The Groups of Real Genus 4

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1. Introduction

A finite group G can be represented as a group of automorphisms of a compact bordered Klein surface [14]. In other words, there is a bordered surface on which the group G acts. The *real genus* $\rho(G)$ [14] is the minimum algebraic genus of any bordered Klein surface on which G acts. This parameter is called the "real" genus because of the important correspondence between compact Klein surfaces and real algebraic curves [1]; the bordered surfaces correspond to curves with real points.

The real genus parameter was introduced in [14], and numerous basic results about the parameter were obtained there. In particular, the groups with real genus $\rho \leq 3$ were classified. There are infinite families of groups with $\rho \leq 1$. The groups of real genus 0 are the cyclic and dihedral groups [14, Thm. 3]. The group G has real genus 1 if and only if G is $Z_2 \times D_n$ with n even or $Z_2 \times Z_n$ with n even, $n \geq 4$ [14, Thm. 4]. Interestingly, there are no groups of real genus 2 [14, Thm. 5], and exactly two groups, S_4 and A_4, have real genus 3 [14, Thm. 6]; also see [4], [2], and [3].

Here we classify the groups with real genus 4. Let G_{18} denote the nonabelian group of order 18 that is not D_9 and not $Z_3 \times D_3$. Our main result is the following.

THEOREM 1. *The finite group G has real genus 4 if and only if G is $D_3 \times D_3$, $Z_3 \times D_3$, G_{18}, or $Z_3 \times Z_3$.*

We also develop some general ideas about large groups of automorphisms of bordered surfaces. One consequence is a useful lower bound for the real genus of a group; this lower bound applies to groups with order not divisible by 4 and to groups that cannot be generated by involutions. In addition, we calculate the real genus of two infinite families of supersolvable groups.

2. Preliminaries

We shall assume that all surfaces are compact. Let X be a bordered surface; X is characterized topologically by orientability, the number k of components

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of the boundary ∂X and the topological genus p. The surface X can carry a dianalytic structure [1, p. 46] and be considered a Klein surface or a non-singular algebraic curve over R. Thus the bordered surface X has an algebraic genus g, which is given by the following relation:

$$g=\begin{cases} 2p+k-1 & \text{if } X \text{ is orientable,} \\ p+k-1 & \text{if } X \text{ is non-orientable.} \end{cases}$$

The algebraic genus appears naturally in bounds for the order of an automorphism group of a Klein surface ([10] and [12], among others), and the real genus of a group is defined in terms of the algebraic genus.

Group actions on Klein surfaces have often been studied using non-Euclidean crystallographic (NEC) groups. Let $\mathcal{L}$ denote the full group of automorphisms of the open upper half-plane U. An NEC group is a discrete subgroup Γ of $\mathcal{L}$ (with the quotient space U/Γ compact). Associated with the NEC group Γ is its *signature*, which has the form

$$(2.1) \qquad (p;\pm;[m_1,\dots,m_r];\{(n_{11},\dots,n_{1s_1}),\dots,(n_{k1},\dots,n_{ks_k})\}).$$

The quotient space $X=U/\Gamma$ is a surface with topological genus p and k holes. The surface is orientable if the plus-sign is used and non-orientable otherwise. The *ordinary periods* $m_1,\dots,m_r$ are the ramification indices of the natural quotient mapping from U to X in fibers above interior points of X. The *link periods* $n_{i1},\dots,n_{is_i}$ are the ramification indices in fibers above points on the ith boundary component of X. Associated with the signature (2.1) is a presentation for the NEC group Γ. For more information about signatures, see [9] and [15].

Let Γ be an NEC group with signature (2.1) and assume $k\geq 1$ so that the quotient space U/Γ is a bordered surface. Then the non-Euclidean area $\mu(\Gamma)$ of a fundamental region for Γ can be calculated directly from its signature [15, p. 235]:

$$(2.2) \qquad \frac{\mu(\Gamma)}{2\pi}=\gamma-1+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)+\sum_{i=1}^{k}\sum_{j=1}^{s_i}\frac{1}{2}\left(1-\frac{1}{n_{ij}}\right),$$

where γ is the algebraic genus of the quotient space U/Γ. If Λ is a subgroup of finite index in Γ, then

$$(2.3) \qquad [\Gamma:\Lambda]=\mu(\Lambda)/\mu(\Gamma).$$

An NEC group K is called a *surface group* if the quotient map from U to U/K is unramified. If the quotient space U/K has a nonempty boundary, then K is called a *bordered surface group*. Bordered surface groups contain reflections but no other elements of finite order.

