational p/q, g.c.d.$(p,q) = 1$, and radius $1/q^2$. Another example is given, in $\mathbf{R}^2$, by the balls of center z/w and radius $1/|w|^2$, where z and w are Gaussian integers and $w \neq 0$. However, the collection of intervals in $\mathbf{R}$ with center a dyadic number $r + p/2^n$ (with $n \in \mathbf{N}$, $r \in \mathbf{Z}$, and p an odd integer) and radius $1/2^{2n}$ is not a well-distributed system in $\mathbf{R}$.

Using the notion of well-distributed system we obtain the following results.

THEOREM 2.1. *Let* $\{\mathcal{W}^l\}_{l=1}^n$ *be a collection of well-distributed systems of balls,* $\mathcal{W}^l = \{B(a_{l,i}, R_{l,i})\}_{i=1}^{\infty}$, *in* $\mathbf{R}^d$ *with constants* Θ_l. *Let*

$$\Theta = \min\{\Theta_1, \Theta_2, \dots, \Theta_n\}.$$

Then, for $0 < \alpha < \tau < 1$ *and* $\mathcal{B}$ *a ball in* $\mathbf{R}^d$, *the* $(d\alpha)$*-dimensional content of the set*

$$H = \{\xi \in \mathcal{B} : \|\xi - a_{l,i}\| < C(\xi) R_{l,i}^{1/\tau} \text{ for infinitely many } i \text{ and for all } l \in \mathcal{L}\},$$

where $\mathcal{L} \subset \{1, 2, \dots, n\}$, *is at least* $C(\Theta, \alpha)(m(\mathcal{B}))^{\alpha}$.

COROLLARY 2.2. *If* $\{\mathcal{W}^l\}_{l=1}^n$, $\mathcal{L}$ *and* $\mathcal{B}$ *are as above, and if* $0 < \tau < 1$, *then the Hausdorff dimension of the set of points* $\xi \in \mathcal{B}$ *such that*

$$\|\xi - a_{l,i}\| < C(\xi) R_{l,i}^{1/\tau} \text{ for infinitely many } i \text{ and for all } l \in \mathcal{L}$$

is at least τd.

In [BS] Baker and Schmidt proved Corollary 2.2 in the case $d = 1$, refining some ideas of Besicovitch [Be]. Our argument is an extension of theirs.

In the proof of Theorem 2.1 we will need the following lemma.

LEMMA 2.3. *Let* ϵ, R *be positive numbers such that* $\epsilon \geq 2R$, *and let* $\mathcal{F}$ *be a family of balls in* $\mathbf{R}^d$ *of radius* R *such that, for all* $B(a_i, R), B(a_j, R) \in \mathcal{F}$ $(i \neq j)$, *we have that* $\|a_i - a_j\| > \epsilon$. *Let* $\mathcal{S} = \{S_j\}$ *be a countable family of balls in* $\mathbf{R}^d$ *such that*

(i) $\sum_j (\operatorname{diam}(S_j))^{\alpha d} < \delta$, *and*
(ii) $\operatorname{diam}(S_j) < \omega$ *for all* $S_j \in \mathcal{S}$,

where α, δ, ω *are positive numbers and* diam(A) *denotes the diameter of the ball* A.

If $\mathcal{F}' \subseteq \mathcal{F}$ *denotes the set of balls* B *in* $\mathcal{F}$ *such that there exists a ball* $S_j \in \mathcal{S}$ *whose intersection with* B *contains a ball of diameter at least* $R/2$, *then*

$$\#\mathcal{F}' \leq \frac{6^d \delta \omega^{d(1-\alpha)}}{\epsilon^d}.$$

Proof. Let $\mathcal{D}$ be the collection of balls $S_j \in \mathcal{S}$ whose intersection with some $B \in \mathcal{F}$ contains a ball of diameter at least $R/2$. For all $D \in \mathcal{D}$, we denote by $\mathcal{G}_D$ the collection of balls of $\mathcal{F}$ which intersect D as we have just described.

We will obtain an upper bound of $\#\mathcal{G}_D$, and since

$$\#\mathcal{F}' \leq \sum_{D \in \mathcal{D}} \#\mathcal{G}_D \tag{2.1}$$

we will get an upper bound of $\#\mathcal{F}'$.

