Factorization of Blaschke Products

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1. Introduction

Let H^∞ be the space of bounded analytic functions in the open unit disc D. Identifying these functions with boundary functions, we can consider that H^∞ is the essentially supremum norm closed subalgebra of L^∞, the space of bounded measurable functions on ∂D with respect to the Lebesgue measure. Sarason [14] proved that $H^\infty + C$ is a closed subalgebra of L^∞, where C is the set of continuous functions on ∂D. We denote by $M(H^\infty + C)$ the maximal ideal space of $H^\infty + C$. In [6], Guillory and Sarason proved that there is a positive integer N such that if $f \in H^\infty + C$ and b is an inner function with $|f| \le |b|$ on $M(H^\infty + C)$, then $f^N/b = f^N \bar{b}$ belongs to $H^\infty + C$, and we cannot take $N = 1$. In [12], the author and Y. Izuchi proved that we can take $N = 2$. In this paper, we assume that b is a Blaschke product and study the cases $|f| \le |b|$ on $M(H^\infty + C)$ and $f\bar{b} \notin H^\infty + C$. Our aim is to investigate the kind of small changes of f or b, say g and ψ respectively, that make $g\bar{b} \in H^\infty + C$ or $f\bar{\psi} \in H^\infty + C$. To prove several of our theorems, Hoffman's factorization theorem for Blaschke products [9, Thm. 5.2] plays an important role.

In Section 3, we shall give an additional property in Hoffman's factorization theorem that zero sets of its factors having zeros of infinite order coincide with each other. In Section 4, we prove that if $f \in H^\infty + C$ and b is a Blaschke product with $|f| \le |b|$ on $M(H^\infty + C)$, then there is a subproduct ψ of b such that $f\bar{\psi} \in H^\infty + C$ and $Z(\psi) = Z(b)$, and there is a function g in $H^\infty + C$ such that $|g| = |f|$ on $M(H^\infty + C)$ and $g\bar{b} \in H^\infty + C$. In Section 5, we shall give a sufficient condition for which the absolute moduli of two Blaschke products coincide on $M(H^\infty + C)$.

2. Preliminaries

For a sequence $\{z_n\}_n$ of points in D with $\sum_{n=1}^\infty 1 - |z_n| < \infty$, the function

$$b(z) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}, \quad z \in D,$$

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is called a *Blaschke product* with zero sequence $\{z_n\}_n$. If $\{z_n\}_n$ satisfies in addition the condition

$$\inf_k \prod_{n:n\neq k} \left| \frac{z_k - z_n}{1-\bar{z}_n z_k} \right| > 0,$$

then $\{z_n\}_n$ and $b(z)$ are called *interpolating*. If $\{z_n\}_n$ is interpolating, then for every bounded sequence $\{a_n\}_n$ there is a function f in H^∞ such that $f(z_n) = a_n$ for every n (see [8]). A function I in H^∞ is called *inner* if $|I| = 1$ in L^∞. A Blaschke product is a typical inner function. An essentially supremum norm closed algebra between H^∞ and L^∞ is called a *Douglas algebra*. By [3; 13], every Douglas algebra is generated by H^∞ and the complex conjugates of some interpolating Blaschke products. For a subset Λ of L^∞, we denote by $[H^\infty, \Lambda]$ the Douglas algebra generated by H^∞ and Λ. For a Douglas algebra B, $M(B)$ denotes the maximal ideal space of B. Then $M(B)$ can be considered to be a closed subset of $M(H^\infty)$ and $M(H^\infty + C) = M(H^\infty) \setminus D$. Also, $M(L^\infty)$ can be considered to be the Shilov boundary for every B. We identify a function with its Gelfand transform. For a subset E of $M(H^\infty)$, $\operatorname{cl} E$ denotes the closure of E in $M(H^\infty)$.

For points x and y in $M(H^\infty)$, put

$$\rho(x,y) = \sup\{|f(y)|;\ f \in H^\infty,\ \|f\| \le 1,\ f(x) = 0\}.$$

It is well known that $\rho(z,w) = |z-w|/|1-\bar{z}w|$ for $z, w \in D$. A set

$$P(x) = \{y \in M(H^\infty);\ \rho(x,y) < 1\}$$

is called a *Gleason part*. In [9], Hoffman studied Gleason parts extensively. He showed that if $P(x) \neq \{x\}$, there is a one-to-one continuous map L_x from D onto $P(x)$ such that $f \circ L_x \in H^\infty$ for every $f \in H^\infty$ and $(f \circ L_x)(0) = f(x)$. For a function f in $H^\infty + C$, we put

$$Z(f) = \{\zeta \in M(H^\infty + C);\ f(\zeta) = 0\}.$$

For $x \in Z(f)$, we put

$$\operatorname{Ord}(f,x) = \begin{cases} \operatorname{Ord}(f \circ L_x, 0) & \text{if } P(x) \neq \{x\}, \\ \infty & \text{if } P(x) = \{x\}, \end{cases}$$

where $\operatorname{Ord}(f \circ L_x, 0)$ is the usual order of the zero of the analytic function $f \circ L_x$ at 0. We note that $\operatorname{Ord}(f,x) = \infty$ if and only if $f = 0$ on $P(x)$. We denote by $Z_\infty(f)$ the set of points x in $Z(f)$ with $\operatorname{Ord}(f,x) = \infty$. By [8, p. 205], if b is an interpolating Blaschke product with zeros $\{z_n\}_n$ then $Z(b) = \operatorname{cl}\{z_n\}_n \setminus \{z_n\}_n$, and $\operatorname{Ord}(b,x) = 1$ for $x \in Z(b)$.

In this paper, we mainly study whether or not f/I is in $H^\infty + C$ when f is in $H^\infty + C$ and I is inner. We consider f/I as a function on the Shilov boundary $M(L^\infty)$, so that $f/I = f\bar{I}$.

3. Hoffman's Factorization Theorem

The following theorem is an additional property of Hoffman's factorization theorem [9, Thm. 5.2]. For a Blaschke product b with zeros $\{z_n\}_n$, let $K_\sigma(b) = \bigcap_{n=1}^\infty \{z;\ \rho(z, z_n) \ge \sigma\}$ for $\sigma > 0$.

THEOREM 3.1. *Let b be a Blaschke product with $Z_\infty(b) \neq \emptyset$. Then b admits a factorization $b = b_1 b_2$ such that $Z_\infty(b_1) = Z_\infty(b_2) = Z(b)$.*

Proof. We recall the construction of b_1 and b_2 in Garnett's book [5, p. 411]. We work in the upper half-plane H^+. Let $\{z_n\}_n$ be the zero sequence of b. Fix λ, $0<\lambda<1$, and form strips $T_k = \{x+iy \in H^+;\ \lambda^{k+1} \le y < \lambda^k\}$, k an integer. Write the z_n in T_k as the (possibly two-sided) sequence $z_{k,j}$, j an integer, so that $x_j < x_i$ if $j<i$. Put $z_{k,j} \in S_1$ if j is odd, $z_{k,j} \in S_2$ if j is even. Let b_1 and b_2 be Blaschke products with zeros S_1 and S_2 respectively. Then Hoffman proved the following inequalities:

$$c|b_1(z)|^{1/d} \le |b_2(z)| \le |b_1(z)|^d/c, \quad z \in K_\sigma(b), \tag{1}$$

where the factors b_1 and b_2 do not depend on σ, $0<\sigma<1$, and constants c and d depend on σ. For $x \in Z(b)\backslash \mathrm{cl}\{z_n\}_n$, there is $\sigma>0$ such that $x \in \mathrm{cl}\, K_\sigma(b)$. Hence by (1) we have

$$b_1 = b_2 = 0 \quad \text{on} \quad Z(b)\backslash \mathrm{cl}\{z_n\}_n.$$

By [9, Thm. 5.3], $Z_\infty(b_i) \supset Z_\infty(b)\backslash \mathrm{cl}\{z_n\}_n$, so that to prove our theorem we need to prove $Z_\infty(b_i) \supset Z(b) \cap \mathrm{cl}\{z_n\}_n$ for $i=1,2$.

Let ζ be a point in $Z_\infty(b) \cap \mathrm{cl}\{z_n\}_n$. Then $P(\zeta) \subset Z_\infty(b)$. If $P(\zeta) \not\subset \mathrm{cl}\{z_n\}_n$ then there is a point x in $P(\zeta)$ with $\mathrm{Ord}(b_i, x) = \infty$, so that $P(\zeta) \subset Z_\infty(b_i)$ for $i=1,2$. Therefore we may assume that $P(\zeta) \subset \mathrm{cl}\{z_n\}_n$.

To prove $P(\zeta) \subset Z_\infty(b_i)$ for $i=1,2$, suppose the contrary. Here we may assume that $P(\zeta) \subset Z_\infty(b_2)$ and $P(\zeta) \not\subset Z_\infty(b_1)$. Since b_1 is not zero identically on $P(\zeta)$, we may assume moreover that $b_1(\zeta) \neq 0$. Under these conditions, we shall reach a contradiction.

Let $\{w_\alpha\}_{\alpha\in\Lambda}$ be a net in $\{z_n\}_n$ such that $w_\alpha \to \zeta$. Since $b_1(\zeta) \neq 0$, we may assume that $b_2(w_\alpha) = 0$ for every $\alpha \in \Lambda$. For $0<\delta<1$, put $V_\alpha = \{z \in H^+;\ \rho(z, w_\alpha) < \delta\}$ and $w_\alpha = x_\alpha + iy_\alpha$, where $\rho(z,w) = |z-w|/|z-\bar{w}|$. Then we have

$$V_\alpha = \{x+iy \in H^+;\quad (x-x_\alpha)^2 + [y - y_\alpha(1+\delta^2)/(1-\delta)]^2 < [2y_\alpha\delta/(1-\delta^2)]^2\}. \tag{2}$$

In what follows, we choose δ sufficiently small so that the following conditions are satisfied:

$$\rho(i\lambda^k, i\lambda^{k+1}) = (1-\lambda)/(1+\lambda) > 2\delta \quad \text{for every } k; \tag{3}$$

$$b_1 \text{ does not vanish on } \{x \in P(\zeta);\ \rho(x,\zeta) \le \delta\}. \tag{4}$$

By (3), for each $\alpha \in \Lambda$ there exists a unique integer k $(=k(\alpha))$ such that

$$V_\alpha \subset T_k \cup T_{k+1} \quad \text{and} \quad V_\alpha \cap T_k \neq \emptyset. \tag{5}$$

Put $t_\alpha = x_\alpha - 2y_\alpha\delta/(1-\delta^2)$, $s_\alpha = x_\alpha + 2y_\alpha\delta/(1-\delta^2)$, and

$$W_\alpha = \{x+iy \in H^+;\ t_\alpha \le x \le s_\alpha,\ \lambda^{k+2} \le y < \lambda^k\}.$$

By (2) and (5), $V_\alpha \subset W_\alpha$. Now we need the following two sublemmas.