Let X be a bordered Klein surface of algebraic genus $g\geq 2$. Then X can be represented as U/K, where K is a bordered surface group with $\mu(K)=2\pi(g-1)$. Let G be a group of dianalytic automorphisms of the Klein surface X. Then there are an NEC group Γ and a homomorphism $\phi:\Gamma\to G$ onto G such that kernel $\phi=K$.

Involutions play an important role in group actions on bordered surfaces. Let S be a generating set for the finitely presented group G, and let $t(S)$ be the number of generators of order larger than 2. We define

$$\tau(G) = \text{minimum}\{t(S) \mid S \text{ a generating set for } G\}.$$

3. Large Groups

Especially important in the study of automorphisms of bordered Klein surfaces are the quadrilateral groups. An *extended quadrilateral group* is an NEC group with signature

$$(0; +; [\ ]; \{(l, m, n, t)\}),$$

where

$$1/l + 1/m + 1/n + 1/t < 2.$$

The group is generated by the reflections in the sides of the non-Euclidean quadrilateral with angles π/l, π/m, π/n, and π/t. We shall denote a group with this signature $\Gamma[l, m, n, t]$.

Let X be a bordered Klein surface of algebraic genus $g \geq 2$. Then the automorphism group G of X has order at most $12(g-1)$ [10]. This general upper bound was established by considering all possible ramification indices of the covering $\pi: X \to X/G$ and applying the Riemann–Hurwitz formula. A careful examination of [10, §3] shows that $o(G) \leq 6(g-1)$ in all but three cases. Here we also use the ideas of Wilkie [17] to state this result in the language of NEC groups.

PROPOSITION 1. *Let G be a group of automorphisms of a bordered Klein surface X of genus $g \geq 2$. If $o(G) > 6(g-1)$, then $o(G)$ is one of the following* (*in each case G is a quotient of the NEC group listed*):

(1) $o(G) = 12(g-1)$, $\Gamma[2, 2, 2, 3]$;
(2) $o(G) = 8(g-1)$, $\Gamma[2, 2, 2, 4]$;
(3) $o(G) = 20(g-1)/3$, $\Gamma[2, 2, 2, 5]$.

Proof. Represent X as U/K where K is a bordered surface group, and obtain an NEC group Λ with signature (2.1) and a homomorphism $\alpha: \Lambda \to G$ onto G such that kernel $\alpha = K$. Then $G = \Lambda/K$ acts on X with quotient space $Y = X/G = (U/K)/(\Lambda/K) = U/\Lambda$. Let $\pi: X \to Y$ be the natural quotient mapping. Then the calculations of [10, §3] show that $o(G) \leq 6(g-1)$ unless the quotient space Y is the disc D and π is ramified above exactly four boundary points of D; further, the ramification index in the fiber above two of these points is 2. Applying Wilkie's results [17, p. 96] as in the proof of Theorem 1 of [11], it follows that the group Λ has signature

$$(0; +; [\ ]; \{(2, 2, n, t)\}).$$

Now $\mu(K)=2\pi(g-1)$ and, from (2.2), $\mu(\Lambda)=\pi(1-1/n-1/t)$. But $o(G)=\mu(K)/\mu(\Lambda)$. Now $n=2$, and $t=3,4,5$ give the three largest orders. Also, easily, if $n=2$ and $t\geq 6$ or $t\geq n\geq 3$, then $o(G)\leq 6(g-1)$. □

This result places an obvious restriction on a large group of automorphisms. Its order must be divisible by 4.

THEOREM 2. *Let G be a finite group with* $\rho(G)\geq 2$. *If the order of G is not divisible by* 4, *then*

$$\rho(G)\geq 1+o(G)/6.$$

Proof. Let G act on a bordered surface of genus $\rho=\rho(G)$. Then, by Proposition 1, $o(G)\leq 6(\rho-1)$. □

If $\rho(G)\geq 2$, then we always have $\rho(G)\geq 1+o(G)/12$ [14, §4].