Let r_D be the radius of a ball $D \in \mathcal{D}$ and let $\tilde{D}$ be the ball with the same center as D and radius $r_D + R/2$. It is clear that the centers c_G of the balls G in $\mathcal{G}_D$ belong to $\tilde{D}$ and, since the distance between them is at least ϵ, we have that there exists a constant $C > 1/2^d$ such that

$$m(\tilde{D}) \geq C \sum_{G \in \mathcal{G}_D} m(B(c_G, \epsilon/2)) \geq \frac{\epsilon^d \Omega_d}{2^{2d}} \# \mathcal{G}_D.$$

Hence,

$$\# \mathcal{G}_D \leq \frac{2^{2d}}{\epsilon^d \Omega_d} m(\tilde{D}) = \frac{2^{2d}}{\epsilon^d} \left(r_D + \frac{R}{2} \right)^d.$$

But $R/2 \leq 2r_D$ and so we have that

$$\# \mathcal{G}_D \leq \frac{6^d}{\epsilon^d} (\operatorname{diam}(D))^d.$$

Therefore, by (2.1),

$$\# \mathcal{F}' \leq \frac{6^d}{\epsilon^d} \sum_{D \in \mathfrak{D}} (\operatorname{diam}(D))^d.$$

But, by (i) and (ii),

$$\sum_{D \in \mathfrak{D}} (\operatorname{diam}(D))^{d(1-\alpha)} (\operatorname{diam}(D))^{d\alpha} < \omega^{d(1-\alpha)} \delta,$$

and so we conclude that

$$\# \mathcal{F}' \leq 6^d \frac{\omega^{d(1-\alpha)} \delta}{\epsilon^d}. \qquad \square$$

Proof of Theorem 2.1. We can suppose, by rearrangement, that $\mathcal{L} = \{1, 2, \dots, p\}$ $(p \leq n)$. If $\mathcal{B}$ is a ball of radius 1 in $\mathbf{R}^d$, we let $\tilde{H}$ denote the set of $\xi \in \mathcal{B}$ such that there exists a sequence $K_j(\xi)$ tending to infinity and a subsequence $\{B_{i(j)}\}$ of $\bigcup_{l \in \mathcal{L}} \mathcal{W}^l$, which also depends on ξ, such that for all j there exists a ball $B(a_{t(j), i(j)}, R_{t(j), i(j)})$ in $\mathcal{W}^{t(j)}$, where $t(j) \in \mathcal{L}$ and $t(j) \equiv j \pmod{p}$, satisfying

$$\|\xi - a_{t(j), i(j)}\| < \frac{1}{K_j^{1/\tau}} \quad \text{and} \quad R_{t(j), i(j)} \geq \frac{1}{K_j}.$$

Then, we will see that $M_{d\alpha}(\tilde{H}) \geq C(\Theta, \alpha)$, and since

$$\tilde{H} = \bigcap_{l \in \mathcal{L}} \Bigg\{ \xi \in \mathcal{B} : \|\xi - a_{l, i(pk+l)}\| < \frac{1}{K_{pk+l}^{1/\tau}} \\ \text{and } R_{l, i(pk+l)} \geq \frac{1}{K_{pk+l}} \text{ for } k = 0, 1, \dots \Bigg\} \subset H,$$

the theorem follows for balls of radius 1.

In the general case, with $\mathcal{B}$ a ball in $\mathbf{R}^d$ with center h and radius r, we have that

$$M_{d\alpha}\left(\left\{ \xi \in \mathcal{B} : \|\xi - a_{l,i}\| < r \left(\frac{R_{l,i}}{r} \right)^{1/\tau} \text{ for infinitely many } i, \text{ for all } l \in \mathcal{L} \right\} \right)$$

$$= r^{d\alpha} M_{d\alpha}\left(\left\{ \eta \in B\left(\frac{h}{r}, 1 \right) : \left\| \eta - \frac{a_{l,i}}{r} \right\| < \left(\frac{R_{l,i}}{r} \right)^{1/\tau} \\ \text{for infinitely many } i, \text{ for all } l \in \mathcal{L} \right\} \right)$$

It is easy to see that the families $\{B(a_{l,i}/r, R_{l,i}/r)\}_{i=1}^{\infty}$ $(l \in \mathcal{L})$ are also well-distributed systems, with constants Θ_l respectively, and so the theorem follows.