SUBLEMMA 1. $\sup_\alpha \sup\{\rho(z,w); z,w\in W_\alpha\}<1$.

Proof. To study the value of the left side, we may take $x_\alpha=0$. Then we have the following three inequalities since $y_\alpha<\lambda^k$:

$$\rho(t_\alpha+i\lambda^{k+2}, s_\alpha+i\lambda^{k+2})^2=\left|\frac{t_\alpha-s_\alpha}{t_\alpha-s_\alpha+i2\lambda^{k+2}}\right|^2=\frac{[4y_\alpha\delta/(1-\delta^2)]^2}{[4y_\alpha\delta/(1-\delta^2)]^2+4\lambda^{2(k+2)}}$$
$$\le\frac{[4\lambda^k\delta/(1-\delta^2)]^2}{[4\lambda^k\delta/(1-\delta^2)]^2+4\lambda^{2(k+2)}}=\frac{[4\delta/(1-\delta^2)]^2}{[4\delta/(1-\delta^2)]^2+4\lambda^2};$$

$$\rho(t_\alpha+i\lambda^{k+2}, s_\alpha+i\lambda^k)\le(1-\lambda^2)/(1+\lambda^2);$$

$$\rho(t_\alpha+i\lambda^{k+2}, s_\alpha+i\lambda^k)^2=\frac{[4y_\alpha\delta/(1-\delta^2)]^2+(\lambda^k-\lambda^{k+2})^2}{[4y_\alpha\delta/(1-\delta^2)]^2+(\lambda^k+\lambda^{k+2})^2}$$
$$\le\frac{[4\lambda^k\delta/(1-\delta^2)]^2+(\lambda^k-\lambda^{k+2})^2}{[4\lambda^k\delta/(1-\delta^2)]^2+(\lambda^k+\lambda^{k+2})^2}$$
$$=\frac{[4\delta/(1-\delta^2)]^2+(1-\lambda^2)^2}{[4\delta/(1-\delta^2)]^2+(1+\lambda^2)^2}.$$

Consequently we have our assertion. □

SUBLEMMA 2. *Let N_α be the number of zeros of b in V_α ($\alpha\in\Lambda$). Then there is a subnet Γ of Λ such that $N_\beta\to\infty$ ($\beta\in\Gamma$).*

Proof. To prove this, suppose the contrary. Then there is α_0 in Λ and a constant $K>0$ such that $N_\alpha\le K$ for every $\alpha\in\Lambda$, $\alpha\ge\alpha_0$. Therefore there exist σ, $0<\sigma<\delta/2$, and $\xi_\alpha\in V_\alpha$ such that for $\alpha\ge\alpha_0$, $\rho(\xi_\alpha,w_\alpha)<\delta/2$ and $\rho(\xi_\alpha,z_j)>\sigma$ for every $z_j\in V_\alpha\cap\{z_n\}_n$. Let $\{\xi_\beta\}_\beta$ ($\beta\in\Gamma$) be a subnet of $\{\xi_\alpha\}_\alpha$ such that $\xi_\beta\to\zeta_1$ for some $\zeta_1\in M(H^\infty+C)$. Since $w_\beta\to\zeta$ and $\rho(\xi_\beta,w_\beta)<\delta/2$, we have $\rho(\zeta_1,\zeta)\le\delta/2$ by the semicontinuity of ρ [9, p. 103]. Since $b_2=0$ on $P(\zeta)$, $b_2(\xi_\beta)\to0$. We note that if $z_j\in V_\alpha$ then $\rho(\xi_\alpha,z_j)>\sigma$, and if $z_j\notin V_\alpha$ then

$$\rho(\xi_\alpha,z_j)\ge\rho(w_\alpha,z_j)-\rho(w_\alpha,\xi_\alpha)\ge\delta-\delta/2=\delta/2>\sigma.$$

By (1), we have $b_1(\zeta_1)=0$. But this contradicts (4). □

Now we return to the proof of Theorem 3.1. By the construction of Blaschke products b_1 and b_2, if we denote by $N_{2,\alpha}$ the number of zeros (counting multiplicities) of b_2 in V_α then the number of zeros of b_1 in W_α is bigger than $N_{2,\alpha}-2$. By Sublemma 2 and the fact that $V_\alpha\subset W_\alpha$, if we denote by $N_{1,\alpha}$ the number of zeros of b_1 in W_α, we have $N_{1,\beta}\to\infty$ ($\beta\in\Gamma$). By Sublemma 1, there is a constant A such that $\rho(\xi,w_\alpha)\le A<1$ for every $\zeta\in W_\alpha$. Hence we have $|b_1(w_\beta)|\le A^{N_{1,\beta}}\to0$. This implies $b_1(\zeta)=0$, which is the desired contradiction. □

For $f\in H^\infty$, put $Z_0(f)=\{x\in M(H^\infty); f(x)=0\}$. Then $Z_0(f)$ is a closed G_δ-subset of $M(H^\infty)$.

COROLLARY 3.1. *Let b be a Blaschke product. Then the set $Z_\infty(b)$ is a closed G_δ-subset of $M(H^\infty)$.*

Proof. Let $\Lambda=\{(i_1,i_2,\dots,i_k);\ i_j=0 \text{ or } 1,\ k=1,2,\dots\}$. Then Λ is a countable set. Using Theorem 3.1, we can define a sequence of Blaschke products $\{b_\alpha;\ \alpha\in\Lambda\}$ as follows:

$$b=b_0b_1 \quad \text{and} \quad b_\alpha=b_{\alpha 0}b_{\alpha 1}.$$

Then $Z_0(b_\alpha)\supset Z_\infty(b_\alpha)=Z_\infty(b)$, so that $Z_\infty(b)\subset\bigcap\{Z_0(b_\alpha);\ \alpha\in\Lambda\}$. Let $x\in Z_0(b)\setminus Z_\infty(b)$. Then $b_\alpha(x)\neq 0$ for some $\alpha\in\Lambda$. Hence $\bigcap\{Z_0(b_\alpha);\ \alpha\in\Lambda\}\subset Z_\infty(b)$. Therefore $Z_\infty(b)=\bigcap\{Z_0(b_\alpha);\ \alpha\in\Lambda\}$ is a closed G_δ-subset of $M(H^\infty)$. □

4. Division Problems in $H^\infty+C$

In [12], the author and Y. Izuchi proved that if b is a Blaschke product and $f\in H^\infty+C$ with $|f|\le|b|$ on $M(H^\infty+C)$, then $f^2\bar{b}\in H^\infty+C$. In this section, we prove that under the above conditions there is a subproduct ψ of b such that $f\bar{\psi}\in H^\infty+C$ and $Z(\psi)=Z(b)$. To prove this, we will use some lemmas. The following lemma is a direct corollary of the theorem in [6, p. 176]; see also [15].

LEMMA 4.1. *Let B be a Douglas algebra and let I be an inner function. Then $IB\subset H^\infty+C$ if and only if $I=0$ on $M(H^\infty+C)\setminus M(B)$.*

The following lemma is a corollary of the main theorem in [12].

LEMMA 4.2. *Let b be a Blaschke product. If $f\in H^\infty+C$ and $|f|\le|b|$ on $M(H^\infty+C)$, then* $\mathrm{Ord}(b,x)=\infty$ *for every $x\in M(H^\infty+C)\setminus M([H^\infty,f\bar{b}])$.*

Proof. By [12],

$$(f\bar{b})^{k+1}b\in H^\infty+C$$

for $k=1,2,\dots$. This implies $b[H^\infty,f\bar{b}]\subset H^\infty+C$. By Lemma 4.1, $b=0$ on $M(H^\infty+C)\setminus M([H^\infty,f\bar{b}])$. Let $x\in M(H^\infty+C)\setminus M([H^\infty,f\bar{b}])$. Then $P(x)\subset M(H^\infty+C)\setminus M([H^\infty,f\bar{b}])$ [12, Lemma 2]. Since $b=0$ on $P(x)$, $\mathrm{Ord}(b,x)=\infty$. □

COROLLARY 4.1. *Let b be a Blaschke product, and let $b=b_1b_2$ be a factorization given in Theorem 3.1. If $f\in H^\infty+C$ and $|f|\le|b|$ on $M(H^\infty+C)$, then $f\bar{b}_i\in H^\infty+C$ for $i=1,2$.*

Proof. By Lemma 4.2, $\mathrm{Ord}(b,x)=\infty$ for $x\in M(H^\infty+C)\setminus M([H^\infty,f\bar{b}])$. By Theorem 3.1, $\mathrm{Ord}(b_1,x)=\infty$. By Lemma 4.1, $b_1[H^\infty,f\bar{b}]\subset H^\infty+C$, so that $f\bar{b}_2=b_1(f\bar{b})\in H^\infty+C$. □

We note that generally $Z(b_i)\neq Z(b)$ in Corollary 4.1, so that in order to find a subproduct with $Z(b_i)=Z(b)$, our job is to move points x in $Z(b_2)$ with

$\mathrm{Ord}(b_2, x) < \infty$ into $Z(b_1)$ so that $f\bar{b}_1 \in H^\infty + C$. The first part of the following lemma is proved by Hoffman [9, Thm. 3.2].

LEMMA 4.3. *Let b be a Blaschke product with distinct zeros $\{z_n\}_n$. Then b admits a factorization $b = b_0 b_1$ such that*

(i) *if* $b_0(z_n) = 0$ *then* $(1-|z_n|^2)|b_0'(z_n)| \geq |b_1(z_n)|$;
(ii) *if* $b_1(z_n) = 0$ *then* $(1-|z_n|^2)|b_1'(z_n)| \geq |b_0(z_n)|$.

Moreover, if x is a point in $Z(b)$ with $2 \leq \mathrm{Ord}(b, x) < \infty$ then $b_0(x) = b_1(x) = 0$.

