The Equivariant Spivak Normal Bundle and Equivariant Surgery

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1. Introduction

The main purpose of this paper is to define equivariant Poincaré complexes, and to show that our definition is good in the sense that its implications for equivariant surgery are similar to those in the classical nonequivariant theory. In particular, we show the following: (1) every G-manifold is an equivariant Poincaré duality complex; (2) every finite G-Poincaré complex has an equivariant spherical Spivak normal fibration; and (3) under suitable gap hypotheses, the π–π Theorem holds for G-Poincaré pairs.

The paper is organized as follows. The remainder of this section reviews some results from [CW3], mainly the existence of ordinary equivariant homology and cohomology theories for which we can prove equivariant Thom isomorphism and Poincaré duality theorems. In Section 2 we prove the existence and uniqueness of the equivariant spherical Spivak normal fibration. In Section 3 we discuss the implications for equivariant surgery, including normal maps and the π–π Theorem. Throughout this paper G is to be a finite group.

The following definitions from [CMW] are fundamental to the theory of equivariant orientations. If X is a G-space, the *fundamental groupoid* $\pi(X;G)$ (or just πX if G is understood) *of* X is the category whose objects are the G-maps $x: G/H \to X$, where H ranges over the subgroups of G; equivalently, x is a point in X^H. A morphism $x \to y$, $y: G/K \to X$, is the equivalence class of a pair (σ, ω), where $\sigma: G/H \to G/K$ is a G-map and where $\omega: G/H \times I \to X$ is a G-homotopy from x to $y \circ \sigma$. Two such maps are equivalent if there is a G-homotopy $k: \omega \simeq \omega'$ such that $k(\alpha, 0, t) = x(\alpha)$ and $k(\alpha, 1, t) = y \circ \sigma(\alpha)$ for $\alpha \in G/H$ and $t \in I$.

Let $\mathcal{G}$ be the category of G-orbits and G-maps between them. There is a functor $\phi: \pi X \to \mathcal{G}$, given by $\phi(x: G/H \to X) = G/H$ on objects and by $\phi(\sigma, \omega) = \sigma$ on morphisms. This turns πX into a groupoid over $\mathcal{G}$ in the sense of [CMW]. If $f: X \to Y$ is a G-map, then there is an induced map $f_*: \pi X \to \pi Y$ over $\mathcal{G}$.

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Let $h\mathcal{O}_n$ be the category of n-dimensional orthogonal G-bundles over G-orbits and G-homotopy classes of linear maps, so there is again a functor $\phi: h\mathcal{O}_n \to \mathcal{G}$, giving the base space. An *n-dimensional representation of* πX is a functor $\rho: \pi X \to h\mathcal{O}_n$ such that $\phi\rho = \phi$; that is, it is a functor over $\mathcal{G}$. A *map* of representations of πX is then a natural transformation over the identity. More generally, if $f: X \to Y$ is a G-map, ρ is a representation of πX, and ρ' is a representation of πY, then a map $\rho \to \rho'$ covering f is given by a natural transformation $\eta: \rho \to \rho' \circ f_*$ over the identity. If ξ is an n-dimensional G-bundle over the G-space X, then ξ determines a representation $\rho(\xi)$ of πX given by $\rho(\xi)(x: G/H \to X) = x^*(\xi)$ on objects. $\rho(\xi)$ is defined on maps using the covering homotopy property for G-bundles. Similarly, a map of G-bundles gives rise to a map of induced representations.

If V is a representation of G, then there is a representation ρ of πX given by letting $\rho(x) = \phi(x) \times V$. We call this representation V again. If M is any smooth G-manifold, then its *tangent representation* μ is defined to be the representation of πM associated with the tangent bundle of M.

We also need the following variations defined in [CMW]. There is a category $v\mathcal{O}_n$ of *virtual* bundles over orbits, for every integer n, positive or negative. Its objects are pairs of bundles, and its morphisms are virtual maps of bundles. A *virtual representation* of πX is then a functor $\pi X \to v\mathcal{O}_n$ over $\mathcal{G}$; we call a map $\pi X \to h\mathcal{O}_n$ an *actual* representation to distinguish it from a virtual one. Maps of virtual representations are defined in the same way as maps of actual representations. The set of isomorphism classes of virtual representations of πX of all dimensions forms a group under direct sum, called $\mathrm{RO}(\pi X)$. If X is compact, or more generally has only finitely many components for each of its fixed sets, then $\mathrm{RO}(\pi X)$ is isomorphic to the Grothendieck group of the monoid of isomorphism classes of actual representations of πX, under direct sum.