Let δ be a real number such that

$$\delta < \left(\frac{\Theta\, m(\mathcal{B})}{2.12^d}\right)^{\alpha}, \tag{2.2}$$

and let $\mathcal{U} = \{U_j\}$ be a countable family of balls in $\mathbf{R}^d$ such that

$$\sum_j (\operatorname{diam}(U_j))^{d\alpha} < \delta. \tag{2.3}$$

We will now prove that $\mathcal{U}$ cannot be a covering of $\tilde{H}$ and, consequently, that $M_{\alpha d}(\tilde{H}) \geq \delta$. In order to see this, we will construct by induction a sequence $\{K_j\}_{j=1}^{\infty}$ of positive numbers tending to infinity and a sequence $\mathcal{V} = \{V_j\}_{j=1}^{\infty}$ of finite unions of nonempty and disjoint closed balls, $V_j = \bigcup_{s \in I_j} V_{j,s}$, contained in $\mathcal{B}$. We will have the following conditions on K_j, $V_j = \bigcup_{s\in I_j} V_{j,s}$, and the positive number λ_j defined as

$$\lambda_j = \frac{C}{K_j^{1/\alpha}(m(\frac{1}{2}V_{j-1}))^{1/d\alpha}} \quad \text{with } C = \left(\frac{2^{2d+2}3^d\delta}{\Theta^2\Omega_d}\right)^{1/d\alpha}$$

(in this proof, if A is a set which is a union of balls, $A = \bigcup_k B(p_k, r_k)$, then we will denote the set $\bigcup_k B(p_k, r_k/2)$ by $\frac{1}{2}A$):

- (I.1) $V_j \subseteq V_{j-1}$;
- (I.2) for each $V_{j,s}$, there exists a ball $B(a,R)$ belonging to $\mathcal{W}^{t(j)}$ with $R \geq 1/K_j$ such that $V_{j,s} = \bar{B}(a, \lambda_j)$;
- (I.3) $V_j \cap U_k = \emptyset$ for all $U_k \in \mathcal{U}$, with $\operatorname{diam}(U_k) > \lambda_j$;
- (I.4) $\lambda_j < \min\{1/(4K_j), \lambda_{j-1}/4, 1/K_j^{1/\tau}\}$;
- (I.5) for all $V_{j,s}, V_{j,s'}$ with $s, s' \in I_j$ $(s \neq s')$, the distance between them is at least $3/(4K_j)$;
- (I.6) $m(\frac{1}{2}V_j) \geq (1/2^{d+1})\Theta\Omega_d\lambda_j^d K_j^d m(\frac{1}{2}V_{j-1})$.

Since the balls in V_j are disjoint and with radii λ_j (by (I.2) and (I.5)), condition (I.6) simply means that the number of balls in V_j is at least

$$\frac{1}{2}\Theta K_j^d m\left(\frac{1}{2}V_{j-1}\right).$$

Notice that by (I.1), (I.2), and (I.4) we get that $\emptyset \neq \bigcap_{j=0}^{\infty} V_j \subset \tilde{H}$ and, since by (I.4) the sequence $\{\lambda_j\}_{j=0}^{\infty}$ tends to zero as $j \to \infty$, we have by (I.3) that $(\bigcap_{j=0}^{\infty} V_j) \cap U_k = \emptyset$ for all $U_k \in \mathcal{U}$.

Here is the inductive construction of $\mathcal{V}$.

Initial step: We take $V_0 = \mathcal{B}$. Notice that, by (2.2), there exists a number β such that

$$\delta < \beta \leq \left(\frac{\Theta\, m(\mathcal{B})}{2.12^d}\right)^{\alpha}.$$

We define λ_0 by the condition $\lambda_0^{d\alpha(1-\alpha)}\delta^{\alpha} = \beta$. Then, it is easy to see that

(2.4) $$\lambda_0^{d(1-\alpha)} \leq \frac{\Theta}{2.12^d \delta} m(\tfrac{1}{2}V_0);$$