Proof. Let x be in $Z(b)$ with $2 \leq \mathrm{Ord}(b, x) < \infty$. Suppose that $b_1(x) \neq 0$. Then $\mathrm{Ord}(b_0, x) = \mathrm{Ord}(b, x) \geq 2$. By [9, Thm. 5.3], there is a net $\{w_\alpha\}_\alpha$ in D such that $b_0(w_\alpha) = 0$ and $w_\alpha \to x$. By [9, Thm. 5.4], $\lim_\alpha (1-|w_\alpha|^2)|b_0'(w_\alpha)| = 0$. By (i), $b_1(w_\alpha) \to 0$, so that $b_1(x) = 0$; this is a contradiction. □

LEMMA 4.4. *Let b be a Blaschke product with zeros $\{z_n\}_n$. If* $\mathrm{Ord}(b, x) = 1$ *for every $x \in \mathrm{cl}\{z_n\}_n$, then b is an interpolating Blaschke product.*

Proof. Suppose not. Then there is a subsequence $\{z_{n_j}\}_j$ of $\{z_n\}_n$ such that

$$\lim_{j\to\infty} (1-|z_{n_j}|^2)|b'(z_{n_j})| = \lim_{j\to\infty} \prod_{n:\, n\neq n_j} \left| \frac{z_{n_j} - z_n}{1 - \bar{z}_n z_{n_j}} \right| = 0.$$

Passing to a subsequence, we may assume that $\{z_{n_j}\}_j$ is interpolating. Let b_1 be the Blaschke product with zeros $\{z_{n_j}\}_j$ and $b = b_1 b_2$. Then

$$\begin{aligned} 0 &= \lim_{j\to\infty} (1-|z_{n_j}|^2)|b'(z_{n_j})| \\ &= \lim_{j\to\infty} (1-|z_{n_j}|^2)|b_1'(z_{n_j})||b_2(z_{n_j})|. \end{aligned}$$

Since b_1 is interpolating, $(1-|z_{n_j}|^2)|b_1'(z_{n_j})| \geq \epsilon > 0$ for every j, so that $b_2(z_{n_j}) \to 0$. Therefore $b_1 = b_2 = 0$ on $\mathrm{cl}\{z_{n_j}\}_j \setminus \{z_{n_j}\}_j$. This contradicts our assumption. □

LEMMA 4.5. *Let b be a Blaschke product with zeros $\{z_n\}_n$. If* $\mathrm{Ord}(b, x) < \infty$ *for every $x \in \mathrm{cl}\{z_n\}_n$, then b is a product of finitely many interpolating Blaschke products.*

Proof. Let x be in $\mathrm{cl}\{z_n\}_n$ and let $m = \mathrm{Ord}(b, x)$. By Hoffman [9, Thm. 5.3], there is a factorization $b = q\prod_{j=1}^m b_j$ such that $q(x) \neq 0$ and b_j is an interpolating Blaschke product with $b_j(x) = 0$ for $j = 1, 2, \ldots, m$. Since $\mathrm{Ord}(b_j, y) = 1$ for every zero point y of b_j, there is a neighborhood V of x in $\mathrm{cl}\{z_n\}_n$ such that $\mathrm{Ord}(b, \zeta) \leq m$ for $\zeta \in V$. Since $\mathrm{cl}\{z_n\}_n$ is a compact subset, there is a positive integer p such that

$$\mathrm{Ord}(b, \zeta) \leq p \quad \text{for every } \zeta \in \mathrm{cl}\{z_n\}_n.$$

Hence b is a product of finitely many Blaschke products which have distinct zero sequences. Instead of working on each factor, we assume that b has a distinct zero sequence.

Let $\Lambda_k=\{(i_1,i_2,\ldots,i_k);\ i_j=0 \text{ or } 1\}$ and $\Lambda=\bigcup_k \Lambda_k$. Using a factorization in Lemma 4.3, we can derive a sequence of Blaschke products $\{b_\alpha;\ \alpha\in\Lambda\}$ as follows:

$$b=b_0b_1 \quad \text{and} \quad b_\alpha=b_{\alpha 0}b_{\alpha 1}.$$

Then $b=\prod\{b_\alpha;\ \alpha\in\Lambda_p\}$. For each $\alpha\in\Lambda_p$, by the last part of Lemma 4.3 we have $\operatorname{Ord}(b_\alpha,\zeta)=0$ or 1 for every $\zeta\in\operatorname{cl}\{z_n\}_n$. By Lemma 4.4, b_α is interpolating for every $\alpha\in\Lambda_p$. □

LEMMA 4.6. *Let b be a Blaschke product. Then there is a sequence of interpolating Blaschke products $\{b_n\}_n$ such that*

(i) $b=\prod_{n=1}^\infty b_n$, *and*

(ii) *if $x\in Z(b)\setminus Z_\infty(b)$ then* $\operatorname{Ord}(\prod_{n=1}^k b_n,x)=\operatorname{Ord}(b,x)$ *for some k depending on x.*

Proof. Let $\{z_j\}_j$ be the zero sequence of b. By Corollary 3.1, $Z_\infty(b)$ is a closed G_δ-subset of $M(H^\infty)$. Let $\{U_n\}_n$ be a decreasing sequence of open subsets of $M(H^\infty)$ such that $\bigcap_{n=1}^\infty U_n=Z_\infty(b)$. Put $\{z_{n,j}\}_j=\{z_j\}_j\cap[U_{n-1}\setminus U_n]$, where $U_0=M(H^\infty)$. Let ψ_n be the Blaschke product with zeros $\{z_{n,j}\}_j$. Then $b=\prod_{n=1}^\infty\psi_n$. Since $\{z_{n,j}\}_j\cap U_n=\emptyset$ and U_n is open, we have

$$\operatorname{cl}\{z_{n,j}\}_j\cap Z_\infty(b)\subset\operatorname{cl}\{z_{n,j}\}_j\cap U_n=\emptyset,$$

so that

$$\operatorname{Ord}(\psi_n,\zeta)\le\operatorname{Ord}(b,\zeta)<\infty \quad \text{for } \zeta\in\operatorname{cl}\{z_{n,j}\}_j.$$

By Lemma 4.5, $\psi_n=\prod_{i=1}^{k_n}\phi_{n,i}$, where $\phi_{n,i}$ is interpolating. Since $\{\phi_{n,i};\ i=1,2,\ldots,k_n,\ n=1,2,\ldots\}$ is a countable set, we can rewrite them as $\{b_n\}_n$. Of course we have $b=\prod_{n=1}^\infty b_n$.

To prove (ii), let x be in $Z(b)\setminus Z_\infty(b)$ and let $m=\operatorname{Ord}(b,x)$. There is an open subset V of $M(H^\infty)$ such that $x\in V$ and $\operatorname{cl}V\cap Z_\infty(b)=\emptyset$. By Hoffman [9, Thm. 5.3], there is a factorization $b=q\prod_{j=1}^m h_j$ such that $q(x)\ne 0$ and h_j is interpolating with $h_j(x)=0$ for $j=1,2,\ldots,m$. Here we may assume that the zero sequence of h_j is contained in V. Since $\operatorname{cl}V\cap Z_\infty(b)=\emptyset$, there is a positive integer t such that $V\cap U_t=\emptyset$. Then $\prod_{j=1}^m h_j$ is a subproduct of $\prod_{n=1}^t\psi_n$. Therefore

$$\begin{aligned}\operatorname{Ord}(b,x)=m&=\operatorname{Ord}\left(\prod_{j=1}^m h_j,x\right)\\&\le\operatorname{Ord}\left(\prod_{n=1}^t\psi_n,x\right)\le\operatorname{Ord}(b,x).\end{aligned}$$

From this we can obtain (ii). □

For a Blaschke product b with zeros $\{z_j\}_{j=1}^\infty$, subproducts with zeros $\{z_j\}_{j=n}^\infty$, $n=1,2,\ldots$, are called *tails* of b. We note that $|b_n|\to 1$ uniformly on each compact subset of D. The following lemma plays a key role in this section.

LEMMA 4.7. *Let $\{V_{s,n}\}_{s,n=1}^{\infty}$ be a family of compact subsets of D. Let $\{I_j\}_j$ be a sequence of Blaschke products. Moreover we assume that $b=\prod_{j=1}^{\infty} I_j$ is a Blaschke product. Then we have:*

(i) *if $\sup_{\zeta\in V_{s,n}}|(b\prod_{j=1}^{k}\bar{I}_j)(\zeta)|\to 0$ as $n\to\infty$ for each s and k, then there is a sequence of tails J_j of I_j such that $\sup_{\zeta\in V_{s,n}}|(b\prod_{j=1}^{\infty}\bar{J}_j)(\zeta)|\to 0$ as $n\to\infty$ for each s;*

(ii) (the dual version) *if $\inf_{\zeta\in V_{s,n}}|(\prod_{j=1}^{k} I_j)(\zeta)|\to 1$ as $n\to\infty$ for each s and k, then there is a sequence of tails J_j of I_j such that*

$$\inf_{\zeta\in V_{s,n}}\left|\left(\prod_{j=1}^{\infty} J_j\right)(\zeta)\right|\to 1$$

as $n\to\infty$ for each s.

