Common Fixed Points of Commuting Holomorphic Mappings in the Product of n Hilbert Balls

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1. Introduction

Let B denote the open unit ball of a complex Hilbert space H. The hyperbolic metric of B is given by the formula,

$$\rho(x,y)=\operatorname{th}^{-1}(1-\sigma(x,y))^{1/2},$$

where $\sigma(x,y)=(1-|x|^2)(1-|y|^2)/|1-(x,y)|^2$ for all $x,y\in B$. More details on the metric space (B,ρ) can be found in the books of Franzoni and Vesentini [FR] and Goebel and Reich [GR].

For $n\geq 1$ consider the hyperball B^n, equipped with its hyperbolic metric,

$$\rho_n(x,y)=\max\{\rho(x_i,y_i);\ 1\leq i\leq n\},$$

for all $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ in B^n. Holomorphic self-mappings of B^n, and more generally ρ_n-nonexpansive mappings, were studied by Kuczumow and Stachura [K1; K2; KS1; KS2], Vigué [V], and Abd-Alla [A1; A2]. In this paper we shall establish the existence of a common fixed point for a family of commuting continuous self-mappings of $\overline{B^n}$ that are holomorphic on B^n. The result provides a positive answer to an open problem of Kuczumow and Stachura [KS2]. Finite-dimensional cases of this result can be found in [S], [E], [HS], and [KS2]. For the result in B ($n=1$), see [K1] or [Si].

2. Preliminaries

In order to understand the geometry of the metric space (B^n,ρ_n), it is useful to study first the space (B,ρ). For each pair of points x,y in B there exists a unique metric segment passing through them. The midpoint of that segment will be denoted by $\frac{1}{2}x\oplus\frac{1}{2}y$; see [GR]. The proof of the next lemma can be found in [Sh].

LEMMA 2.1. *For x,y,z in B,*

$$\rho(\tfrac{1}{2}x\oplus\tfrac{1}{2}y,z)^2\leq\tfrac{1}{2}\rho(x,z)^2+\tfrac{1}{2}\rho(y,z)^2-\tfrac{1}{4}\rho(x,y)^2.$$

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The next "cosine rule" is useful when dealing with behavior near the boundary of B.

LEMMA 2.2. *For nonzero x and y in B,*

$$\operatorname{ch}\rho(x,y)=|\operatorname{ch}\rho(0,x)\operatorname{ch}\rho(0,y)-\operatorname{sh}\rho(0,x)\operatorname{sh}\rho(0,y)\cdot(x,y)/(|x|\cdot|y|)|.$$

Proof. Since $\operatorname{th}\rho(x,y)=(1-\sigma(x,y))^{1/2}$ and $\operatorname{ch}^2 t=1/(1-\operatorname{th}^2 t)$, we have

$$\operatorname{ch}^2\rho(x,y)=|1-(x,y)|^2/((1-|x|^2)(1-|y|^2)).$$

The result follows easily by noting that $\operatorname{ch}^2\rho(0,x)=1/(1-|x|^2)$, $\operatorname{sh}^2\rho(0,x)=|x|^2/(1-|x|^2)$, and the corresponding formulas for y. □

The next proposition provides a useful criterion for checking convergence to a point on the boundary of a given net in B.

PROPOSITION 2.3. *Let $\{x_\alpha\}_{\alpha\in D}$ be a ρ-unbounded net in B satisfying*

$$\sup_{\beta\ge\alpha}\{\rho(x_\alpha,x_\beta)-\rho(0,x_\beta)\}=R<\infty. \tag{2.1}$$

Then there is a point $u\in\partial B$ such that $u=\lim_\alpha x_\alpha$.