(2.5) $$\delta < \lambda_0^{d\alpha}.$$

Now, by (2.3) and (2.5), it is clear that

(2.6) $$\operatorname{diam}(U_k) < \lambda_0 \quad \text{for all } k.$$

Inductive step: We now fix j in the rest of the argument. If $K_1, \dots, K_{j-1}$ and $V_0, V_1, \dots, V_{j-1}$ have already been constructed, then we take K_j large enough so that (I.4) is verified and K_j also satisfies the following two conditions:

(2.7) $$K_j \geq K^{t(j)}(V_{j-1,s}) \quad \text{for all } s \in I_{j-1},$$

where $K^{t(j)}(V_{j-1,s})$ is the constant given for the ball $V_{j-1,s}$ in the definition of the well-distributed system $\mathcal{W}^{t(j)}$; and

(2.8) $$\frac{3}{4K_{j-1}} \geq \frac{1}{K_j}.$$

Notice that (I.4) can be satisfied since $\alpha < \tau < 1$.

Now, let $\mathfrak{I}_j$ be the finite collection given by

$$\mathfrak{I}_j = \bigcup_{s \in I_{j-1}} \mathcal{W}^{t(j)}(K_j, \tfrac{1}{2}V_{j-1,s}).$$

We recall that $\mathcal{W}^{t(j)}(K_j, \frac{1}{2}V_{j-1,s})$ is the subset of the well-distributed system $\mathcal{W}^{t(j)}$ obtained by applying the definition to each $\frac{1}{2}V_{j-1,s}$ and the number K_j.

Let $a_1, \dots, a_m$ be the centers of the balls in $\mathfrak{I}_j$, and let $\mathfrak{F}_j$ be the collection of closed balls $\bar{B}(a_i, 2\lambda_j)$ $(i = 1, 2, \dots, m)$. Let us observe that

$$m = \#\mathfrak{I}_j = \#\mathfrak{F}_j = \sum_{s \in I_{j-1}} \#\mathcal{W}^{t(j)}(K_j, \tfrac{1}{2}V_{j-1,s});$$

using (W3) (for the well-distributed system $\mathcal{W}^{t(j)}$) and the fact that, by induction, V_{j-1} is a union of disjoint balls, we obtain

(2.9) $$\#\mathfrak{F}_j \geq \sum_{s \in I_{j-1}} \Theta K_j^d m(\tfrac{1}{2}V_{j-1,s}) = \Theta K_j^d m(\tfrac{1}{2}V_{j-1}).$$

We note that if two balls in the collection $\mathfrak{F}_j$ have their centers in different balls $\frac{1}{2}V_{j-1,s}$, then, by (I.5) for $j-1$ and (2.8), the distance between them is at least $1/K_j$. On the other hand, if the centers belong to the same ball $\frac{1}{2}V_{j-1,s}$, then applying (W1) and (W2) (for the well-distributed system $\mathcal{W}^{t(j)}$) we get the same conclusion. So, in any case, by (I.4) the balls in $\mathfrak{F}_j$ are disjoint. Also it is clear, from (I.2) for $j-1$ and (I.4) for j, that the balls in $\mathfrak{F}_j$ are contained in V_{j-1}. Hence if $j > 1$ then, by (I.3) (which holds for $j-1$ by induction), for all $\bar{B}(a_i, 2\lambda_j) \in \mathfrak{F}_j$ we have that

(2.10) $$\bar{B}(a_i, 2\lambda_j) \cap U_k = \emptyset \quad \text{for all } U_k \in \mathcal{U} \text{ with } \operatorname{diam}(U_k) > \lambda_{j-1}.$$

Next we split $\mathfrak{F}_j$ into two disjoint families $\mathfrak{F}_j'$ and $\mathfrak{F}_j''$. $\mathfrak{F}_j'$ consists of those balls Q of $\mathfrak{F}_j$ such that there exists a ball $U_k \in \mathfrak{U}$ whose intersection with Q contains a ball of diameter at least λ_j. By Lemma 2.3 with $\mathfrak{F}=\mathfrak{F}_j$, $R=2\lambda_j$, $\epsilon=1/K_j$, $\omega=\lambda_{j-1}$, and $\mathcal{S}=\{U\in\mathfrak{U}: \operatorname{diam}(U)\le\lambda_{j-1}\}$, we get that