An extended quadrilateral group is generated by reflections, of course. Then a large group of automorphisms of a bordered Klein surface must be generated by involutions. Thus Proposition 1 also yields an improved lower bound for the real genus of a group that is not generated by involutions.

THEOREM 3. *Let G be a finite group with* $\rho(G)\geq 2$. *If* $\tau(G)\geq 1$, *then*

$$\rho(G)\geq 1+o(G)/6.$$

Theorem 3 has some immediate applications. If the group G is not a 2-group and the Sylow 2-subgroup of G is normal, then G is not generated by involutions.

COROLLARY 1. *Let G be a finite group that is not a 2-group and has* $\rho(G)\geq 2$. *If the Sylow* 2-*subgroup of G is normal in G, then*

$$\rho(G)\geq 1+o(G)/6.$$

COROLLARY 2. *Let G be a finite nilpotent group that is not a 2-group and has* $\rho(G)\geq 2$. *Then*

$$\rho(G)\geq 1+o(G)/6.$$

Proof. The Sylow 2-subgroup S of G is normal but $S\neq G$. □

Proposition 1 also improves in most cases the general upper bound $8(g-1)$ for the order of a nilpotent group acting on a bordered Klein surface of genus g [12, Thm. 1].

COROLLARY 3. *Let G be a nilpotent group of automorphisms of a bordered Klein surface of genus* $g\geq 2$. *If G is not a 2-group, then* $o(G)\leq 6(g-1)$.

The automorphism groups of largest possible order have received the most attention, of course. These groups are called *M^*-groups* [11]. The first important result about M^*-groups was that they must have a certain partial presentation [11, p. 5]. This was established by considering an M^*-group as a quotient of a quadrilateral group $\Gamma[2,2,2,3]$. We can extend the ideas in [11].

PROPOSITION 2. *Let G be a finite group and* $\Gamma = \Gamma[2,2,2,n]$ *an extended quadrilateral group. If there is a homomorphism* $\phi: \Gamma \to G$ *onto G such that* $K = \text{kernel}\,\phi$ *is a bordered surface group, then G is generated by three distinct nontrivial elements T, U, V satisfying the relations*

$$T^2 = U^2 = V^2 = (TU)^2 = (TV)^n = 1. \tag{3.1}$$

Proof. Here $n \geq 3$. The group Γ is generated by the four reflections t, u, j, v with defining relations

$$t^2 = u^2 = j^2 = v^2 = (tu)^2 = (uj)^2 = (jv)^2 = (tv)^n = 1.$$

Now the bordered surface group K contains a reflection, and since K is normal in Γ, it follows that one of the four generators must be in Γ [9, p. 1198].

Suppose $t \in K = \text{kernel}\,\phi$. If n is odd then clearly $v \in K$ and also $tv \in K$. If n is even, $n \geq 4$, then $(tv)^2 \in K$. In either case, K would contain an analytic element of finite order, a contradiction. Thus $t \notin K$. Similarly, $v \notin K$.

Assume then that $j \in K$, and set $T = \phi(t)$, $U = \phi(u)$, and $V = \phi(v)$. Since K contains no analytic elements of finite order, it is easy to see that T, U, V are distinct and nontrivial. These three elements clearly generate G and satisfy the relations (3.1).

The argument in the other case is similar. If $u \in K$, then choose $T = \phi(v)$, $U = \phi(j)$, and $V = \phi(t)$. □

COROLLARY 4. *The group G contains a subgroup isomorphic to the dihedral group* D_n.

Proof. Let $H = \langle T, V \rangle$. Then H is a dihedral group and $o(TV)$ must be n, again since K contains no analytic elements of finite order. □

4. Applications

Before concentrating on groups of genus 4, we briefly consider some applications of these general ideas. There is a nice connection with supersolvable groups.

The especially tractable supersolvable M^*-groups were studied in [13]. Among the M^*-groups, the supersolvable ones are completely determined by their order. An M^*-group G is supersolvable if and only if $o(G) = 4 \cdot 3^r$ for some positive integer r [13, Thm. 1].