Proof. We mainly prove (i). By small changes, we can prove (ii) as a dual version. Let $\{a_n\}_n$ and $\{\epsilon_n\}_n$ be sequences of positive numbers such that

$$a_n\to 0 \quad\text{and}\quad \sum_{n=1}^{\infty}\epsilon_n<\infty; \tag{1}$$

or

$$a_n\to 1 \quad\text{and}\quad \sum_{n=1}^{\infty}\epsilon_n<\infty \tag{1'}$$

for the dual version. By induction, we shall choose a family of positive integers $\{N_{s,n}\}_{n\ge s}$ and a sequence of Blaschke products $\{J_n\}_n$ which satisfy the following conditions:

$$\text{for each } s,\ N_{s,n}<N_{s,n+1},\ \text{so that } N_{s,n}\to\infty \text{ as } n\to\infty, \tag{2}$$

$$J_n \text{ is a tail of } I_n, \tag{3}$$

$$\sup_{\zeta\in V_{s,n}}\left|\left(b\prod_{j=1}^{k}\bar{J}_j\right)(\zeta)\right|<a_k \quad\text{for } n\ge N_{s,k} \text{ and } 1\le s\le k, \tag{4}$$

$$\sup_{\zeta\in V_{s,n}}\left|\left(b\prod_{j=1}^{k}\bar{J}_j\right)(\zeta)\right|<\sup_{\zeta\in V_{s,n}}\left|\left(b\prod_{j=1}^{k-1}\bar{J}_j\right)(\zeta)\right|+\epsilon_k \quad \text{for } 1\le n<N_{s,k} \text{ and } 1\le s\le k; \tag{5}$$

and

$$\inf_{\zeta\in V_{s,n}}\left|\left(\prod_{j=1}^{k} J_j\right)(\zeta)\right|>a_k \quad\text{for } n\ge N_{s,k} \text{ and } 1\le s\le k, \tag{4'}$$

$$\inf_{\zeta\in V_{s,n}}\left|\left(\prod_{j=1}^{k} J_j\right)(\zeta)\right|>\inf_{\zeta\in V_{s,n}}\left|\left(\prod_{j=1}^{k-1} J_j\right)(\zeta)\right|-\epsilon_k \quad \text{for } 1\le n<N_{s,k} \text{ and } 1\le s\le k. \tag{5'}$$

First we shall choose $N_{1,1}$ and J_1. By our assumption,

$$\sup_{\zeta\in V_{1,n}}|(b\bar{I}_1)(\zeta)|\to 0 \quad\text{as } n\to\infty.$$

Take $N_{1,1}$ such that

$$\sup_{\zeta \in V_{1,n}} |(b\bar{I}_1)(\zeta)| < a_1 \quad \text{for every } n \ge N_{1,1}.$$

Since $\bigcup\{V_{1,n};\ 1 \le n < N_{1,1}\}$ is compact in D, there is a tail J_1 of I_1 such that

$$\sup_{\zeta \in V_{1,n}} |(b\bar{J}_1)(\zeta)| < \sup_{\zeta \in V_{1,n}} |b(\zeta)| + \epsilon_1$$

for $1 \le n < N_{1,1}$. Of course we have

$$\sup_{\zeta \in V_{1,n}} |(b\bar{J}_1)(\zeta)| < a_1 \quad \text{for every } n \ge N_{1,1}.$$

Our induction works on k. Suppose that $\{N_{s,n}\}_{s \le n \le k}$ and J_s, $1 \le s \le k$, are already chosen so that they satisfy (2)–(5). We shall choose J_{k+1} and $N_{s,k+1}$, $s = 1, 2, \ldots, k+1$. By our assumption,

$$\sup_{\zeta \in V_{s,n}} \left| \left(b \prod_{j=1}^{k} \bar{J}_j \bar{I}_{k+1} \right)(\zeta) \right| \le \sup_{\zeta \in V_{s,n}} \left| \left(b \prod_{j=1}^{k+1} \bar{I}_j \right)(\zeta) \right| \to 0$$

as $n \to \infty$. Take $N_{s,k+1}$, $s = 1, 2, \ldots, k+1$, so that $N_{s,k} < N_{s,k+1}$ $(s \le k)$ and

$$\sup_{\zeta \in V_{s,n}} \left| \left(b \prod_{j=1}^{k} \bar{J}_j \bar{I}_{k+1} \right)(\zeta) \right| < a_{k+1}$$

for every $n \ge N_{s,k+1}$ and $1 \le s \le k+1$. Since $\bigcup\{V_{s,n};\ 1 \le n < N_{s,k+1},\ 1 \le s \le k+1\}$ is a compact subset of D, there is a tail J_{k+1} of I_{k+1} such that

$$\sup_{\zeta \in V_{s,n}} \left| \left(b \prod_{j=1}^{k+1} \bar{J}_j \right)(\zeta) \right| < \sup_{\zeta \in V_{s,n}} \left| \left(b \prod_{j=1}^{k} \bar{J}_j \right)(\zeta) \right| + \epsilon_{k+1}$$

for $1 \le n < N_{s,k+1}$ and $1 \le s \le k+1$. Then we also have

$$\sup_{\zeta \in V_{s,n}} \left| \left(b \prod_{j=1}^{k+1} \bar{J}_j \right)(\zeta) \right| < a_{k+1}$$

for $n \ge N_{s,k+1}$ and $1 \le s \le k+1$. This completes the induction for (i). By almost the same argument, we get the dual version.

Let $s \ge 1$ and $n \ge N_{s,s}$. Then by (2) there is an integer $k_n \ge s$ such that $N_{s,k_n} \le n < N_{s,k_n+1}$. Let $i \ge k_n + 1$. Then $n < N_{s,i}$. By (4) and (5), we have

$$\begin{aligned}
\sup_{\zeta \in V_{s,n}} \left| \left(b \prod_{j=1}^{i} \bar{J}_j \right)(\zeta) \right| &< \sup_{\zeta \in V_{s,n}} \left| \left(b \prod_{j=1}^{i-1} \bar{J}_j \right)(\zeta) \right| + \epsilon_i \\
&< \sup_{\zeta \in V_{s,n}} \left| \left(b \prod_{j=1}^{k_n} \bar{J}_j \right)(\zeta) \right| + \sum_{j=k_n+1}^{i} \epsilon_j \\
&< a_{k_n} + \sum_{j=k_n+1}^{i} \epsilon_j.
\end{aligned}$$

Since $b \prod_{j=1}^{i} \bar{J}_j \to b \prod_{j=1}^{\infty} \bar{J}_j$ as $i \to \infty$ uniformly on each compact subset of D, we get

$$\sup_{\zeta \in V_{s,n}} \left| \left(b \prod_{j=1}^{\infty} \bar{J}_j \right)(\zeta) \right| \le a_{k_n} + \sum_{j=k_n+1}^{\infty} \epsilon_j.$$

Now let $n\to\infty$; then $k_n\to\infty$. By (1), we have

$$\sup_{\zeta\in V_{s,n}}\left|\left(b\prod_{j=1}^{\infty}\bar{J}_j\right)(\zeta)\right|\to 0\quad\text{as } n\to\infty.$$

This completes the proof of (i).

For (ii), we can obtain

$$\inf_{\zeta\in V_{s,n}}\left|\left(\prod_{j=1}^{\infty}J_j\right)(\zeta)\right|\geq a_{k_n}-\sum_{j=k_n+1}^{\infty}\epsilon_j.$$

By (1′), $a_{k_n}-\sum_{j=k_n+1}^{\infty}\epsilon_j\to 1$ as $n\to\infty$. Hence

$$\inf_{\zeta\in V_{s,n}}\left|\left(\prod_{j=1}^{\infty}J_j\right)(\zeta)\right|\to 1\quad\text{as } n\to\infty.\qquad\square$$

Applying Lemma 4.7(i), we can prove the following theorem.

THEOREM 4.1. *Let b be a Blaschke product. If $f\in H^\infty+C$ and $|f|\leq|b|$ on $M(H^\infty+C)$, then there is a subproduct ψ of b such that $f\bar{\psi}\in H^\infty+C$, $Z(\psi)=Z(b)$, and* $\mathrm{Ord}(b,x)=\mathrm{Ord}(\psi,x)$ *for every $x\in Z(b)$.*

Proof. Let $b=b_1b_2$ be a factorization in Theorem 3.1. Then, by Lemma 4.2,

$$Z_\infty(b_1)=Z_\infty(b_2)\supset M(H^\infty+C)\backslash M([H^\infty,f\bar{b}]).$$

We shall show that there is a subproduct b_3 of b_2 such that

(1) $$|b_3|>0\ \text{ on } Z(b_2)\backslash Z_\infty(b_2),$$

(2) $$b_3=0\ \text{ on } M(H^\infty+C)\backslash M([H^\infty,f\bar{b}]).$$

By [11, Lemma 2.2], there is a sequence of interpolating Blaschke products $\{q_j\}_j$ such that

$$[H^\infty,f\bar{b}]=[H^\infty,\bar{q}_j;\ j=1,2,\ldots].$$

By Chang and Marshall's theorem [3; 13],

$$M([H^\infty,f\bar{b}])=\{x\in M(H^\infty);\ |q_j(x)|=1\ \text{ for every } j\}.$$

Put

$$q_0(x)=\sum_{j=1}^{\infty}(1/2)^j|q_j(x)|\quad\text{for } x\in M(H^\infty).$$

Then q_0 is a continuous function on $M(H^\infty)$, $0\leq q_0\leq 1$, and

$$\{x\in M(H^\infty+C);\ q_0(x)<1\}=M(H^\infty+C)\backslash M([H^\infty,f\bar{b}]).$$

Hence

(3) $$Z_\infty(b_2)\supset\{x\in M(H^\infty+C);\ q_0(x)<1\}.$$

For positive integers s and n, put

$$V_{s,n}=\{z\in D;\ q_0(z)\leq 1-1/s,\ 1-1/n\leq|z|\leq 1-1/(n+1)\}.$$

Then $V_{s,n}$ is a compact subset of D: By Lemma 4.6, there is a sequence of interpolating Blaschke products $\{I_n\}_n$ such that $b_2 = \prod_{n=1}^{\infty} I_n$, and if $x \in Z(b_2)\backslash Z_\infty(b_2)$ then $\mathrm{Ord}(\prod_{n=1}^{k} I_n, x) = \mathrm{Ord}(b_2, x)$ for some k. By (3), we have

$$b_2 \prod_{j=1}^{k} \bar{I}_j = 0 \quad \text{on} \quad \{x \in M(H^\infty + C);\, q_0(x) < 1\}.$$

By the definition of $V_{s,n}$, we have

$$\bigcap_{k=1}^{\infty} \mathrm{cl}\left[\bigcup_{n=k}^{\infty} V_{s,n}\right] \subset M(H^\infty + C);$$

$$q_0 \le 1 - 1/s \quad \text{on} \quad \bigcap_{k=1}^{\infty} \mathrm{cl}\left[\bigcup_{n=k}^{\infty} V_{s,n}\right].$$

Therefore we obtain

$$\sup_{\zeta \in V_{s,n}} \left|\left(b_2 \prod_{j=1}^{k} \bar{I}_j\right)(\zeta)\right| \to 0 \quad \text{as} \quad n \to \infty$$

for each s and k. Hence we can apply Lemma 4.7(i). Then there is a sequence of tails J_n of I_n such that

$$\text{(4)} \qquad \sup_{\zeta \in V_{s,n}} \left|\left(b_2 \prod_{j=1}^{\infty} \bar{J}_j\right)(\zeta)\right| \to 0 \quad \text{as} \quad n \to \infty$$

for each s. We note that $\prod_{j=1}^{\infty} J_i$ is a subproduct of b_2. We shall show that $b_3 = b_2 \prod_{j=1}^{\infty} \bar{J}_j$ satisfies (1) and (2).