$$\#\mathfrak{F}_j' < 6^d K_j^d \lambda_{j-1}^{d(1-\alpha)}\delta.$$

So, for case $j=1$, using (2.4), we obtain

$$\#\mathfrak{F}_1' < \tfrac{1}{2}\Theta K_1^d m(\tfrac{1}{2}V_0);$$

for case $j>1$, using (I.6) (which holds for $j-1$ by induction), we have

$$\#\mathfrak{F}_j' < \frac{6^d\delta K_j^d}{\lambda_{j-1}^{d\alpha}}\,\frac{2^{d+1}m(\tfrac{1}{2}V_{j-1})}{\Theta\Omega_d K_{j-1}^d m(\tfrac{1}{2}V_{j-2})}.$$

By the definition of λ_{j-1} we obtain that

$$\#\mathfrak{F}_j' < \tfrac{1}{2}\Theta K_j^d m(\tfrac{1}{2}V_{j-1}).$$

Hence, using (2.9),

$$\#\mathfrak{F}_j' < \tfrac{1}{2}\#\mathfrak{F}_j,$$

and so

$$\#\mathfrak{F}_j'' \ge \tfrac{1}{2}\#\mathfrak{F}_j \ge \tfrac{1}{2}\Theta K_j^d m(\tfrac{1}{2}V_{j-1}) > 0. \tag{2.11}$$

If $\mathfrak{F}_j''=\{Q_s: s\in I_j\}$, then we define $V_{j,s}=\tfrac{1}{2}Q_s$ and $V_j=\bigcup_{s\in I_j}V_{j,s}$.

We need to check that the conditions (I.1)–(I.6) hold for K_j and V_j: (I.1)–(I.4) follow by construction; (I.5) follows from (I.4) because the distance between the centers of the balls $V_{j,s}$ is at least $1/K_j$ and the radii are λ_j. Finally, since

$$m(\tfrac{1}{2}V_j)=\#\mathfrak{F}_j'' m(\tfrac{1}{2}V_{j,s})=\#\mathfrak{F}_j''\left(\frac{\lambda_j}{2}\right)^d\Omega_d,$$

using (2.11) we get

$$m(\tfrac{1}{2}V_j)\ge\frac{1}{2^{d+1}}\Theta\Omega_d\lambda_j^d K_j^d m(\tfrac{1}{2}V_{j-1}),$$

and so (I.6) holds too. □

3. Proof of Theorems

LEMMA 3.1. *Let $\mathcal{S}$ be a countable collection of balls $B_j=B(c_j,r_j)$ (with $r_j\le 1$) in $\mathbf{R}^d$ such that for all i,j with $i\ne j$,*

$$\|c_i-c_j\|>\min\{r_i,r_j\} \tag{3.1}$$

Then, given a number τ, $0<\tau<1$, the Hausdorff dimension of the set of points ξ such that

$$\|\xi-c_j\|<C(\xi)r_j^{1/\tau}\quad \textit{for infinitely many } c_j$$

is at most τd.

Proof. Let $\mathcal{B}$ be a ball in $\mathbf{R}^d$ of radius r, and let M be a positive real number. Consider the set $\mathcal{H}$ defined as

$$\mathcal{H}=\{\xi\in\mathcal{B}:\|\xi-c_i\|<Mr_j^{1/\tau}\text{ for infinitely many }B_j\text{ with }c_j\in\mathcal{B}\}.$$

To prove the lemma it is enough to show that $\operatorname{Dim}(\mathcal{H})$ is at most τd.

Given a number $\mu\in(0,1)$, let $\mathcal{A}_n$ denote the set

$$\{B_j\in\mathcal{S}\mid c_j\in\mathcal{B}\text{ and }r_j\in(\mu^{n+1},\mu^n]\}$$

It is clear that for every $B_i, B_j\in\mathcal{A}_n$, $i\neq j$,

$$B\left(a_i,\frac{\mu^{n+1}}{2}\right)\cap B\left(a_j,\frac{\mu^{n+1}}{2}\right)=\emptyset.$$

