The Inclusion of Classical Families in the Closure of the Universal Teichmüller Space

STEPHEN D. CASEY

Introduction

A major problem of Teichmüller theory is to characterize the closure of Bers's model of the universal Teichmüller space $\mathfrak{T}$. In this paper, we give new geometric conditions that if satisfied by the range of a conformal mapping from a disk guarantee that the Schwarzian derivative of that mapping is in the closure of $\mathfrak{T}$. The condition is described by defining a subclass of close-to-convex functions. This result extends the work of Gehring and Astala [3].

Let $\mathbf{B}$ denote the Banach space of functions ϕ holomorphic on the unit disk $\mathbb{D}$ with finite norm

$$\|\phi\| = \sup_{z \in \mathbb{D}} |\phi(z)|[1-|z|^2].$$

If f is conformal on $\mathbb{D}$ and

$$\phi = \frac{d}{dz} \log f'(z) = \frac{f''(z)}{f'(z)},$$

then $\|\phi\| \leq 8$ (see [13]). The development of injectivity criteria for functions on $\mathbb{D}$ led Becker and Pommerenke (see [4; 5]) to study the space

$$\mathbf{T} = \{\phi = (f''/f') : f \text{ conformal on the unit disk } \mathbb{D} \text{ with quasiconformal extension to the Riemann sphere } \hat{\mathbb{C}}\}.$$

This subset of $\mathbf{B}$ is an alternative model for the universal Teichmüller space (see [3]).

In [3], Astala and Gehring develop a characterization of the closure of $\mathbf{T}$ in the $\|\cdot\|$ norm. They apply this characterization to prove that $f''/f' \in \mathrm{cl}(\mathbf{T})$ if f is convex in the direction θ. We introduce a new method which shows that $f''/f' \in \mathrm{cl}(\mathbf{T})$ provided that $\mathbb{C} \setminus f(\mathbb{D})$ is the union of disjoint half-lines l, where the angle of l is continuous in the prime end topology.

Let $\mathbf{L}$ denote the class of linearly accessible functions; that is, $\mathbf{L} = \{f$ conformal on $\mathbb{D}$ with $f(0)=0$, $f'(0)=1$: $\mathbb{C} \setminus f(\mathbb{D})$ is the union of closed half-lines such that the corresponding open half-lines are disjoint$\}$. Lewandowski [10; 11] (also see [6]) shows that $\mathbf{L}$ is equivalent to the class of close-to-convex functions introduced by Kaplan [9]. Thus, given $f \in \mathbf{L}$, there exists

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a conformal map h such that $h(\mathbb{D})$ is a convex region and $\mathrm{Re}\{f'/h'\}>0$. We refer to h as the *associated convex function* of f.

The derivatives of $\log(f'(z))$ for general functions f in $\mathbf{L}$ are not necessarily in $\mathrm{cl}(\mathbf{T})$. Astala and Gehring [3] provide a counterexample, the starlike function

$$f(z)=\frac{z}{(1-z^2)}.$$

Writing $f=g\circ h$ for h convex, we have that

$$h(z)=\int_0^z \frac{f(\zeta)}{\zeta}\,d\zeta=\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)=\operatorname{arctanh} z$$

and that $h(\mathbb{D})$ is a strip. In particular, $\partial h(\mathbb{D})$ is not smooth. This example leads us to examine the following class of functions.

DEFINITION. Let $f\in\mathbf{L}$. We say that f is *linearly accessible smoothly* (denoted $f\in\mathbf{L}_0$) if the associated convex domain $h(\mathbb{D})$ has a boundary with a continuously turning tangent (see Pommerenke [15, p. 295]).

We prove the following theorem.

THEOREM 1. *Suppose that f is linearly accessible smoothly. Then*

$$\frac{f''}{f'}\in\mathrm{cl}(\mathbf{T}).$$

By observing that $\mathbf{L}_0$ contains some important subclasses, we also obtain the following.

COROLLARY 1. *Let f satisfy* $\mathrm{Re}\, f'>0$ *or be a bounded starlike function. Then*

$$\frac{f''}{f'}\in\mathrm{cl}(\mathbf{T}).$$

The map $\psi(\phi)=\phi'-\frac{1}{2}(\phi)^2$ is a continuous imbedding of $\mathbf{T}$ into Bers's model of the universal Teichmüller space $\mathfrak{T}$, mapping $(f''(z)/f'(z))$ of f conformal onto their Schwarzian derivatives

$$\{f,z\}=\left(\frac{f''(z)}{f'(z)}\right)'-\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

(see [3; 7]). We therefore obtain the next corollary.