Proof. We note first that $\lim_\alpha \rho(0,x_\alpha)=\infty$. Otherwise there would exist M, and for each $i\in D$ an $\alpha_i\ge i$, such that $\rho(0,x_{\alpha_i})\le M$. But then, for all i,

$$\rho(0,x_i)\le\rho(0,x_{\alpha_i})+\rho(x_{\alpha_i},x_i)\le 2\rho(0,x_{\alpha_i})+R\le 2M+R,$$

contradicting the ρ-unboundedness of $\{x_\alpha\}$. By (2.1) there exists a $\gamma_0\in D$ and a constant c such that $\operatorname{ch}\rho(x_\alpha,x_\beta)/\operatorname{sh}\rho(0,x_\beta)\le c$ whenever $\beta\ge\alpha\ge\gamma_0$. By Lemma 2.2 we have, for $\beta\ge\alpha\ge\gamma_0$,

$$\operatorname{Re}(x_\alpha,x_\beta)/(|x_\alpha||x_\beta|)\ge\coth\rho(0,x_\alpha)\coth\rho(0,x_\beta)-c/\operatorname{sh}\rho(0,x_\alpha).$$

For a subnet $\{x_{\beta_i}\}$ converging weakly to u we have

$$\operatorname{Re}(x_\alpha,u)/|x_\alpha|\ge\coth\rho(0,x_\alpha)-c/\operatorname{sh}\rho(0,x_\alpha).$$

Now, for a subnet $\{x_{\alpha_j}\}$ converging weakly to v, we get $\operatorname{Re}(u,v)\ge 1$. Hence $u=v$ and $|u|=1$. Since the subnets were arbitrary, we conclude that the net $\{x_\alpha\}_{\alpha\in D}$ converges strongly to u. □

In a similar manner the following more general result can be verified. We omit the proof.

PROPOSITION 2.4. *Let $\{x_\alpha\}_{\alpha\in D}$ be a net in B. Then $\{x_\alpha\}_{\alpha\in D}$ converges to a point on the boundary of B if and only if*

$$\lim_{\alpha,\beta}\rho(0,x_\alpha)+\rho(0,x_\beta)-\rho(x_\alpha,x_\beta)=\infty.$$

3. Main results

Next we shall examine ρ_n-nonexpansive mappings.

DEFINITION 3.1. A mapping $T: B^n \to B^n$ is ρ_n-nonexpansive if

$$\rho_n(Tx, Ty) \le \rho_n(x, y), \quad \forall x, y \in B^n.$$

$N(B^n)$ will denote the class of all such mappings.

It is known (see [FV; GR]) that $N(B^n)$ contains all holomorphic self-mappings of B^n. The fixed point set of a mapping T will be denoted by $F(T)$.

THEOREM 3.2. *Let $\{T_\alpha\}_{\alpha \in I} \subset N(B^n)$ be a commuting family with a ρ_n-bounded invariant subset C. Then $\bigcap_{\alpha \in I} F(T_\alpha) \ne \phi$.*

Proof. We shall use induction on n. The case $n=1$, which is known (see [Si]), will be examined in the course of the proof. Let $\{S_s\}_{s \in D}$ denote the semigroup generated by $\{T_\alpha\}_{\alpha \in I}$ via composition. Each $s \in D$ may be identified with a function f_s from I to the nonnegative integers which is zero except for a finite number of entries. That is, if $S_s = T_{\alpha_1}^{n_1} \cdots T_{\alpha_k}^{n_k}$ then $f_s(\alpha_i) = n_i$, $i = 1, \dots, k$, and $f_s(\alpha) = 0$ for $\alpha \in I \setminus \{\alpha_1, \dots, \alpha_k\}$. This identification induces a natural order on D. Fix $x \in C$ and consider the functional $h: B^n \to [0, \infty)$ defined by

$$h(y) = \lim_{t \in D} \sup_{s \ge t} \rho_n(y, S_s x)^2.$$

It is easy to see that $h(T_\alpha y) \le h(y)$ for all $\alpha \in I$ and $y \in B^n$. In addition,

$$\rho_n(\tfrac{1}{2}y_1 + \tfrac{1}{2}y_2, S_s x) \le \max\{\rho_n(y_1, S_s x), \rho_n(y_2, S_s x)\}.$$

Let $a = \inf\{h(y);\ y \in B^n\}$. It follows that for all $b > a$ the set $\{y \in B^n;\ h(y) \le b\}$ is a nonempty closed and convex invariant subset for $\{T_\alpha\}_{\alpha \in I}$. A weak compactness argument shows that $K = \{y \in B^n;\ h(y) = a\}$ is a nonempty invariant subset.