Comparing volumes, we have that

$$\sum_{i\in I} m\left(B\left(a_i,\frac{\mu^{n+1}}{2}\right)\right)\le m(B'),$$

where $I=\{i: B_i\in\mathcal{A}_n\}$ and B' is the ball with the same center as $\mathcal{B}$ and radius $r+\mu^{n+1}/2$. Thus, we get

$$\begin{aligned}(3.2)\qquad \#\mathcal{A}_n&\le\frac{2^d}{\Omega_d\mu^d}\left(\frac{1}{\mu^n}\right)^d m(B')\\ &=\frac{2^d}{\Omega_d\mu^d}\left(1+\frac{\mu^{n+1}}{2r}\right)^d\left(\frac{1}{\mu^n}\right)^d m(B).\end{aligned}$$

If $2r\ge 1$, then using (3.2) we obtain

$$(3.3)\qquad \#\mathcal{A}_n\le\frac{2^{2d}}{\Omega_d\mu^d}\left(\frac{1}{\mu^n}\right)^d m(B)\quad\text{for all }n\in\mathbf{N}.$$

If $2r\in(\mu^{n_0+1},\mu^{n_0}]$ with $n_0\in\mathbf{N}$, then we also obtain (3.3) for $n\ge n_0$. Furthermore, if there exist $a_l\in\mathcal{B}$ such that $B(a_l,r_l)\in\mathcal{S}$ and $r_l>\mu^{n_0}$, then for all a_j such that $r_j>\mu^{n_0}$ we have that

$$\|a_l-a_j\|>\min\{r_l,r_j\}>\mu^{n_0},$$

and since $2r\le\mu^{n_0}$ we conclude that $a_j\notin\mathcal{B}$. Hence, if $2r\in(\mu^{n_0+1},\mu^{n_0}]$ then

$$(3.4)\qquad \sum_{t=0}^{n_0+1}\#\mathcal{A}_t\le 1.$$

Notice that, since $\#A_n<\infty$ for all $n\in\mathbf{N}$, we have that for all ξ in $\mathcal{H}$ there exists a sequence $\{r_j(\xi)\}$ such that r_j tends to zero as $j\to\infty$ and $\|\xi-c_j\|<Mr_j^{1/\tau}$. Hence we get that $\mathcal{H}$ is covered by the collection of balls

$$\tilde{\mathcal{S}}_k=\{\tilde{B}_j=B(c_j,\tilde{r}_j)\mid\tilde{r}_j=Mr_j^{1/\tau},\ c_j\in\mathcal{B},\ r_j\le\mu^k\}$$

for each positive integer k. Since

$$\sum_{\substack{j\\ \tilde{B}_j\in\tilde{\mathcal{S}}_k}}\tilde{r}_j^{\beta}=M^\beta\sum_{\substack{j\\ c_j\in\mathcal{B}\\ r_j\le\mu^k}} r_j^{\beta/\tau}\le M^\beta\sum_{n=k}^{\infty}\sum_{\substack{r_j\in(\mu^{n+1},\mu^n]\\ c_j\in\mathcal{B}}} r_j^{\beta/\tau},$$

using (3.3) and (3.4) we have that, for all $k \geq n_0$,

$$\sum_{\substack{j \\ \tilde{B}_j \in \tilde{\mathcal{S}}_k}} \tilde{r}_j^{\beta} \leq C(M) \sum_{n=k}^{\infty} \frac{\mu^{n\beta/\tau}}{\mu^{nd}}.$$

So, if $\beta/\tau > d$ then $\sum_{j, \tilde{B}_j \in \tilde{\mathcal{S}}_k} \tilde{r}_j^{\beta}$ tends to zero as $k \to \infty$, because $\sum_n \mu^{n(\beta/\tau - d)}$ is convergent. Hence $M_\beta(\mathcal{K}) = 0$ and consequently Dim $\mathcal{K} \leq \tau d$. □

Proof of Theorem 1. Let $\{\mathcal{E}_l\}_{l=1}^{n}$ be the set of all cusps of $\mathfrak{M} = \mathbf{H}^{d+1}/\Gamma$. For each l, let $\{H_i^l\}_{i=1}^{\infty}$ denote the set of horoballs corresponding to the cusp $\mathcal{E}_l$.