Using virtual bundles, we define $G\mathfrak{R}\mathfrak{U}$ to be the category whose objects are pairs (X, γ), where X is a G-space and γ is a virtual representation of πX. A morphism $(X, \gamma) \to (Y, \delta)$ is given by a G-map $f: X \to Y$ and a map of representations $\gamma \to \delta$ covering f. For technical reasons explained in [CW3], we make the following restriction: We only consider as objects in $G\mathfrak{R}\mathfrak{U}$ those (X, γ) for which there exists a G-representation V such that, for all objects $x \in \pi X$, if $\phi(x) = G/H$ then $(\gamma \oplus V)(x) \cong G \times_H W - \mathbf{R}^n$ for some representation W of H and some n. If X is compact this is no restriction at all; in general one could probably do without this condition by using more sophisticated techniques. If $(X, \gamma) \in G\mathfrak{R}\mathfrak{U}$, then define $(X, \gamma) \times I$ to be the pair $(X \times I, \gamma')$ where $\gamma' = \gamma \circ p_*$, $p: X \times I \to X$ the projection. This gives us the notion of homotopy and the homotopy category $hG\mathfrak{R}\mathfrak{U}$.

Finally, we can define *spherical* representations by repeating all of the above using the category $h\mathfrak{F}_n$ of spherical bundles over G-orbits and spherical maps between them. Likewise, there is the category $v\mathfrak{F}_n$ of virtual spherical bundles, which gives us virtual spherical representations. These categories are all related by a commutative diagram

$$\begin{array}{ccc} h\mathcal{O}_n & \to & h\mathfrak{F}_n \\ \downarrow & & \downarrow \\ v\mathcal{O}_n & \to & v\mathfrak{F}_n \end{array}$$

of categories over $\mathcal{G}$.

In order to define homology and cohomology, we need to specify coefficients. The usual coefficients used are *Mackey functors*, which are contravariant functors from the stable orbit category $\hat{\mathcal{G}}$ to the category $\mathcal{A}b$ of abelian groups. As explained in [CW3], there is a similar construction possible with fundamental groupoids: $\hat{\pi}X$ is a category with the same objects as πX, and with $\hat{\pi}X(x,y)$ the free abelian group generated by equivalence classes of diagrams $x \leftarrow z \rightarrow y$ in πX. A *local coefficient system on* X, or a $\hat{\pi}X$*-group*, is a contravariant additive functor $\hat{\pi}X \rightarrow \mathcal{A}b$. The coefficient system we are most interested in is the *Burnside ring* system. This is the Mackey functor $\mathcal{A}_G: \hat{\mathcal{G}} \rightarrow \mathcal{A}b$ given by $\mathcal{A}_G(G/H) = A(H)$, the Burnside ring of H. We can consider this a coefficient system on any X by composing with the functor $\hat{\pi}X \rightarrow \hat{\mathcal{G}}$.

In [CW3] we constructed the following functors on $hG\mathcal{R}\mathcal{U}$.

THEOREM 1.1. *Let (X,γ) be an object in $hG\mathcal{R}\mathcal{U}$, let $A \subset X$, and let T be a local coefficient system on X. Then there are abelian groups $H^G_\gamma(X,A;T)$ and $H_G^\gamma(X,A;T)$. These are functors on the homotopy category of pairs of objects in $G\mathcal{R}\mathcal{U}$ and coefficient systems. Moreover, they satisfy the following properties.*

(i) *These functors extend Bredon's ordinary homology and cohomology with twisted coefficients* [B1].

(ii) *There are the expected long exact sequences.*

(iii) *There are isomorphisms*

$$\sigma_V: H^G_\gamma(X,A;T) \cong H^G_{\gamma+V}((X,A)\times(D(V),S(V));T)$$

and

$$\sigma_V: H_G^\gamma(X,A;T) \cong H_G^{\gamma+V}((X,A)\times(D(V),S(V));T)$$

for any G-representation V. These satisfy $\sigma_W\sigma_V = \sigma_{V+W}$.