If $n = 1$, Lemma 2.1 shows that K is a singleton and we are done, so we may assume $n > 1$. For $x, y \in K$ denote $\frac{1}{2}x \oplus \frac{1}{2}y = (\frac{1}{2}x_1 \oplus \frac{1}{2}y_1, \dots, \frac{1}{2}x_n \oplus \frac{1}{2}y_n)$. By Lemma 2.1 we have

$$h(\tfrac{1}{2}x \oplus \tfrac{1}{2}y) \le \tfrac{1}{2}h(x) + \tfrac{1}{2}h(y) - \min\{\rho(x_i, y_i)^2/4;\ 1 \le i \le n\},$$

so $x_{i_0} = y_{i_0}$ for some i_0. For $x, y, z \in K$ we consider $w = \frac{1}{2}z \oplus \frac{1}{2}(\frac{1}{2}x \oplus \frac{1}{2}y)$, and applying Lemma 2.1 twice we obtain

$$h(w) \le \tfrac{1}{2}h(z) + \tfrac{1}{4}h(x) + \tfrac{1}{4}h(y) - \min_i\{\tfrac{1}{8}\rho(x_i, y_i)^2 + \tfrac{1}{4}\rho(\tfrac{1}{2}x_i \oplus \tfrac{1}{2}y_i, z_i)^2\}.$$

Hence $x_{i_1} = y_{i_1} = z_{i_1}$ for some i_1. Continuing inductively we see that each finite subset of K has a common coordinate. This, for subsets of order $2n$, is enough to imply the existence of a common coordinate for all the members of K. After a possible reordering of indices we may assume that $K = \{x_1\} \times K'$ where $x_1 \in B$ and $K' \subset B^{n-1}$. For each $\alpha \in I$ define $T'_\alpha: B^{n-1} \to B^{n-1}$ by

$$T'_\alpha y = ((T_\alpha(x_1, y))_2, \dots, (T_\alpha(x_1, y))_n) \quad \forall y \in B^{n-1}.$$

K' is invariant under $\{T'_\alpha\}_{\alpha \in I}$, so by the induction hypothesis there is a fixed point y' for $\{T'_\alpha\}_{\alpha \in I}$ in B^{n-1}. Hence (x_1, y') is the desired common fixed point for $\{T_\alpha\}_{\alpha \in I}$. □

REMARK 3.3. Theorem 3.2 can be generalized to a wider class of semigroups, such as left reversible semigroups.

We quote the next result from [K1]. We remark that the existence of a common fixed point can also be deduced from Theorem 3.2, while the existence of a ρ_n-nonexpansive retraction follows from a modification of Bruck's retraction method; see [B].

THEOREM 3.4. *Let $T_1, \ldots, T_m$ be commuting mappings in $N(B^n)$ such that $F(T_j) \neq \phi$, $1 \le j \le m$. Then $\bigcap_{j=1}^m F(T_j)$ is a (nonempty) ρ_n-nonexpansive retract of B^n.*

In order to deal with mappings in $CN(B^n)$—that is, those mappings in $N(B^n)$ which have a continuous extension to the boundary—it will be convenient to consider a slightly more general class of mappings; see [K2].

DEFINITION 3.5. $N(\overline{B^n})$ is the set of all continuous mappings $T: \overline{B^n} \to \overline{B^n}$ such that $tT|_{B^n} \in N(B^n)$ for all t in $(0,1)$.

Note that we may have $Tx \in \partial B^n$ for $x \in B^n$ if $T \in N(\overline{B^n})$. But (as one can easily check) if $Tx = v$, where $|v_{i_1}| = \cdots = |v_{i_k}| = 1$, then $(Ty)_{i_1} = v_{i_1}, \ldots, (Ty)_{i_k} = v_{i_k}$ for all $y \in \overline{B^n}$.

The next lemma is essential for the proof of our main theorem.