Let $x \in Z(b_2)\backslash Z_\infty(b_2)$. Then $\mathrm{Ord}(b_2, x) = \mathrm{Ord}(\prod_{j=1}^{k} I_j, x)$ for some k. Since J_j is a tail of I_j, $\mathrm{Ord}(\prod_{j=1}^{k} J_j, x) = \mathrm{Ord}(\prod_{j=1}^{k} I_j, x)$. Hence $\mathrm{Ord}(b_3, x) = \mathrm{Ord}(b_2 \prod_{j=1}^{\infty} \bar{J}_j) = 0$; that is, $b_3(x) \ne 0$. Thus we have (1).

Next let $y \in M(H^\infty + C)\backslash M([H^\infty, f\bar{b}]) = \{\zeta \in M(H^\infty + C);\ q_0(\zeta) < 1\}$. Take a positive integer s_0 such that $q_0(y) < 1 - 1/s_0$. Then

$$y \in \mathrm{cl}\left[\bigcup_{n=k}^{\infty} V_{s_0,n}\right] \quad \text{for every } k.$$

By (4), we have $b_3(y) = 0$. Hence we get (2).

To prove our assertion, we put $\psi = b\bar{b}_3 = b_1(b_2\bar{b}_3)$. By Lemma 4.1 and (2), $f\bar{\psi} = b_3(f\bar{b}) \in H^\infty + C$. By (1), $\mathrm{Ord}(b_2\bar{b}_3, x) = \mathrm{Ord}(b_2, x)$ for $x \notin Z_\infty(b) = Z_\infty(b_2)$; hence

$$\begin{aligned}\mathrm{Ord}(\psi, x) &= \mathrm{Ord}(b_1, x) + \mathrm{Ord}(b_2\bar{b}_3, x)\\ &= \mathrm{Ord}(b_1, x) + \mathrm{Ord}(b_2, x)\\ &= \mathrm{Ord}(b, x).\end{aligned}$$

If $x \in Z_\infty(b)$, then $x \in Z_\infty(b_1)$ and $\mathrm{Ord}(\psi, x) = \infty$. As a consequence, we have $\mathrm{Ord}(\psi, x) = \mathrm{Ord}(b, x)$ for every $x \in Z(b)$. □

Using Lemma 4.7(ii), we can prove the following theorem.

THEOREM 4.2. *Let b be a Blaschke product. If $f \in H^\infty + C$ and $|f| \le |b|$ on $M(H^\infty + C)$, then there is a function g in $H^\infty + C$ such that $|g| = |f|$ on $M(H^\infty + C)$ and $g\bar{b} \in H^\infty + C$.*

Proof. We shall prove that there is a Blaschke product $J = \prod_{j=1}^\infty J_j$ such that

$$J = 0 \text{ on } M(H^\infty + C) \backslash M([H^\infty, f\bar{b}]), \tag{1}$$

$$|J| = 1 \text{ on } \{x \in M(H^\infty + C); |b(x)| > 0\}. \tag{2}$$

As in the proof of Theorem 4.1, there is a sequence of interpolating Blaschke products $\{q_j\}_j$ such that

$$[H^\infty, f\bar{b}] = [H^\infty, \bar{q}_i; i = 1, 2, \dots],$$

$$b = 0 \text{ on } \{x \in M(H^\infty + C); |q_i(x)| < 1 \text{ for some } i\}. \tag{3}$$

Let $\{I_j\}_j$ be a sequence of interpolating Blaschke products which consist of functions in $\{q_i\}_i$, where each q_i appears in $\{I_j\}_j$ infinitely many times. By considering tails of I_j, we may assume that $\prod_{j=1}^\infty I_j$ is a Blaschke product. For positive integers s and n, put

$$V_{s,n} = \{z \in D; |b(z)| \ge 1/s, 1 - 1/n \le |z| \le 1 - 1/(n+1)\}.$$

Then $V_{s,n}$ is a compact subset of D, and

$$\bigcap_{k=1}^\infty \mathrm{cl}\left[\bigcup_{n=k}^\infty V_{s,n}\right] \subset M(H^\infty + C); \ |b| \ge 1/s \text{ on } \bigcap_{k=1}^\infty \mathrm{cl}\left[\bigcup_{n=k}^\infty V_{s,n}\right].$$

By (3), for every s and j we have

$$\inf_{\zeta \in V_{s,n}} |I_j(\zeta)| \to 1 \quad \text{as } n \to \infty.$$

For suppose the contrary; then there is a sequence $\{\zeta_{n_i}\}_i$, $\zeta_{n_i} \in V_{s,n_i}$, such that $|I_j(\zeta_{n_i})| < \epsilon < 1$ for every i. Let ζ_0 be a cluster point of $\{\zeta_{n_i}\}_i$. Then $\zeta_0 \in M(H^\infty + C)$ and $|b(\zeta_0)| \ge 1/s$. Since $I_j = q_t$ for some t, $|q_t(\zeta_0)| = |I_j(\zeta_0)| \le \epsilon < 1$. By (3), $b(\zeta_0) = 0$ and this is a contradiction. Therefore for every s and k,

$$\inf_{\zeta \in V_{s,n}} \left| \left(\prod_{j=1}^k I_j \right)(\zeta) \right| \to 1 \quad \text{as } n \to \infty.$$

By Lemma 4.7(ii), there is a sequence of tails J_j of I_j such that

$$\inf_{\zeta \in V_{s,n}} \left| \left(\prod_{j=1}^\infty J_j \right)(\zeta) \right| \to 1 \quad \text{as } n \to \infty \tag{4}$$

for each s. Since $J = \prod_{j=1}^\infty J_j$ and $\{J_j\}_j$ contains infinitely many tails of each q_i, we have

$$J = 0 \text{ on } \{x \in M(H^\infty + C); |q_i(x)| < 1\}$$

for every i; that is,

$$J=0 \text{ on } M(H^\infty+C)\setminus M([H^\infty, f\bar{b}]).$$

Thus we have (1).

To prove (2), let $x \in M(H^\infty+C)$ with $|b(x)|>0$. Take a positive integer s_0 such that $|b(x)|>1/s_0$. Then

$$x \in \operatorname{cl}\left[\bigcup_{n=k}^{\infty} V_{s_0,n}\right] \quad \text{for every } k.$$

By (4), we have $|J(x)|=1$. Hence we obtain (2).

Set $g=fJ$. Since $Z(b)\subset Z(f)$, by (2) we get $|g|=|f|$ on $M(H^\infty+C)$. By (1) and Lemma 4.1, $J[H^\infty, f\bar{b}]\subset H^\infty+C$, so that $g\bar{b}=J(f\bar{b})\in H^\infty+C$. This completes the proof. □

We have the following problem.

PROBLEM 4.1. In Theorem 4.1, is there a subproduct ϕ of b such that $f\bar{\phi}\in H^\infty+C$ and $|\phi|=|b|$ on $M(H^\infty+C)$?

The following theorem is a partial answer to Problem 4.1.

THEOREM 4.3. *Let b be the Blaschke product*

$$b(z)=\prod_{n=1}^{\infty}\left(\frac{-\bar{z}_n}{|z_n|}\frac{z-z_n}{1-\bar{z}_n z}\right)^{k_n},$$

where $k_n\to\infty$ *as* $n\to\infty$. *If* $f\in H^\infty+C$ *and* $|f|\le|b|$ *on* $M(H^\infty+C)$, *then there is a subproduct* ψ *of b such that* $f\bar{\psi}\in H^\infty+C$ *and* $|\psi|=|b|$ *on* $M(H^\infty+C)$.

Proof. By the proof of Theorem 4.2, there is a Blaschke product $J=\prod_{n=1}^{\infty}J_n$ such that

(1) $$J=0 \text{ on } M(H^\infty+C)\setminus M([H^\infty, f\bar{b}])$$

and $|J|=1$ on $\{x\in M(H^\infty+C);\ |b(x)|>0\}$. By [15], there is an interpolating Blaschke product ϕ with zeros $\{w_n\}_n$ such that

(2) $$\{x\in M(H^\infty+C);\ |\phi(x)|<1\}=\{x\in M(H^\infty+C);\ |J(x)|<1\}.$$

Then $b=0$ on $\{x\in M(H^\infty+C);\ |\phi(x)|<1\}$, so that $b(w_n)\to 0$ as $n\to\infty$. Here we can choose a sequence of positive integers $\{N_n\}_n$ satisfying the following conditions:

(3) $$N_n\le k_n,\ N_n\to\infty \text{ and } k_n/N_n\to\infty \text{ as } n\to\infty;$$

(4) $$\psi_0(w_n)\to 0 \text{ as } n\to\infty, \quad \text{where } \psi_0(z)=\prod_{n=1}^{\infty}\left(\frac{-\bar{z}_n}{|z_n|}\frac{z-z_n}{1-\bar{z}_n z}\right)^{N_n}.$$

The detailed proof is left for the reader.

By (4), $\psi_0=0$ on $Z(\phi)$. By (3), $Z(\psi_0)=Z_\infty(\psi_0)$. Then, by [2; 7], $\psi_0\bar{\phi}^n\in H^\infty+C$ for every n. Hence we have

(5) $$\psi_0=0 \text{ on } \{x\in M(H^\infty+C);\ |\phi(x)|<1\}.$$

For any positive number M, by (3) there exists n_0 such that $k_n/N_n>M$ for $n\geq n_0$. Then on $M(H^\infty+C)$ we have

$$\begin{aligned}|b| &= \left|\prod_{n=n_0}^{\infty}\left(\frac{-\bar{z}_n}{|z_n|}\frac{z-z_n}{1-\bar{z}_n z}\right)^{k_n}\right| \\ &\leq \left|\prod_{n=n_0}^{\infty}\left(\frac{-\bar{z}_n}{|z_n|}\frac{z-z_n}{1-\bar{z}_n z}\right)^{N_n}\right|^M \\ &= |\psi_0|^M.\end{aligned}$$

This implies that $|\psi_0|=1$ on $\{x\in M(H^\infty+C);\ |b(x)|>0\}$. Since ψ_0 is a subproduct of b, $\psi=b\bar{\psi}_0$ is also a subproduct of b and $|\psi|=|b|$ on $M(H^\infty+C)$. By (1), (2), and (5),

$$\psi_0=0 \text{ on } M(H^\infty+C)\backslash M([H^\infty, f\bar{b}]).$$

By Lemma 4.1, we have $f\bar{\psi}=\psi_0(f\bar{b})\in H^\infty+C$. This completes the proof. □

5. Absolute Moduli of Blaschke Products on $M(H^\infty+C)$

First we shall give a sufficient condition for the absolute values of the moduli of two Blaschke products to coincide on $M(H^\infty+C)$.