(iv) *If $K \subset G$ then there is a restriction homomorphism*

$$\rho: H^G_\gamma(X,A;T) \rightarrow H^K_{\gamma|K}(X,A;T|K)$$

and a similar one in cohomology; we will usually write $\rho(a) = a|K$. The composite

$$H^G_\gamma(G\times_K X, G\times_K A;T) \rightarrow H^K_{\gamma|K}(G\times_K X, G\times_K A;T|K) \rightarrow H^K_{\gamma|K}(X,A;T|K)$$

is an isomorphism, as is the similar map in cohomology.

(v) *If $K \subset G$ then there is a restriction to fixed sets*

$$\zeta: H^G_\gamma(X,A;T) \rightarrow H^{NK/K}_{\gamma^K}(X^K,A^K;T^K)$$

and similarly in cohomology. We will write a^K for $\zeta(a)$ and sometimes also for $\zeta(a)|e \in H_{\gamma^K}(X^K, A^K; T^K)$.

(vi) *There is a cup product*

$$-\cup- : H_G^\gamma(X,A;S)\otimes H_G^\delta(Y,B;T)\to H_G^{\gamma+\delta}((X,A)\times(Y,B);S\square T).$$

If T is a $\hat{\pi}X$-ring, then there is a cup product

$$-\cup- : H_G^\gamma(X,A;T)\otimes H_G^\delta(X,B;T)\to H_G^{\gamma+\delta}(X,A\cup B;T).$$

This product satisfies $(\alpha\cup\beta)|K=(\alpha|K)\cup(\beta|K)$ and $(\alpha\cup\beta)^K=\alpha^K\cup\beta^K$.

(vii) *There is a cap product*

$$-\cap- : H_G^\delta(X,B;S)\otimes H^G_{\gamma+\delta}(X,A\cup B;T)\to H^G_\gamma(X,A;S\otimes_{\hat{\pi}X}\Delta_*T)$$

satisfying $(\alpha\cup\beta)\cap a=\alpha\cap(\beta\cap a)$, $(\alpha\cap a)|K=(\alpha|K)\cap(a|K)$, and $(\alpha\cap a)^K=\alpha^K\cap a^K$.

The algebra involving the coefficients is explained in [CW3]. It suffices for our purposes to know that $\mathfrak{A}_G$ is a ring, every coefficient system is a module over $\mathfrak{A}_G$, and that $\mathfrak{A}_G|K=\mathfrak{A}_K$, $(\mathfrak{A}_G)^K|e=\mathbf{Z}$, $\mathfrak{A}_G\square\mathfrak{A}_G=\mathfrak{A}_G$, and $\mathfrak{A}_G\otimes_{\hat{\pi}X}\Delta_*\mathfrak{A}_G=\mathfrak{A}_G$. We must also note a convention used in discussing fixed sets: γ^K has only locally constant dimension, so $H_{\gamma^K}(X^K)$ really stands for a sum over the components of X^K of homology groups of the indicated dimensions.

The following is proved in [CW3] and also in [CW2] with a different definition.

THEOREM 1.2 (Thom isomorphism). *If ξ is a G-vector bundle over X, and ρ is the corresponding representation of πX, then there is a class $t_\xi\in H_G^\rho(X;\mathfrak{A}_G)$ such that*

$$-\cup t_\xi : H_G^\gamma(X;T)\to H_G^{\gamma+\rho}(D(\xi),S(\xi);T)$$

is an isomorphism. t_ξ is called a Thom class *for ξ. Moreover, $t_\xi|K$ is a Thom class for ξ as a K-bundle, and t_ξ^K is a Thom class for ξ^K as an NK/K-bundle.*

In [CW3] we show the following.