LEMMA 3.6. *Let $\{z_\alpha\}_{\alpha \in I}$ be a ρ_n-unbounded net in B^n such that*

$$\sup_{\alpha \le \beta}\{\rho_n(z_\alpha, z_\beta) - \rho_n(0, z_\beta)\} < \infty.$$

Then there are indices $1 < i_1 < i_2 < \cdots < i_r \le n$ $(1 \le r \le n)$ and points $\{e_j\}_{j=1}^r$ in ∂B such that, for any $T \in CN(B^n)$ for which there is α_0 with $\{\rho_n(z_\alpha, Tz_\alpha)\}_{\alpha \ge \alpha_0}$ bounded, the face $K = \{y \in \partial B^n;\ y_{i_1} = e_1, \ldots, y_{i_r} = e_r\}$ is T-invariant.

Proof. By passing to a subnet and reordering indices if necessary, we may assume that for some r with $1 \le r \le n$ we have:

$$\sup_\alpha\{\rho_n(0, z_\alpha) - \rho(0, (z_\alpha)_i)\} < \infty \quad \text{for } 1 \le i \le r;$$

$$\sup_\alpha\{\rho_n(0, z_\alpha) - \rho(0, (z_\alpha)_i)\} = \infty \quad \text{for } r+1 \le i \le n.$$

For $1 \le i \le r$ and $\beta \ge \alpha$ we have

$$\rho((z_\alpha)_i, (z_\beta)_i) - \rho(0, (z_\beta)_i) \le \rho_n(z_\alpha, z_\beta) - \rho_n(0, z_\beta) + M$$

for some M. Hence by Proposition 2.3 there are $e_1, \ldots, e_r$ in ∂B such that $\lim_\alpha (z_\alpha)_i = e_i$ for $1 \le i \le r$. If $r = n$ then $\lim_\alpha z_\alpha = (e_1, \ldots, e_n)$, $\lim_\alpha Tz_\alpha = (e_1, \ldots, e_n)$, and $(e_1, \ldots, e_n)$ is a fixed point of T for each T as in the statement of the lemma. So we may assume $r < n$.

We shall show that $K = \{y \in \partial B^n;\ y_1 = e_1, \ldots, y_r = e_r\}$ is T-invariant for each T as above. Fix $(x_{r+1}, \ldots, x_n)$ in B^{n-r}, and for each α let $v_\alpha = ((z_\alpha)_1, \ldots,$

$(z_\alpha)_r, x_{r+1}, \dots, x_n)$. We have $\lim_\alpha v_\alpha = (e_1, \dots, e_r, x_{r+1}, \dots, x_n) = v$, and hence $\lim_\alpha Tv_\alpha = Tv = w$ exists. We claim that $w_i = e_i$ for $1 \le i \le r$. Indeed, for $\alpha \ge \alpha_0$ we have, for some R,

$$\begin{aligned}\max_{1\le i\le r} \rho((Tv_\alpha)_i, (z_\alpha)_i) &\le \rho_n(Tv_\alpha, z_\alpha)\\ &\le \rho_n(Tv_\alpha, Tz_\alpha) + R\\ &\le \max_{r+1\le i\le n} \rho(x_i, (z_\alpha)_i) + R.\end{aligned}$$

By the definition of r, we shall face a contradiction unless $w_i = e_i$ for $1 \le i \le r$. ☐

Next we state and prove our main theorem.

THEOREM 3.7. *A commuting family of mappings $\{T_\alpha\}_{\alpha\in I}$ in $N(\overline{B^n})$ has a common fixed point in $\overline{B^n}$.*

Proof. We shall use induction on n. The case $n=1$ is known (see [K1]), but will be verified with a different proof for the sake of completeness. The proof is divided into several steps.

(1) Assume first that there is $T = T_{\alpha_0}$ for which $Tx \in \partial B^n$ for some $x \in B^n$. Without loss of generality, $Tx = v = (v_1, \dots, v_n)$, where $|v_1| = \cdots = |v_r| = 1$ and $|v_{r+1}|, \dots, |v_n| < 1$ for some $1 \le r \le n$. It follows that for all $y \in B^n$, $(Ty)_j = v_j$ for $1 \le j \le r$. Hence, for all $\alpha \in I$,

$$(T_\alpha(Tx))_j = (T(T_\alpha x))_j = v_j, \quad 1 \le j \le r.$$

If $r = n$ (this is clearly the case if $n=1$), then v is a common fixed point, so assume $r < n$. For $(z_{r+1}, \dots, z_n) \in B^{n-r}$ denote $\tilde{z} = (v_1, \dots, v_r, z_{r+1}, \dots, z_n)$. For all s, $0 < s < 1$, and $\alpha \in I$ we have

$$\begin{aligned}\rho_n(sT_\alpha(sTx), sT_\alpha(s\tilde{z})) &\le \rho_n(sTx, s\tilde{z})\\ &= \max\{\rho(sv_j, sz_j);\ r+1 \le j \le n\}\\ &\le \max\{\rho(v_j, z_j);\ r+1 \le j \le n\}.\end{aligned}$$