Let $\gamma_v(t)$ be a geodesic in $\mathfrak{M}$ emanating from p with direction v and such that

$$\limsup_{t \to \infty} \frac{\operatorname{dist}(\gamma_v(t), p)}{t} \geq \alpha. \tag{3.5}$$

Then we have a sequence t_i tending to infinity such that $\gamma_v(t_i)$ is inside some cusp $\mathcal{E}_{l(i)}$ of $\mathfrak{M}$ ($l(i) \in \{1, 2, \ldots, n\}$) and $d_i \geq \alpha t_i$, where

$$d_i = \max\{\operatorname{dist}(\gamma_v(t), p) : t \in [0, t_i]\}.$$

Now, let $\bar{\gamma}_v$ be a lifting to $\mathbf{H}^{d+1}$ of γ_v. Without loss of generality we can suppose that $\bar{\gamma}_v$ is a vertical ray ending at a point $\xi \in \mathbf{R}^d$. We have that

$$d_i = C_{l(i)} + \log \frac{R_{k(i)}}{r_{k(i)}} \quad (k(i) \in \mathbf{N}),$$

where $R_{k(i)}$ is the radius of the horoball $H_{k(i)}^{l(i)}$ corresponding to the cusp $\mathcal{E}_{l(i)}$ which contains $\bar{\gamma}_v(t_i)$, and $r_{k(i)}$ is the radius of the horoball, with the same base-point $a_{k(i)}$ as $H_{k(i)}^{l(i)}$, whose projection on $\mathfrak{M}$ is the region of $\mathcal{E}_{l(i)}$ not attained by γ_v before the time t_i. $C_{l(i)}$ denotes a constant which depends only on the cusp $\mathcal{E}_{l(i)}$. For the sake of simplicity, hereafter we will write r_i and R_i instead of $r_{k(i)}$ and $R_{k(i)}$.

It is clear that $r_i = Ce^{-t_i}$, and so

$$\frac{R_i}{r_i} \geq C_{l(i)} \left(\frac{1}{r_i} \right)^{\alpha}.$$

Therefore

$$\|\xi - a_i\| = r_i \leq C(\xi) R_i^{1/(1-\alpha)}, \tag{3.6}$$

where $C = \max\{C_1, \ldots, C_n\}$.

Thus, if ξ is not a base-point of a horoball corresponding to some cusp $\mathcal{E}_l$, then there are infinitely many solutions a_i of the inequality (3.6). On the other hand, if (3.6) has infinitely many solutions a_i, where each a_i is the base-point of a horoball corresponding to some cusp $\mathcal{E}_{l(i)}$, then the geodesic $\bar{\gamma}_v$ in $\mathbf{H}^{d+1}$ with endpoint $\xi \in \mathbf{R}^d$ projects on a geodesic γ_v in $\mathfrak{M}$ which satisfies (3.5).

Hence, the set appearing in Theorem 1 has the same Hausdorff dimension as the set of points $\xi \in \mathbf{R}^d$ such that the inequality (3.6) holds for infinitely many a_i's. Thus, Theorem 1 follows from Theorem 2. □

REMARK. We can prove more than stated in Theorem 1 by using a similar argument and Theorem 3 instead of Theorem 2.

Given a cusp $\mathcal{E}_l$, let T_l be the set of times t such that $\gamma_v(t) \in \mathcal{E}_l$. Then the Hausdorff dimension of the set of $v \in S(p)$ such that

$$\limsup_{\substack{t \to \infty \\ t \in T_l}} \frac{\operatorname{dist}(\gamma_v(t), p)}{t} \geq \alpha \quad \text{for all } l \in \mathcal{L} \subset \{1, 2, \ldots, n\}$$

is $d(1-\alpha)$.

PROOF OF THEOREM 3. We will prove that the system $\mathcal{W}$ of balls $B(a_i, R(a_i))$ in $\mathbf{R}^d$, where a_i and $R(a_i)$ are respectively the base-points and the radii of the horoballs corresponding to a fixed cusp $\mathcal{E}$ of $\mathbf{H}^{d+1}/\Gamma$, is a well-distributed system. Thus the inequality $\operatorname{Dim} \geq \tau d$ follows from Corollary 2.2, and the opposite inequality is a consequence of Lemma 3.1.