THEOREM 1.3 (Poincaré duality). *If M is a compact G-manifold and μ is the representation of M associated with the tangent bundle of M, then there is a class $[M,\partial M]\in H_\mu^G(M,\partial M;\mathfrak{A}_G)$ such that*

$$-\cap[M,\partial M] : H_G^\gamma(M;\mathfrak{A}_G)\to H^G_{\mu-\gamma}(M,\partial M;\mathfrak{A}_G)$$

and

$$-\cap[M,\partial M] : H_G^\gamma(M,\partial M;\mathfrak{A}_G)\to H^G_{\mu-\gamma}(M;\mathfrak{A}_G)$$

are isomorphisms. $[M,\partial M]$ is called a fundamental class *for M. Moreover, $[M,\partial M]|K$ is a fundamental class for M as a K-manifold, and $[M,\partial M]^K$ is a fundamental class for M^K as an NK/K-manifold.*

2. The Spivak Normal Bundle

We now construct the equivariant Spivak normal bundle. All homology and cohomology will be taken with coefficients in $\mathfrak{A}_G$. Let τ be a representation of the fundamental groupoid of a given G-space X.

DEFINITION 2.1. X is a *G-Poincaré duality space of dimension* τ if there exists a class $[X] \in H_\tau^G(X)$ such that:

(a) $-\cap [X]: H_G^*(X) \to H_{\tau-*}^G(X)$ is an isomorphism;
(b) for each $K \subset G$, $-\cap [X]^K: H^*(X^K) \to H_{\tau^K-*}(X^K)$ is an isomorphism.

More generally, the pair of G-spaces (X, Y) is a *G-Poincaré duality pair of dimension* τ if there exists a class $[X, Y] \in H_\tau^G(X, Y)$ such that:

(a) the maps $-\cap [X, Y]: H_G^*(X) \to H_{\tau-*}^G(X, Y)$ and $-\cap [X, Y]: H_G^*(X, Y) \to H_{\tau-*}^G(X)$ are isomorphisms;
(b) for each $K \subset G$, $-\cap [X, Y]^K: H^*(X^K) \to H_{\tau^K-*}(X^K, Y^K)$ and $-\cap [X, Y]^K: H^*(X^K, Y^K) \to H_{\tau^K-*}(X^K)$ are isomorphisms.

The Poincaré duality theorem (1.3) shows that a closed G-manifold M is a G-Poincaré duality space of dimension μ, where μ is the tangent representation of M, and more generally that the pair $(M, \partial M)$ is a G-Poincaré duality pair for any compact G-manifold M.

Fix a G-Poincaré duality space X of dimension τ, and assume that X is a finite G-CW complex. Choose a G-embedding of X in the G-representation V with regular neighborhood U; then one has the projection $p: U \to X$ and the composite $q: \partial U \hookrightarrow U \to X$. One can replace p and q by a pair of G-fibrations $\Gamma p: E = \Gamma U \to X$ and $\Gamma q: E_0 = \Gamma \partial U \to X$. Let $E/_X E_0$ denote the fiberwise quotient over X, so that $r: E/_X E_0 \to X$ is a sectioned G-fibration.

THEOREM 2.2. *For sufficiently large V, the map $r: E/_X E_0 \to X$ is fiber G-homotopy equivalent to a spherical G-fibration of dimension $V - \tau$.*

The first step in the proof of Theorem 2.2 is the construction of a suitable candidate for the Thom class of r.

LEMMA 2.3. *There exists a class $t \in H_G^{V-\tau}(U, \partial U) \cong H_G^{V-\tau}(E, E_0)$ such that, for each $K \subset G$, if $t^K \in H^{V^K - \tau^K}(U^K, \partial U^K)$ denotes the restriction of t then*

$$-\cup t^K|: H^*(x) \to H^{|V^K - \tau^K| + *}(\Gamma p^{-1}(x)^K, \Gamma q^{-1}(x)^K)$$

is an isomorphism for each $x \in X^K$.