THEOREM 5.1. *Let b_1 and b_2 be Blaschke products with zeros $\{z_n\}_n$ and $\{w_n\}_n$ respectively. If $\rho(z_n, w_n)\to 0$ as $n\to\infty$, then $|b_1|=|b_2|$ on $M(H^\infty+C)$.*

We use the following lemma.

LEMMA 5.1 [5, pp. 310, 404]. *Let $\{\zeta_n\}_n$ be an interpolating sequence with $\inf_k \prod_{n:n\neq k}\rho(\zeta_n,\zeta_k)\geq\delta>0$. Then there exists $\lambda=\lambda(\delta)$, $0<\delta<1$, such that $V_n=\{z\in D;\ \rho(z,\zeta_n)<\lambda\}$ are pairwise disjoint domains, and such that if $w_n\in V_n$ then $\{w_n\}_n$ is an interpolating sequence. Moreover, if $\delta\to 1$ then $\lambda\to 1$.*

For a Blaschke product b with zeros $\{z_n\}_n$, let $\delta(b)=\inf_k\prod_{n:n\neq k}\rho(z_n,z_k)$. Let $\{\zeta_j\}_j$ be a sparse sequence; that is, $\lim_{k\to\infty}\prod_{j:j\neq k}\rho(\zeta_j,\zeta_k)=1$. Let ψ be a Blaschke product with zeros $\{\zeta_j\}_j$. Let ψ_m be the mth tail of ψ; that is, ψ_m is the Blaschke product with zeros $\{\zeta_j\}_{j\geq m}$. Then $Z(\psi_m)=Z(\psi)$, $\delta(\psi_m)\leq\delta(\psi_{m+1})$, and $\delta(\psi_m)\to 1$ as $m\to\infty$.

Proof of Theorem 5.1. It is sufficient to prove that if $b_1(x)\neq 0$, $x\in M(H^\infty+C)$, then $|b_1(x)|=|b_2(x)|$. Let $x\in M(H^\infty+C)$ with $b_1(x)\neq 0$. Take a sequence $\{\zeta_j\}_j$ in D such that $b_1(\zeta_j)\to b_1(x)$ and $b_2(\zeta_j)\to b_2(x)$. Then

$$b_1=b_1(x) \text{ and } b_2=b_2(x) \text{ on } \mathrm{cl}\{\zeta_j\}_j\backslash\{\zeta_j\}_j.$$

Passing to a subsequence, we may assume moreover that $\{\zeta_j\}_j$ is sparse. Let ψ be the Blaschke product with zeros $\{\zeta_j\}_j$. For δ, $1/\sqrt{2}<\delta<1$, take m such that $\delta<\lambda(\delta(\psi_m))$, where $\lambda(\delta(\psi_m))$ is a number given in Lemma 5.1. If

$$V_n=\{z\in D;\ \rho(z,\zeta_n)<\delta\} \quad \text{for } n\geq m \text{ and } V=\bigcup_{n\geq m} V_n,$$

then $V_i\cap V_j=\emptyset$ if $i\neq j$ and $i,j\geq m$. Let N_n be the number of elements in $\{z_i\}_i\cap V_n$. If $\lim_n\sup N_n=\infty$, then $|b_1(\zeta_n)|\leq\delta^{N_n}$, so that b_1 vanishes somewhere on $Z(\psi)=\mathrm{cl}\{\zeta_j\}_j\setminus\{\zeta_j\}_j$. But $b_1=b_1(x)\neq 0$ on $\mathrm{cl}\{\zeta_j\}_j\setminus\{\zeta_j\}_j$. Hence $\{N_n\}_n$ is a bounded sequence. Put $K=\max\{N_n;\ n\geq m\}$. For the sake of simplicity, we assume $K=N_n$ for every $n\geq m$. Then $\{z_i\}_i\cap V_n$ has K elements, so that we let

$$\{\xi_{k,n}\}_{k=1}^{K}=\{z_i\}_i\cap V_n \quad \text{for each } n\geq m.$$

By Lemma 5.1, for each fixed k, $1\leq k\leq K$, $\{\xi_{k,n}\}_{n\geq m}$ is interpolating. Since $\xi_{k,n}\in\{z_i\}_i\cap V_n$, $\xi_{k,n}=z_s$ for some s. For the corresponding point w_s, we rename it as $\eta_{k,n}$; that is, $\eta_{k,n}=w_s$. Then $\{\xi_{k,n}\}_{n\geq m}$ is an interpolating subsequence of $\{z_i\}_i$ and $\{\eta_{k,n}\}_{n\geq m}$ is a subsequence of $\{w_i\}_i$.

Since $\rho(z_i,w_i)\to 0$ as $i\to\infty$, by Lemma 5.1 again, $\{\eta_{k,n}\}_{n\geq m}$ is interpolating except for a finite set for each k. Let ϕ_k and ψ_k be the Blaschke products with zeros $\{\xi_{k,n}\}_{n\geq m}$ and $\{\eta_{k,n}\}_{n\geq m}$ respectively. Since $\rho(\xi_{k,n},\eta_{k,n})\to 0$ as $n\to\infty$, $Z(\phi_k)=Z(\psi_k)$. By [2; 7], $|\phi_k|=|\psi_k|$ on $M(H^\infty+C)$ for $k=1,2,\dots,K$, so we may assume that $\{z_n\}_n\cap V=\emptyset$.

Set $\epsilon_n=\rho(z_n,w_n)$,

$$A_n=\inf_{j\geq n}\frac{1-\epsilon_j}{1+\epsilon_j} \quad\text{and}\quad B_n=\sup_{j\geq n}\frac{1+\epsilon_j}{1-\epsilon_j}.$$

Then $A_n,B_n\to 1$ as $n\to\infty$. By [5, p. 4], we have

$$\frac{\rho(\zeta_j,z_n)-\rho(z_n,w_n)}{1-\rho(\zeta_j,z_n)\rho(z_n,w_n)}\leq\rho(\zeta_j,w_n)\leq\frac{\rho(\zeta_j,z_n)+\rho(z_n,w_n)}{1+\rho(\zeta_j,z_n)\rho(z_n,w_n)},$$

so that

$$A_n\leq\frac{1-\rho^2(\zeta_j,w_n)}{1-\rho^2(\zeta_j,z_n)}\leq B_n. \tag{1}$$

Choose a positive number c_δ so that

$$1-t\leq-\log t\leq c_\delta(1-t) \quad \text{for } 2\delta^2-1\leq t\leq 1. \tag{2}$$

Here we can take c_δ such that $c_\delta\to 1$ as $\delta\to 1$. Since

$$|\rho(\zeta_j,z_n)-\rho(\zeta_j,w_n)|\leq\rho(z_n,w_n)\to 0 \quad \text{as } n\to\infty$$

and

$$\rho^2(\zeta_j,z_n)\geq\delta^2>\delta^2-(1-\delta^2)=2\delta^2-1$$

for $j\geq m$, there exists a positive integer N such that

$$\rho^2(\zeta_j,w_n)>2\delta^2-1$$

for every $n \geq N$ and $j \geq m$. By (2), for $j \geq m$,

$$\sum_{n \geq N} 1-\rho^2(\zeta_j, w_n) \leq -\log \prod_{n \geq N} \rho^2(\zeta_j, w_n) \leq c_\delta \sum_{n \geq N} 1-\rho^2(\zeta_j, w_n);$$

$$\sum_{n \geq N} 1-\rho^2(\zeta_j, z_n) \leq -\log \prod_{n \geq N} \rho^2(\zeta_j, z_n) \leq c_\delta \sum_{n \geq N} 1-\rho^2(\zeta_j, z_n).$$

By (1), we have

$$\frac{A_N}{c_\delta}\left[-\log \prod_{n \geq N} \rho^2(\zeta_j, z_n)\right] \leq -\log \prod_{n \geq N} \rho^2(\zeta_j, w_n) \leq c_\delta B_N\left[-\log \prod_{n \geq N} \rho^2(\zeta_j, z_n)\right].$$

Let

$$b_{1,N} = \prod_{n \geq N} \frac{-\bar{z}_n}{|z_n|}\frac{z-z_n}{1-\bar{z}_n z} \quad \text{and} \quad b_{2,N} = \prod_{n \geq N} \frac{-\bar{w}_n}{|w_n|}\frac{z-w_n}{1-\bar{w}_n z}.$$

Then

$$(A_N/c_\delta)[-\log|b_{1,N}(\zeta_j)|^2] \leq -\log|b_{2,N}(\zeta_j)|^2 \leq c_\delta B_N[-\log|b_{1,N}(\zeta_j)|^2],$$

so that

$$|b_{1,N}(\zeta_j)|^{A_N/c_\delta} \geq |b_{2,N}(\zeta_j)| \geq |b_{1,N}(\zeta_j)|^{c_\delta B_N}$$

for $j \geq m$. Hence

$$|b_{1,N}|^{A_N/c_\delta} \geq |b_{2,N}| \geq |b_{1,N}|^{c_\delta B_N} \text{ on } Z(\psi) = Z(\psi_m).$$

Since $|b_1| = |b_{1,N}|$ and $|b_2| = |b_{2,N}|$ on $M(H^\infty + C)$, we have

$$|b_1|^{A_N/c_\delta} \geq |b_2| \geq |b_1|^{c_\delta B_N} \text{ on } Z(\psi) = Z(\psi_m).$$

Let $N \to \infty$ and $\delta \to 1$. Since $A_N, B_N \to 1$ and $c_\delta \to 1$,

$$|b_1| \geq |b_2| \geq |b_1| \text{ on } Z(\psi).$$

Consequently we have $|b_1(x)| = |b_2(x)|$. □

The following example shows that the condition in Theorem 5.1 does not imply $b_1 \bar{b}_2 \in H^\infty + C$ generally.