THEOREM 4. *Let G be a supersolvable M^*-group of order* $4 \cdot 3^r$, $r \geq 2$. *If H is a subgroup of index* 2 *in G, then*

$$\rho(H) = 1 + o(H)/6.$$

Proof. The group G and its subgroup H act on a bordered surface of genus g, where $o(G)=12(g-1)$. Thus $\rho(H)\le g=1+o(H)/6$. But $o(H)=2\cdot 3^t$ is not divisible by 4. That $\rho(H)$ is not zero or unity follows from the structure of a supersolvable M^*-group [13, §3]. Hence, by Theorem 2, $\rho(H)\ge 1+o(H)/6$. □

If the M^*-group G is not supersolvable, then it might happen that $\rho(H)<\rho(G)=1+o(H)/6$. For example, $Z_2\times S_4$ is an M^*-group [10] and thus has real genus 5, but $\rho(S_4)=3$.

We apply Theorem 4 to an infinite family of groups. Let $G^{n,q,r}$ be the group with generators A, B, C and defining relations

$$A^n=B^q=C^r=(AB)^2=(BC)^2=(CA)^2=(ABC)^2=1.$$

Also, let $(l,m,n;k)$ denote the group with generators R, S, T and defining relations

$$R^l=S^m=(RS)^n=(R^{-1}S^{-1}RS)^k=1.$$

These related families contain many important groups, and were studied in [5]; see also [6].

Any finite group $G^{3,q,r}$ is an M^*-group, and any M^*-group is a quotient of some group $G^{3,q,r}$ [7, §5]. In particular, the group $G^{3,6,2t}$ is an M^*-group of order $12t^2$ [6, p. 139]. Thus we have

$$\rho(G^{3,6,2t})=1+t^2.$$

In general, the group $G^{3,6,2t}$ has two subgroups of index 2, $(2,3,6;t)$ and $(2,3,2t;3)$ [5, p. 109]. Now assume $t=3^n$. Then the M^*-group $G^{3,6,2t}$ has order $12t^2=4\cdot 3^{2n+1}$ and therefore is supersolvable.

COROLLARY 5. *If* $t=3^n$, $n\ge 1$, *then*

$$\rho(2,3,6;t)=1+t^2 \quad \textit{and} \quad \rho(2,3,2t;3)=1+t^2.$$

5. Genus 4

Theorem 4 is also relevant to our classification problem, since the M^*-group $D_3\times D_3$ is supersolvable [13]. The group $H=D_3\times D_3$ has subgroups of index 2 isomorphic to $Z_3\times D_3$ and G_{18}. (Let K be the Sylow 3-subgroup of H, and apply the correspondence theorem to the quotient map $\pi: H\to H/K\cong Z_2\times Z_2$. The subgroup isomorphic to G_{18} corresponds to $\langle(1,1)\rangle$.) The group G_{18} is denoted $((3,3,3;2))$ in [6, p. 134]. By Theorem 4, $\rho(Z_3\times D_3)=\rho(G_{18})=4$. In addition, $\rho(Z_3\times Z_3)=4$ [14, §7]. Hence $D_3\times D_3$ and its subgroups $Z_3\times D_3$, G_{18}, and $Z_3\times Z_3$ have real genus 4.

We shall consider group actions on bordered surfaces of algebraic genus 4. There are seven topological types (or *species*) of surfaces of this genus; three are orientable. Suppose G is a group with $\rho=\rho(G)=4$. Then $o(G)\le 36=12(\rho-1)$. But the only M^*-group of order 36 is $D_3\times D_3$ [7], so we may assume that $o(G) < 36$. Then, by Proposition 1, if $o(G) > 18$ then $o(G)$ is either 20 or 24. The only group with order less than 16 and real genus 4 is $Z_3 \times Z_3$ [14, §7]. A group of order 17 is cyclic, of course, and has real genus 0. Therefore, to complete the classification of the groups with real genus 4, we need only consider groups of orders 24, 20, 18, and 16. We shall generally use the notation of [6, pp. 134, 135], where there is a table of non-abelian groups of order less than 32.

6. Groups of Order 24

There are 15 groups of order 24; three of these are abelian. No group of this order has real genus 4. Some have lower genus. The groups Z_{24} and D_{12} have real genus 0 [14, Thm. 3], while $Z_2 \times Z_{12}$ and $Z_2 \times D_6$ have real genus 1 [14, Thm. 4]. Also, $\rho(S_4) = 3$ [14, Thm. 6].