Proof. Since $(U, \partial U)$ is a G-manifold of dimension V, one has a fundamental class $[U, \partial U] \in H_V^G(U, \partial U)$. We take t to be the dual of $[X] \in H_\tau^G(X) \cong H_\tau^G(U)$ with respect to this class, so that $t \cap [U, \partial U] = [X]$. Restricting to the nonequivariant cohomology of the fixed sets, it follows that $t^K \cap [U, \partial U]^K = [X]^K$. Since U and X are G-Poincaré duality spaces, $[U, \partial U]^K$ and $[X]^K$

are fundamental classes for the relevant fixed sets, and so can be written as $[U^K, \partial U^K]$ and $[X^K]$, whence $t^K \cap [U^K, \partial U^K] = [X^K]$. By the nonequivariant theory [S], $E_0^K \to X^K$ is equivalent to a spherical fibration with E^K equivalent to the corresponding disc fibration (fiberwise cone). It follows from the above equation that t^K is the nonequivariant Thom class of the fibration $E_0^K \to X^K$, showing the result. □

Choose any point $x \in X$, and let L be its isotropy. Write $\tau(x) = G \times_L \tau_x$. The restriction of the Thom class t to the fiber determines an L-map $F = r^{-1}(x) \to K(W)$, where $W = V - \tau_x$ and where $K(W)$ is the Wth space of the equivariant Eilenberg-MacLane spectrum with coefficients in the Burnside system [LMM; CW1].

LEMMA 2.4. *For sufficiently large W, the unit $S^W \to K(W)$ induces an isomorphism $\pi_i(S^{W^J}) \to \pi_i(K(W)^J)$ for $0 \le i \le \dim W^J$, for every $J \subset W$.*

Proof. The result follows from the following computation of $\pi_i(K(W)^J)$. One has

$$\begin{aligned}
\pi_i(K(W)^J) &\cong [S^i, K(W)^J] \\
&\cong [S^i, K(W)]_J \\
&\cong \tilde{H}_J^{W-i}(S^0) \\
&\cong \tilde{H}_i^J(S^W) \\
&\cong \tilde{H}_{i-n}^J(S^{W-n}), \quad \text{where } n = \dim W^J; \\
&\cong \tilde{H}_{i-n}^J(\tilde{E}\mathcal{P}).
\end{aligned}$$

Here, $\mathcal{P}$ is the family of all proper subgroups of J, $E\mathcal{P}$ is the classifying space of $\mathcal{P}$ [P1], and $\tilde{E}\mathcal{P}$ is the cofiber of the projection $E\mathcal{P}^+ \to S^0$. The equivalence $H_{i-n}^J(S^{W-n}) \cong \tilde{H}_{i-n}^J(\tilde{E}\mathcal{P})$ follows from a connectivity argument, for large enough W. Finally, one has $\tilde{H}_{i-n}^J(\tilde{E}\mathcal{P}) \cong \mathbf{Z}$ if $i = n$, and 0 if $i < n$. (This last step may be seen using Bredon's universal coefficients spectral sequence [B1].) □

Proof of Theorem 2.2. Consider again the map $\alpha: F \to K(W)$. If $J \subset L$, then Lemma 2.3 asserts that $t^J|$ is the generator of the cohomology of F^J, and therefore F^J is a cohomology sphere, so that $\pi_i(F^J) \cong \mathbf{Z}$ if $i = n$, and 0 if $i < n$, where $n = \dim W^J$. Further, the fact that $t^J|$ is the generator of the cohomology of F^J, together with Lemma 2.4, implies that the map $\alpha^J: F^J \to K(W)^J$ induces an isomorphism in homotopy up through dimension n. It now follows by the equivariant Whitehead theorem that the unit $u: S^W \to K(W)$ factors through F. If $\beta: S^W \to F$ is any L-map with $\alpha\beta \simeq u$, then β is a homology isomorphism on all fixed sets, and hence an equivariant equivalence. □

We also have the following uniqueness result. Let X be a finite G-Poincaré duality space of dimension τ embedded in V with regular neighborhood

U. Then there is a collapse map $S^V \to U/\partial U \simeq E/E_0$, where $r:(E, E_0) \to X$ is the Spivak normal fibration. This determines a class α in $\pi_V^G(E/E_0) = [S^V, E/E_0]_G$, satisfying $t_r \cap \alpha_*[S^V] = [X]$. We now have the following analogue of [B2, §I.4.19], proved in essentially the same way.

PROPOSITION 2.5. *Let ξ be a $(V-\tau)$-dimensional spherical fibration over the G-Poincaré duality space X, and assume that there exists a class $\beta \in \pi_V^G(T\xi)$ satisfying $t_\xi \cap \beta_*[S^V] = [X]$. Then there is a fiber homotopy equivalence $b: \xi \to r$, unique up to fiber homotopy, such that $Tb_*(\beta) = \alpha$.*

Similar results for a Poincaré pair (X, Y) can be proved in a like manner.