Given a ball $\mathcal{B}$ in $\mathbf{R}^d$, let $\mu \in (0,1)$ and $n_0 \in \mathbf{N}$ be the numbers in Lemma 1.3, and let $K(\mathcal{B}) = 1/\mu^{n_0}$. Then, for $K \geq K(\mathcal{B})$, consider the subcollection

$$\mathcal{W}(K, \mathcal{B}) = \{B(a_i, R(a_i)) \mid a_i \in \mathcal{B} \text{ and } R(a_i) \geq 1/K\}$$

By definition, $\mathcal{W}(K, \mathcal{B})$ satisfies (W1). (W2) follows immediately from the fact that the horoballs in $\mathbf{H}^{d+1}$ with base-points a_i and radii $R(a_i)$ come from a cusp of $\mathbf{H}^{d+1}/\Gamma$ and hence are disjoint. Finally, if $1/K \in (\mu^{n+1}, \mu^n]$ (and so $n \geq n_0$), then $\#\mathcal{W}(K, \mathcal{B})$ is at least the number $\nu_n(\mathcal{E}, \bar{\mathcal{B}})$ appearing in Lemma 1.3 and so (W3) follows from that lemma. □

PROOF OF THEOREM 2. Obviously, any collection of balls which contains a well-distributed system of balls is also a well-distributed system. Therefore, since the family $\mathcal{W}'$ of balls in $\mathbf{R}^d$, $\{B(a_i, R(a_i))\}$ (where a_i and $R(a_i)$ are respectively the base-points and the radii of the horoballs corresponding to any cusp of $\mathfrak{M}$) contains the family $\mathcal{W}$ appearing in the proof of Theorem 3, $\mathcal{W}'$ is a well-distributed system. Hence, the inequality $\operatorname{Dim} \geq \tau d$ follows from Corollary 2.2.

On the other hand, we can get that the horoballs corresponding to different cusps of $\mathfrak{M}$ are disjoint (if they correspond to the same cusp then by construction they are also disjoint), and therefore the balls in $\mathcal{W}'$ satisfy the condition in Lemma 3.1. Thus we obtain the inequality $\operatorname{Dim} \leq \tau d$. □

References

[A] L. V. Ahlfors, *Möbius transformations in several dimensions,* Lecture notes, Univ. of Minnesota, Minneapolis, Minn., 1981.

[BS] A. Baker and W. M. Schmidt, *Diophantine approximation and Hausdorff dimension,* Proc. London Math. Soc. (3) 21 (1970), 1–11.

[B] A. F. Beardon, *The geometry of discrete groups,* Springer, New York, 1983.

[Be] A. S. Besicovitch, *Sets of fractional dimension (IV): On rational approximations to real numbers,* J. London Math. Soc. 9 (1934), 126–131.

[C] L. Carleson, *Selected problems on exceptional sets*, Van Nostrand, Princeton, NJ, 1967.

[J] V. Jarník, *Zur metrischen Theorie der diophantischen Approximationen*, Prace Mat.-Fiz. 36 (1928–1929), 91–106.

[Ka] R. Kaufman, *On the theorem of Jarník and Besicovitch*, Acta Arith. 39 (1981), 265–267.

[K] I. Kra, *Automorphic forms and Kleinian groups*, Benjamin, Reading, Mass., 1972.

[N1] P. Nicholls, *A lattice point problem in hyperbolic space*, Michigan Math. J. 30 (1983), 273–287.

[N2] ———, *The ergodic theory of discrete groups*, London Math. Soc. Lecture Note Ser., 143, Cambridge Univ. Press, Cambridge, 1989.

[P] S. J. Patterson, *The limit set of a Fuchsian group*, Acta Math. 136 (1976), 241–273.

[PD] H. Pollard and H. G. Diamond, *The theory of algebraic numbers*, Carus Math. Monographs, 9, Math. Assoc. America, Washington, DC, 1975.

[R] C. A. Rogers, *Hausdorff measures*, Cambridge Univ. Press, Cambridge, 1970.

[S] D. Sullivan, *Disjoint spheres, approximation by imaginary quadratic numbers and the logarithm law for geodesics*, Acta Math., 149 (1982), 215–237.

[T] M. Tsuji, *Potential theory in modern function theory*, Chelsea, New York, 1975.

Departamento de Matemáticas
Universidad Autónoma de Madrid
28049 Madrid
España