EXAMPLE. We work in the upper half-plane H^+. On the horizontal line $\{x+i \in H^+;\ x \text{ is real}\}$, we consider the following two sequences:

$$\{n^2 + k/n + i;\ 0 \leq k < n,\ n = 1, 2, \dots\};$$

$$\{n^2 + k/n + i;\ 0 < k \leq n,\ n = 1, 2, \dots\}.$$

We denote these sequences by $\{z_j\}_j$ and $\{w_j\}_j$, respectively. The map

$$\phi(n^2 + k/n + i) = n^2 + (k+1)/n + i$$

induces a one-to-one and onto correspondence between $\{z_j\}_j$ and $\{w_j\}_j$. If $z_j = n^2 + k/n + i$ then $\rho(z_j, \phi(z_j)) = 1/2n \to 0$ as $j \to \infty$ by an easy calculation. Also, $\{z_j\}_j$ and $\{w_j\}_j$ satisfy the Blaschke condition

$$\sum_{j=1}^{\infty} \frac{y_j}{1-|z_j|^2} < \infty \quad \text{for } z_j = x_j + i y_j.$$

Let b_1 and b_2 be the Blaschke products with zeros $\{z_j\}_j$ and $\{w_j\}_j$. By Theorem 5.1, $|b_1| = |b_2|$ on $M(H^\infty + C)$. Let ϕ_0, ϕ_1, and ϕ_2 be the Blaschke products with zeros

$$\{n^2 + k/n + i;\, 0 < k < n,\, n = 1, 2, \ldots\},$$

$$\{n^2 + i;\, n = 1, 2, \ldots\}, \quad \text{and}$$

$$\{n^2 + 1 + i;\, n = 1, 2, \ldots\}.$$

Then $b_1 = \phi_0 \phi_1$ and $b_2 = \phi_0 \phi_2$. By [5, p. 288], $\phi_1 \phi_2$ is an interpolating Blaschke product. Hence $Z(\phi_1) \cap Z(\phi_2) = \emptyset$, so that $\phi_1 \bar{\phi}_2 \notin H^\infty + C$ and $\phi_2 \bar{\phi}_1 \notin H^\infty + C$. Therefore $b_1 \bar{b}_2 = \phi_1 \bar{\phi}_2 \notin H^\infty + C$ and $b_2 \bar{b}_1 = \phi_2 \bar{\phi}_1 \notin H^\infty + C$.

The following shows that if $\rho(z_n, w_n)$ approaches zero very rapidly in Theorem 5.1, then $b_2 \bar{b}_1 \in H^\infty + C$.

PROPOSITION 5.1. *Let b_1 be a Blaschke product with distinct zero sequence $\{z_n\}_n$. Then there is a sequence of positive numbers $\{\sigma_n\}_n$ such that if b_2 is a Blaschke product with zero sequence $\{w_n\}_n$ and if $\rho(z_n, w_n) < \sigma_n$, then $b_2 \bar{b}_1 \in H^\infty + C$.*

Proof. For each positive integer k, consider a Blaschke product

$$B_k = \prod_{j: j \neq k} \frac{-\bar{z}_j}{|z_j|} \frac{z - z_j}{1 - \bar{z}_j z}.$$

Put $\delta_k = |B_k(z_k)|$. Then $\delta_k > 0$. Take a sequence $\{\sigma_k\}_k$ such that $0 < \sigma_k < (1/2)^k \delta_k$. Let b_2 be a Blaschke product with zeros $\{w_k\}_k$ such that $\rho(z_k, w_k) < \sigma_k$. By Theorem 5.1, $|b_1| = |b_2|$ on $M(H^\infty + C)$.

We set $a_k = B_k(z_k)^{-1} b_2(z_k)$. Since $\rho(z_k, w_k) < \sigma_k$, we have $|b_2(z_k)| < \sigma_k < (1/2)^k \delta_k$. Hence

$$|a_k| = \delta_k^{-1} |b_2(z_k)| < (1/2)^k.$$

Let

$$f_n(z) = \sum_{k=n}^{\infty} a_k B_k(z) \quad \text{for } z \in D.$$

Then $f_n \in H^\infty$, $\|f_n\| \leq (1/2)^{n-1}$, and

$$f_n(z_k) = a_k B_k(z_k) = b_2(z_k) \quad \text{for } k \geq n.$$

Hence $(f_n - b_2) \bar{b}_1 \in H^\infty + C$. Therefore

$$\begin{aligned} \|b_2 \bar{b}_1 + H^\infty + C\| &= \|(b_2 - f_n) \bar{b}_1 + f_n \bar{b}_1 + H^\infty + C\| \\ &\leq \|f_n\| \\ &\leq (1/2)^{n-1} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Consequently we have $b_2 \bar{b}_1 \in H^\infty + C$. □

Next we prove the following theorem.

THEOREM 5.2. *Let b be a Blaschke product and let q be a product of finitely many interpolating Blaschke products with $|b|\le|q|$ on $M(H^\infty+C)$. Then there is a subproduct b_0 of b such that $|b|=|b_0q|$ on $M(H^\infty+C)$.*

To prove this theorem, we need lemmas.

LEMMA 5.2 [5, p. 439]. *Let E and F be subsets of D such that $\mathrm{cl}\,E\cap\mathrm{cl}\,F$ contains a point x with $P(x)\ne\{x\}$. Then* $\inf\{\rho(z,w);\ z\in E,\ w\in F\}=0$.

LEMMA 5.3. *Let $f\in H^\infty$ and let q be an interpolating Blaschke product with $Z(q)\subset Z(f)$ and $Z(q)\not\subset Z_\infty(f)$. If B is the Blaschke factor of f, then there is an interpolating subproduct b_0 of B such that $Z(b_0)\subset Z(q)$ and $Z(b_0)\supset Z(q)\setminus Z_\infty(f)$.*

Proof. Let $\{w_n\}_n$ be the zero sequence of q, and let

$$\delta=\delta(q)=\inf_k \prod_{n:n\ne k}\rho(w_n,w_k)>0.$$

Let $\lambda=\lambda(\delta)$ be a number given in Lemma 5.1. Then

$$V_n=\{z\in D;\ \rho(z,w_n)<\lambda\},\quad n=1,2,\dots,$$

are a set of disjoint domains. Put $f=BF$, where B is the Blaschke factor and F is a zero-free function on D. Since $Z(F)=Z_\infty(F)$, by Corollary 3.1, $Z_\infty(f)$ is a closed G_δ-subset of $M(H^\infty)$. Since $Z(q)=\mathrm{cl}\{w_n\}_n\setminus\{w_n\}_n$ and $Z(q)$ is a totally disconnected set [8, p. 205], there is a sequence of open and closed subsets $\{W_n\}_n$ of $Z(q)$ such that $W_n\cap W_m=\emptyset$ if $n\ne m$, and $Z(q)\setminus Z_\infty(f)=\bigcup_n W_n$. By [10, Cor. 1] there is a subproduct q_n of q such that $Z(q_n)=W_n$ and $\prod_{n=1}^\infty q_n$ is a subproduct of q. Take a sequence $\{a_n\}_n$ such that $0<a_n<\lambda$ and $a_n\to 0$. We denote by $\{w_{n,j}\}_j$ the zero sequence of q_n, and put

$$V_{n,j}=\{z\in D;\ \rho(z,w_{n,j})<a_n\}.$$

Let $\{z_k\}_k$ be the zero sequence of B. We have

$$Z(q_n)\subset Z(q)\setminus Z_\infty(f)\subset Z(f)\setminus Z_\infty(f)\subset Z(B)\setminus Z_\infty(B).$$

By [9, p. 100], $Z(B)\setminus Z_\infty(B)\subset\mathrm{cl}\{z_k\}_k$. Since q_n is interpolating, $Z(q_n)=\mathrm{cl}\{w_{n,j}\}_j\setminus\{w_{n,j}\}_j$ and every point x in $Z(q_n)$ satisfies $P(x)\ne\{x\}$ [9, Thm. 5.5]. Hence by Lemma 5.2, for every subsequence $\{\xi_j\}_j$ in $\{w_{n,j}\}_j$ we have $\inf\{\rho(\xi_j,z_k);\ j,k=1,2,\dots\}=0$. This means that there exist j_n such that for every $j\ge j_n$ there is a point $z_{n,j}$ in $\{z_k\}_k\cap V_{n,j}$ with

$$\rho(z_{n,j},w_{n,j})\to 0\quad\text{as } j\to\infty.$$

By Lemma 5.1, $\{z_{n,j};\ j\ge j_n,\ n=1,2,\dots\}$ is an interpolating sequence. Let b_0 be the Blaschke product with these zeros. Then

$$Z(q_n)=\mathrm{cl}\{w_{n,j}\}_j\setminus\{w_{n,j}\}_j\subset Z(b_0).$$

Hence $Z(q)\setminus Z_\infty(f)=\bigcup_n W_n\subset Z(b_0)$. Since $z_{n,j}\in V_{n,j}$ we have $\rho(z_{n,j},w_{n,j})< a_n\to\infty$, so that

$$\rho(z_{n,j},w_{n,j})\to 0 \quad \text{as } n\to\infty \text{ or } j\to\infty.$$

Then $Z(b_0)\subset Z(q)$. □

Proof of Theorem 5.2. First suppose that q is interpolating. By our assumption, $Z(q)\subset Z(b)$. If $Z(q)\subset Z_\infty(b)$, by [2] and [7] $b\bar{q}^n\in H^\infty+C$ for $n=1,2,\dots$. Then $b=0$ on $\{x\in M(H^\infty+C);\ |q(x)|<1\}$. Therefore we can take $b_0=b$.