3. Surgery

We can now duplicate many of the initial steps of nonequivariant surgery in the equivariant context. Let X be a finite G-Poincaré duality space of dimension τ; we wish to determine if X is G-homotopy equivalent to a smooth closed G-manifold. As usual, embed X in V with regular neighborhood U so that $U/\partial U \to X$ is equivalent to a spherical fibration of dimension $V-\tau$. We first meet the linearity obstruction; let us assume this vanishes so that there is a linear G-bundle ξ over X of dimension $V-\tau$, spherically equivalent to the Spivak normal bundle. The collapse map $c: S^V \to T\xi$ satisfies $t_\xi \cap c_*[S^V] = [X]$, where t_ξ is the Thom class of ξ. We wish to make c transverse to the zero section of ξ, and here we may meet the first G-transversality obstruction. We shall see below that the assumptions we place on τ in order to do surgery in fact guarantee that this obstruction vanishes. Assuming that c can be made transverse, we let $M = c^{-1}(X)$ and let ν be the normal bundle to the inclusion of M in V. Then we have a map $f: M \to X$ covered by $b: \nu \to \xi$. Let $d: S^V \to T\nu$ be the collapse; this yields

$$f_*[M] = f_*(t_\nu \cap d_*[S^V]) = f_*(f^* t_\xi \cap d_*[S^V]) = t_\xi \cap c_*[S^V] = [X].$$

Here, if μ is the representation of πM associated with the tangent bundle of M, we are implicitly using the virtual map $\mu \cong V - \nu \to V - \xi \cong \tau$ to identify grading. Notice now that

$$f_*^K[M^K] = [X^K]$$

for all $K \subset G$, since $[M^K] = [M]^K$ and similarly for X. Thus f is a degree 1 map on components of the fixed sets. Hence we should make the following definition.

DEFINITION 3.1. Let X be a G-Poincaré duality space of dimension τ, and let ξ be a $(V-\tau)$-dimensional bundle over X. A *normal map* into X is a pair (f, b), where $f: M \to X$ is a G-map from a closed G-manifold M, and $b: \nu \to \xi$ is a stable G-map from the normal bundle of M. This map is required to have degree 1, in the sense that $f_*[M] = [X]$, where, if μ is the dimension of M, we use $\mu \cong V - \nu \to V - \xi \cong \tau$ to identify the gradings.

A similar definition can be made for pairs. We shall show that, under certain assumptions on the fixed-set data, a normal map in this sense is normal in the sense of either [DP] or [LM]. The bundle data in a normal map, as we defined it, is essentially stable, but some unstable data is needed to make surgery work. One solution available to us is alluded to in [DR, §3], and that is to put conditions on τ that allow us to destabilize. Here is such a set of conditions.

DEFINITIONS 3.2. (*Local case*) The virtual representation $V-W$ of G is *ideal* if the following is true. Let $K\subset G$, and decompose $V-W$ into a formal sum of K-irreducibles as

$$V-W=\mathbf{R}^{n_0}\oplus\sum Z_i^{n_i}.$$

Let $d_i=1$, 2, or 4 if Z_i is (respectively) real, complex, or quaternionic. Then we require that

$$n_0<d_i(n_i+1)-1$$

for all $i\neq 0$.

(*Global case*) The representation τ of the fundamental groupoid of X is *ideal* if, for every object $x\in\pi X$, if $\tau(x)=G\times_K(V-W)$ then $(V-W)$ is an ideal representation of K.