The remaining groups have real genus 5 or greater. For $m \geq 2$, let H_m denote the dicyclic group $\langle 2, 2, m \rangle$ of order $4m$. Then $\rho(H_6) = 13$ and $\rho(Z_2 \times H_3) \geq \rho(H_3) = 6$ [14, Thm. 8]. Most of the remaining groups can be handled with the results of Section 3. The remaining abelian group $Z_2 \times Z_2 \times Z_6$ and the groups $Z_3 \times Q$ and $Z_3 \times D_4$ are nilpotent; by Corollary 2, the real genus of each is at least 5. Also the Sylow 2-subgroup of $Z_2 \times A_4$ is normal; thus $\rho(Z_2 \times A_4) \geq 5$ by Corollary 1.

Let B denote the binary tetrahedral group $\langle 2, 3, 3 \rangle$. The group B has A_4 as a quotient group [6, p. 68]. It follows that the Sylow 2-subgroup of B must be normal. Hence $\rho(B) \geq 5$.

Now let G be a group of order 24 that acts on a bordered surface of genus $g = 4$. Then $24 = 8(g-1)$, and G must contain a subgroup isomorphic to D_4 by Proposition 1 and Corollary 4. In other words, a Sylow 2-subgroup of G must be dihedral. A Sylow 2-subgroup of $Z_4 \times D_3$ is isomorphic to $Z_4 \times Z_2$, of course, and the ZS-metacyclic group $\langle -2, 2, 3 \rangle$ has a cyclic Sylow 2-subgroup [6, p. 11]. Hence neither of these two groups has real genus 4.

There is one remaining group of order 24 to consider. Let G_{24} be the group with generators R and S and defining relations

$$R^4 = S^6 = (RS)^2 = (R^{-1}S)^2 = 1.$$

This group is denoted $(4, 6 \mid 2, 2)$ in [6, p. 135]. It is not hard to see that $R^2 S = SR^2$, and in fact the center $Z(G_{24}) = \langle R^2 \rangle$. Also, a Sylow 2-subgroup of G_{24} is dihedral, isomorphic to $\langle R, S^3 \rangle$.

Suppose G_{24} acts on a bordered surface X with algebraic genus 4 and k boundary components. Let H be the subgroup of G_{24} that fixes each component of ∂X; H is either a dihedral group or a cyclic group with $[G_{24} : H] \leq k!$ [14, §6]. If $k = 1$ then G_{24} itself would be cyclic or dihedral. If X were a sphere with five holes or a real projective plane with four holes, then G_{24} would act on a sphere (and have symmetric genus 0) [14, §5]. But G_{24} does not act on the sphere [8, pp. 287–291]. Thus we may assume that topologically X is one of the three remaining species, orientable with $k=3$ or non-orientable with $k=2$ or $k=3$.

First suppose X to be non-orientable with $k=2$. Since H is cyclic or dihedral, H must have index 2 in G_{24}. Let W be the surface obtained from X by attaching a disc to one of the boundary components. Then the group H acts on W, a non-orientable surface with one hole and algebraic genus 3. It follows that $o(H) \leq 8$ [3, p. 42], an obvious contradiction.

Next suppose X to be one of the species with $k=3$. Then the quotient space X/G_{24} is the disc D, and the quotient map $\pi: X \to D$ is ramified above exactly four points of D, all on ∂D, with ramification indices $2, 2, 2, 4$ (see the proof of Proposition 1). Then let the center Z act on X and set $X' = X/Z$. Let g' be the genus of X' and let $\phi: X \to X'$ be the quotient map. Then G_{24}/Z acts on X', and we have the following diagram of quotient maps:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle\nu} \\ & & D. \end{array}$$

Applying the Riemann–Hurwitz formula for coverings of bordered surfaces [10, p. 201] to ν yields

$$2g' - 2 = 12\left[-2 + \sum\left(1 - \frac{1}{e_i}\right)\right],$$

where the e_i's are the ramification indices. The possible values for these indices are quite limited since $\pi = \nu \circ \phi$, and it is easy to see that the only solution is $g' = 1$ (and $e_i = 2$ for $i = 1, \ldots, 4$). Thus the quotient space $X' = X/Z$ has genus 1.