Letting s tend to 1 we conclude that $(T_\alpha(\tilde{z}))_j = v_j$ when $1 \le j \le r$. Hence, the face $\{v_1, \dots, v_r\} \times \overline{B^{n-r}}$ is invariant under $\{T_\alpha\}_{\alpha\in I}$, and we may use the induction hypothesis to establish the existence of a common fixed point.

By (1) we may assume that $\{T_\alpha\}_{\alpha\in I} \subset CN(B^n)$.

(2) Assume there is $T = T_{\alpha_0}$ for which $F(T) \cap B^n = \phi$. Consider the sequence $\{T^n 0\}_{n\ge 1}$. Define $n_1 = 1$, and for $k \ge 1$ let n_{k+1} be the least $m > n_k$ for which $\rho_n(0, T^m 0) = \max\{\rho_n(0, T^j 0);\ 1 \le j \le m\}$. Denote $z_k = T^{n_k} 0$. By definition, for $k \ge m$ we have

$$\begin{aligned}\rho_n(z_m, z_k) - \rho_n(0, z_k) &= \rho_n(T^{n_m}0, T^{n_k}0) - \rho_n(0, T^{n_k}0)\\ &\le \rho_n(0, T^{n_k - n_m}0) - \rho_n(0, T^{n_k}0) \le 0.\end{aligned}$$

For all α we also have $\sup_k \rho_n(z_k, T_\alpha z_k) \le \rho_n(0, T_\alpha 0) < \infty$. By Lemma 3.6 we obtain an invariant face (a common fixed point if $n=1$) for $\{T_\alpha\}_{\alpha\in I}$, and we may use induction.

(3) By the previous steps we may assume that $F(T_\alpha)\cap B^n \neq \phi$ for all α.

Let D be the set of all finite subsets of I. For each $s\in D$ there corresponds a subset $\{\alpha_1,\ldots,\alpha_k\}$, and by Theorem 3.4 there exists a ρ_n-nonexpansive retraction $P_s: B^n \to \bigcap_{i=1}^k (F(T_{\alpha_i})\cap B^n)$. The set D is directed by inclusion. Consider the net $\{P_s 0\}_{s\in D}$. Assume first that it is ρ_n-unbounded. For $s\le t$ we have

$$\rho_n(P_s 0, P_t 0) = \rho_n(P_s 0, P_s P_t 0) \le \rho_n(0, P_t 0).$$

In addition, for each α_0 in I let s_0 correspond to the singleton $\{\alpha_0\}$. For $s\ge s_0$ we clearly have $T_{\alpha_0} P_s 0 = P_s 0$. Hence Lemma 3.6 can be applied once again to produce an invariant face for $\{T_\alpha\}_{\alpha\in I}$, and the result follows from the induction hypothesis.

The remaining case is when $\{P_s 0\}_{s\in D}$ is ρ_n-bounded. In that case, consider the functional $h: B^n \to [0,\infty)$ defined by

$$h(x) = \limsup_{t\in D\, s\ge t} \rho_n(x, P_s 0).$$

For all $\alpha_0\in I$ and $x\in B^n$ we have $h(T_{\alpha_0}x)\le h(x)$, since we may consider only $t\ge s_0$ where s_0 corresponds to the singleton $\{\alpha_0\}$.

Let $a=\inf_{x\in B^n} h(x)$ and consider $C=\{x\in B^n;\ h(x)\le a+1\}$. C is a nonempty, ρ_n-bounded, $(\{T_\alpha\}_{\alpha\in I})$-invariant subset; hence by Theorem 3.2 there is a common fixed point in B^n. □

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