LEMMA 3.3. *Let $(f,b)\colon(M,\nu)\to(X,\xi)$ be a normal map, with X a Poincaré duality G-space of dimension τ where τ is ideal. With TM the tangent bundle of M, let $-b\colon TM\to V-\xi$ be the virtual negative of b. Then, for some $n\geq 0$, there exists a $(\tau\oplus\mathbf{R}^n)$-dimensional G-bundle ζ over X and a G-bundle map $c\colon TM\oplus\mathbf{R}^n\to\zeta$ covering f with $c=-b$ as virtual G-bundle maps.*

Proof of Lemma 3.3. Let $B_GO(k)$ denote the classifying space of k-dimensional orthogonal G-bundles. If W is an orthogonal G-module, then there is a G-map $\sigma_W\colon B_GO(k)\to B_GO(k+|W|)$ given by addition of W. If Y is a γ-dimensional Poincaré complex (where γ is ideal) and if $p\colon E\to Y$ is $(\gamma+V)$-dimensional, let $\delta\colon Y\to B_GO(|\gamma|+|V|)$ classify p. We claim that δ factors through $\sigma_{V_G}\colon B_GO(|\gamma|+|V^G|)\to B_GO(|\gamma|+|V|)$. If $K\subset G$, and C is a component of Y^K, then $\delta^K(C)$ is contained in a component of $B_GO(|\gamma|+|V|)^K$ equivalent to $BO_K(\gamma(y)+V)$, where $y\in Y^K$ and $O_K(W)$ denotes the group of K-equivariant orthogonal automorphisms of W. Decompose $\gamma(y)$ and V into formal sums of K-irreducibles as

$$\gamma(y)=\mathbf{R}^{n_0}\oplus\sum Z_i^{n_i}\quad\text{and}\quad V=\mathbf{R}^{k_0}\oplus\sum Z_i^{k_i}.$$

Then $O_K(\gamma(y)+V)=\Pi_iA_i(n_i+k_i)$, where $A_i=O$, U, or Sp according as Z_i is real, complex, or quaternionic. It follows that the inclusion

$$BO_K(\gamma(y)+V^G)\to BO_K(\gamma(y)+V)$$

is a $\mathrm{Min}_{i0}\{d_i(n_i+1)-1\}$-equivalence. Since for each $K\subset G$, Y^K is $\gamma^K=n_0$-dimensional, the claim follows from the assumption that γ is ideal. Finally, since homotopy classes of maps between bundles correspond to homotopies between their classifying maps, the extra 1-dimensional connectivity in γ permits destabilization of maps as well as bundles. The lemma follows by applying this argument to X to destabilize $V-\xi$ and to M to destabilize the map $-b$. $\square$

REMARKS 3.4. (a) If X is a G-manifold of dimension τ and τ is ideal, then Lemma 3.3 gives us a G-normal map in the sense of [LM]. However, all one really needs for surgery is to be able to destablize restrictions of bundles to embedded spheres of no more than half the fixed-set dimension, and for this we can weaken the condition in Definition 3.2 to $n_0 < 2d_i(n_i+1)-2$.

(b) Let $f: M^\gamma \to E^{\gamma-\tau}$ be a G-map of the γ-dimensional manifold M into the total space of a $(\gamma-\tau)$-dimensional G-vector bundle, and assume that τ is ideal. Then, by [P2, §4.13], f is G-homotopic to a map that is transverse to the zero section. This justifies the claim in the first paragraph of this section: If X is a τ-dimensional G-Poincaré duality space and if τ is ideal, then the collapse map $S^V \to T\xi$ onto the Thom space of an associated $(V-\tau)$-dimensional bundle can be made transverse to the zero section.

The following conditions are standard, and are required to make surgery work.

DEFINITION 3.5. τ satisfies the *gap hypothesis* if the following is true. Suppose that $\phi(x)=G/K$ and that $\tau(x)=G\times_K V$. For every x and every $L\subset K$ we require that either $V^L=V^K$ or $\dim V^L \geq 2\dim V^K$. We say that *τ has fixed sets of dimension $\geq n$* if $\dim V^K \geq n$ for all K and all x.

We now have the following.

THEOREM 3.6 (π-π Theorem). *Let (X, Y) be a G-Poincaré duality pair of dimension τ; suppose that τ is ideal, satisfies the gap hypothesis, and has fixed sets of dimension ≥ 5. Suppose further that $\pi Y \to \pi X$ is an equivalence of groupoids over $\mathcal{G}$. If M is a smooth compact G-manifold and $f: (M, \partial M) \to (X, Y)$ is a degree 1 map covered by a stable G-bundle map $b: \nu \to \xi$, then (f, b) is normally cobordant to a G-homotopy equivalence.*

The proof is by induction on the fixed sets, following [DP], using nonequivariant surgery on each fixed set.

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