Let C be any one of the three components of ∂X, and let

$$H_c = \{f \in G_{24} \mid f(C) = C\}.$$

Since G_{24} is transitive on ∂X (X/G_{24} is the disc), it is basic that $[G_{24}: H_c] = k = 3$. Thus H_c is a Sylow 2-subgroup and $H_c \cong D_4$. Now H_c acts as a dihedral group on C. But $o(Z) = 2$ so that $Z \subset H_c$. Hence $Z = Z(H_c)$ and Z must act on C as a rotation. Then $\phi(C)$ is a component of $\partial X'$, and thus X' would have (at least) three boundary components. This is not possible, since X' has genus 1. Therefore G_{24} does not act on a bordered surface of algebraic genus 4, and there is no group of order 24 with $\rho = 4$.

7. Groups of Order 20, 18, and 16

There are five groups of order 20. None have real genus 4. The groups Z_{20} and D_{10} have real genus 0 and $\rho(Z_2 \times Z_{10}) = 1$. The dicyclic group H_5 has real genus 11 [14, Thm. 8].

The only group of order 20 left to consider is the K-metacyclic group M. The group M has generators S and T and defining relations [6, p. 134]

$$S^5 = T^4 = 1, \quad T^{-1}ST = S^2.$$

Each element of M can be written S^iT^j, where $0 \le i \le 4$ and $0 \le j \le 3$. It is not hard to see that M has exactly five involutions, the elements of the form S^iT^2. But each of these elements is in the normal subgroup $\langle S, T^2 \rangle \cong D_5$. Thus M is not generated by involutions, and $\tau(M) = 1$. Now, by Theorem 3, $\rho(M) \ge 5$.

There are five groups of order 18, and two of these, $Z_3 \times D_3$ and G_{18}, have real genus 4. The groups Z_{18} and D_9 have real genus 0.

Let $G = Z_3 \times Z_6$, the remaining group of order 18, and let G act on a bordered surface X of algebraic genus g. We shall show that $g \ge 7$. We know $g \ge 4$. There are an NEC group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that $X = U/K$, where $K = \text{kernel}\,\phi$ is a bordered surface group with $\mu(K) = 2\pi(g-1)$. Let γ denote the algebraic genus of the quotient space $Y = U/\Gamma$, and simplify the canonical presentation for Γ as in [14, §2]. In this simplified presentation there must be at least two elements with order larger than 2, since $\Gamma/K \cong G$. It follows (as in [14]) that

$$\gamma + r \ge 2. \tag{7.1}$$

From (2.3) we obtain

$$(g-1)/18 = \mu(\Gamma)/2\pi,$$

which is given by (2.2). We obtain a lower bound for this expression. If $\gamma \ge 2$, then easily $g \ge 19$. Suppose $\gamma = 1$. From (7.1) we have $r \ge 1$ with at least one ordinary period larger than 2. Thus $(g-1)/18 \ge 2/3$ and $g \ge 13$. Now assume $\gamma = 0$ so that Y is the disc D. Using (7.1) again, we have $r \ge 2$ and at least two of the ordinary periods are greater than 2. Then $(g-1)/18 \ge -1 + 2 \cdot 2/3$ and $g \ge 7$. Hence $\rho(G) \ge 7$.

Now let G be a group of order 16, and suppose that G acts on a bordered surface X of algebraic genus 4. Then represent X as U/K, where K is a bordered surface group, and proceed as in the proof of Proposition 1. Obtain an NEC group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that $\text{kernel}\,\phi = K$. Let G act on X and consider the quotient mapping $\pi : X \to X/G$. Then a careful examination of the calculations of [10, §3] shows that there is only one possibility for the ramification indices of π, and consequently G is a quotient of an extended quadrilateral group $\Gamma[2,2,2,8]$. Therefore, by Corollary 4, G must be the dihedral group D_8. Thus D_8 is the only group of order 16 that acts on a bordered surface of algebraic genus 4, and $\rho(D_8) = 0$. No group of order 16 has real genus 4.

8. Real Genus 5

It is now natural to consider the problem of classifying the groups of real genus 5. There are at least five such groups. First $Z_2 \times S_4$ is an M^*-group [10]. Hence $Z_2 \times S_4$ and its subgroup $Z_2 \times A_4$ have real genus 5. The quaternion group Q and $(Z_2)^4$ also have this real genus [14, §7]. Using the upper bound for the real genus of a group from [14, §3], it is not hard to see that the group G of order 16 denoted $(4,4 \mid 2,2)$ in [6, p. 134] has $\rho(G) = 5$.

We conjecture that there are other groups (at least one or two) of real genus 5.

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