If $Z(q)\not\subset Z_\infty(b)$, by Lemma 5.3 there is a subproduct ψ of b such that $Z(\psi)\subset Z(q)$ and $Z(\psi)\supset Z(q)\setminus Z_\infty(b)$. Put $b_0=b\bar{\psi}$. By [10, Cor. 1], there is a subproduct ϕ of q such that $Z(\phi)=Z(\psi)$. Put $q_0=q\bar{\phi}$. Then

$$Z(q_0)=Z(q)\setminus Z(\phi)=Z(q)\setminus Z(\psi)\subset Z_\infty(b).$$

Hence, by the same way as above, $b=0$ on $\{x\in M(H^\infty+C);\ |q_0(x)|<1\}$. Since $Z(\phi)=Z(\psi)$, $|\phi\bar{\psi}|=1$ on $M(H^\infty+C)$ by [2; 7]. Consequently,

$$|b_0q|=|b\bar{\psi}q_0\phi|=|bq_0|=|b| \text{ on } M(H^\infty+C).$$

Next let $q=\prod_{j=1}^n q_j$, where q_j is interpolating. Since $|b|\le|q|\le|q_1|$ on $M(H^\infty+C)$, by the first part there is a subproduct b_1 of b such that $|b|=|b_1q_1|$. Then we have $|b_1|\le|\prod_{j=2}^n q_j|$ on $M(H^\infty+C)$. In the same way, we can find a subproduct b_2 of b_1 such that $|b_1|=|b_2q_2|$, so that $|b|=|b_2q_1q_2|$. At the nth step, we have a subproduct b_n of b such that $|b|=|b_nq|$ on $M(H^\infty+C)$. □

PROBLEM 5.1. Is the assertion of Theorem 5.2 true when q is a general Blaschke product?

In the last part of this paper, we prove the following theorem.

THEOREM 5.3. *Let b be a product of finitely many interpolating Blaschke products. Let b_n be the* nth *tail of b. Then for every f in H^∞,*

$$\lim_{n\to\infty}\|f+b_nH^\infty\|=\|f+b(H^\infty+C)\|.$$

To prove this theorem, we need a lemma which comes from Theorem 3.1.

LEMMA 5.4. *Let f be a function in H^∞ and let $q=\prod_{j=1}^n q_j$, where q_j is an interpolating Blaschke product. If* $\mathrm{Ord}(f,x)\ge\mathrm{Ord}(q,x)$ *for every $x\in Z(q)$, then there is a factorization $f=\prod_{j=1}^n f_j$ such that $f_j\in H^\infty$, $\|f_j\|=\|f\|^{1/n}$, and $Z(f_j)\supset Z(q_j)$ for $j=1,2,\dots,n$.*

Proof. For simplicity, we shall prove Lemma 5.4 when $n=2$. Let $f=BF$, where B is a Blaschke factor. Since F is zero-free in D, $h=F^{1/2}\in H^\infty$. We note that

$$Z_\infty(f)=Z_\infty(B)\cup Z(F)=Z_\infty(B)\cup Z(h).$$

Since $Z(F)$ is a closed G_δ-subset of $M(H^\infty+C)$, by Corollary 3.1 $Z_\infty(f)$ is a closed G_δ-subset of $Z(f)$. We separate the following two cases.

Case 1: Suppose that $Z(q_i)\subset Z_\infty(f)$ for $i=1,2$, and let $B=B_1B_2$ be a factorization in Theorem 3.1. Then

$$Z_\infty(B_ih)=Z_\infty(f) \text{ for } i=1,2, \quad \text{and} \quad f=(B_1h)(B_2h).$$

Case 2: Suppose that $Z(q_1)\not\subset Z_\infty(f)$. By Lemma 5.3, we can find an interpolating Blaschke subproduct b_1 of B such that

$$Z(b_1)\subset Z(q_1) \quad \text{and} \quad Z(b_1)\supset Z(q_1)\setminus Z_\infty(f).$$

If $Z(q_2)\subset Z_\infty(f\bar{b}_1)$, let $B\bar{b}_1=B_3B_4$ be a factorization in Theorem 3.1. Then $f=(b_1B_3h)(B_4h)$. By Theorem 3.1, $Z(B_3)\supset Z(q_2)$, so that $Z(b_1B_3h)\supset Z(q_2)$. Since

$$Z_\infty(f\bar{b}_1)=Z_\infty(B\bar{b}_1)\cup Z(F)=Z_\infty(B_4)\cup Z(h),$$

we have $Z(q_2)\subset Z(B_4h)$.

If $Z(q_2)\not\subset Z_\infty(f\bar{b}_1)$, then by Lemma 5.3 again there is an interpolating Blaschke subproduct b_2 of $B\bar{b}_1$ such that

$$Z(b_2)\subset Z(q_2) \quad \text{and} \quad Z(b_2)\supset Z(q_2)\setminus Z_\infty(f\bar{b}_1).$$

Let

$$B\bar{b}_1\bar{b}_2=B_5B_6$$

be a factorization in Theorem 3.1. Then $f=(b_1B_5h)(b_2B_6h)$, $Z(b_1B_5h)\supset Z(q_1)$, and $Z(b_2B_6h)\supset Z(q_2)$. □

For a point x in $M(H^\infty)$, μ_x denotes the representing measure on $M(L^\infty)$ for H^∞. For a function f in L^∞, we denote by $N(f)$ the closure of the union set of support sets of μ_x with $f|_{\operatorname{supp}\mu_x}\notin H^\infty|_{\operatorname{supp}\mu_x}$, $x\in M(H^\infty+C)$.

Proof of Theorem 5.3. Let $b=\prod_{j=1}^k q_j$, where q_j is an interpolating Blaschke product. Hence, in this proof, k is a fixed integer. For each n, put $\phi_n=b\bar{b}_n$. Then $b_nH^\infty=b\bar{\phi}_nH^\infty$ and $\phi_n=b\bar{b}_n\in C$, so that we have

$$\liminf_{n\to\infty}\|f+b_nH^\infty\|\geq\|f+b(H^\infty+C)\|. \tag{1}$$

Next we shall prove

$$\limsup_{n\to\infty}\|f+b_nH^\infty\|\leq\|f+b(H^\infty+C)\|. \tag{2}$$

Since $H^\infty+C$ has the best approximation property [1], there exists a function h in $H^\infty+C$ such that

$$\|f+bh\|=\|f+b(H^\infty+C)\|. \tag{3}$$

By [11, Cor. 2.1], $N(\bar{b})$ is a weak peak set for H^∞ and $(H^\infty+C)|_{N(\bar{b})}=H^\infty|_{N(\bar{b})}$. By [4, p. 58], there is a function g in H^∞ such that $g=f+bh$ on $N(\bar{b})$ and

(4) $$\|g\| = \|f+bh\|_{N(\bar{b})} \leq \|f+bh\|,$$

where $\|f+bh\|_{N(\bar{b})} = \sup\{|(f+bh)(x)|;\ x \in N(\bar{b})\}$. Then

$$\mathrm{Ord}(f-g, x) \geq \mathrm{Ord}(b, x) \quad \text{for every } x \in Z(b).$$

By Lemma 5.4, there exist $f_j \in H^\infty$ $(1 \leq j \leq k)$ such that

(5) $$f - g = \prod_{j=1}^{k} f_j;$$

(6) $$\|f_j\| = \|f-g\|^{1/k} \leq (2\|f\|)^{1/k};$$

(7) $$Z(f_j) \supset Z(q_j) \quad \text{for } j = 1, 2, \ldots, k.$$

Let $\{z_{j,i}\}_i$ be the zero sequence of q_j. Then, by (7), $f_j(z_{j,i}) \to 0$ as $i \to \infty$, so that for each $\epsilon > 0$ there is a positive integer $N = N(\epsilon)$, independent of j, such that

$$|f_j(z_{j,i})| < \epsilon \quad \text{for } i \geq N,\ j = 1, 2, \ldots, k.$$

Since $\{z_{j,i}\}_i$ is an interpolating sequence, there exist an absolute constant M and $F_j \in H^\infty$ such that

(8) $$\|F_j\| < \epsilon M \text{ and } F_j(z_{j,i}) = f_j(z_{j,i}) \quad \text{for } i \geq N,\ j = 1, 2, \ldots, k.$$

Consequently $f_j - F_j \in q_{j,N} H^\infty$, where $q_{j,N}$ is the Nth tail of q_j. Let $h_j \in H^\infty$, so that

(9) $$f_j - F_j = q_{j,N} h_j.$$

We remark that if ϵ changes then h_j changes. Thus, for each positive integer n,

$$\begin{aligned}
\|f + b_n H^\infty\| &= \left\| g + \prod_{j=1}^{k} f_j + b_n H^\infty \right\| && \text{by (5)} \\
&\leq \|g\| + \left\| \prod_{j=1}^{k} (F_j + q_{j,N} h_j) - \prod_{j=1}^{k} q_{j,N} h_j \right\| \\
&\quad + \left\| \prod_{j=1}^{k} q_{j,N} h_j + b_n H^\infty \right\| && \text{by (9)} \\
&\leq \|g\| + \prod_{j=1}^{k} (\epsilon M + \|h_j\|) - \prod_{j=1}^{k} \|h_j\| \\
&\quad + \left\| \prod_{j=1}^{k} q_{j,N} h_j + b_n H^\infty \right\| && \text{by (8).}
\end{aligned}$$

To prove the last inequality, we use the elementary inequality

$$\left| \prod_{j=1}^{k} (a_j + b_j) - \prod_{j=1}^{k} b_j \right| \leq \prod_{j=1}^{k} (|a_j| + |b_j|) - \prod_{j=1}^{k} |b_j|$$

for complex numbers $\{a_j\}$ and $\{b_j\}$. Since $q_{j,N}$ is the Nth tail of q_j, $\prod_{j=1}^k q_{j,N}$ is a tail of $b=\prod_{j=1}^k q_j$. Hence the function $\prod_{j=1}^k q_{j,N}h_j$ is contained in $b_n H^\infty$ for some large integer n. Thus we have

$$\limsup_{n\to\infty}\|f+b_nH^\infty\| \le \|g\| + \prod_{j=1}^k(\epsilon M+\|h_j\|) - \prod_{j=1}^k\|h_j\|. \tag{10}$$

Here we have

$$\begin{aligned} \|h_j\| &= \|f_j - F_j\| && \text{by (9)} \\ &\le \|f_j\| + \epsilon M && \text{by (8)} \\ &\le (2\|f\|)^{1/k} + \epsilon M && \text{by (6).} \end{aligned} \tag{11}$$

Now let $\epsilon\to 0$. Recall that the function h_j depends on the value ϵ. But (11) implies that $\|h_j\|$ is bounded as $\epsilon\to 0$ for each $j=1,2,\dots,k$. Since $\epsilon M\to 0$, by (10) we have

$$\limsup_{n\to\infty}\|f+b_nH^\infty\|\le\|g\|.$$

By (3) and (4), we obtain (2). As a consequence of (1) and (2), we have our assertion. □

PROBLEM 5.2. Is the assertion of Theorem 5.3 true when b is a general Blaschke product?

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