COROLLARY 2. *If $f\in\mathbf{L}_0$ then $\{f,z\}\in\mathrm{cl}(\mathfrak{T})$.*

2. Sequences of Linearly Accessible Functions

Let $\mathbf{H}(\mathbb{D})$ be the set of holomorphic functions on $\mathbb{D}$. Let $\mathbf{P}$ be the set of $p\in\mathbf{H}(\mathbb{D})$ such that

$$\mathrm{Re}\{p\}>0,\quad p(0)=1.$$

Choose a linearly accessible function f. Then $f = g \circ h$, where $\operatorname{Re}\{g'\} > 0$ on $h(\mathbb{D})$. Therefore,

$$\frac{f'}{h'} = p.$$

We define *the approximating family* $\{f_\lambda\}$ $(0 < \lambda < 1)$ of an $f \in \mathbf{L}$ by

$$f_\lambda' = p^\lambda \cdot h'.$$

We normalize each f_λ by an appropriate transformation so that $f_\lambda(0) = 0$ and $f_\lambda'(0) = 1$. Clearly, as $\operatorname{Re} p > 0$, we may choose a branch of the logarithm so that $\operatorname{Re} p^\lambda > 0$. Thus $f_\lambda \in \mathbf{L}$, and by a result of Kaplan [9] we have the following lemma.

LEMMA 1. *For* $0 < \lambda < 1$, f_λ *is univalent.*

REMARK. Lemmas 1–5 hold for the approximating family $\{f_\lambda\}$ of any f in the full class $\mathbf{L}$. The additional smoothness conditions imposed on the subclass $\mathbf{L}_0$ come to play in Lemma 6 and in Section 3.

Next, we show that

$$\phi_\lambda = \frac{f_\lambda''}{f_\lambda'}, \quad 0 \le \lambda \le 1,$$

is a continuous path in $\mathbf{B}$.

LEMMA 2. $\|\phi_{\lambda_1} - \phi_{\lambda_2}\| \to 0$ *as* $\lambda_1 \to \lambda_2$.

Proof. As $f_\lambda' = h' \cdot p^\lambda$,

$$\phi_\lambda = \frac{f_\lambda''}{f_\lambda'} = \frac{h''}{h'} + \lambda \frac{p'}{p}.$$

By the classical distortion bound,

$$\left|\frac{p'}{p}\right| \le \frac{2}{(1-|z|^2)}.$$

Thus,

$$\begin{aligned}\|\phi_{\lambda_1} - \phi_{\lambda_2}\| &= |\lambda_1 - \lambda_2| \sup_{z \in \mathbb{D}} \left|\frac{p'}{p}\right| (1-|z|^2) \\ &\le 2|\lambda_1 - \lambda_2|.\end{aligned}$$

In particular,

$$\|\phi_\lambda - \phi\| \to 0 \quad \text{as } \lambda \to 1.$$ □

Some of the results on the geometry of $f_\lambda(\mathbb{D})$ needed for our result are now demonstrated.

LEMMA 3. *Let f be a linearly accessible function. Then the following are equivalent:*

(i) *There exists a* $\delta = \delta(\lambda) > 0$ *such that*

$$\operatorname{Re}\left\{\frac{f'}{e^{i\theta} h'}\right\} > 0 \quad \textit{for} \ |\theta| < \delta.$$

(ii) *There exists a* $\lambda=\lambda(\delta)$, $0<\lambda<1$, *and a* $p\in P$ *such that*

$$\frac{f'}{h'}=p^{\lambda}.$$

Proof. Let (ii) hold. For the appropriate choice of the logarithm,

$$\left[\frac{f'(z)}{h'(z)}\right]^{1/\lambda}=\exp\left[\frac{1}{\lambda}\log\left[\frac{f'(z)}{h'(z)}\right]\right].$$

Thus

$$\operatorname{Re}\left\{\left[\frac{f'(z)}{h'(z)}\right]^{1/\lambda}\right\}=\left|\frac{f'(z)}{h'(z)}\right|^{1/\lambda}\cos\left(\operatorname{Arg}\left\{\left[\frac{f'(z)}{h'(z)}\right]^{1/\lambda}\right\}\right),$$

and so

$$\operatorname{Re}\left\{\left[\frac{f'(z)}{h'(z)}\right]^{1/\lambda}\right\}>0 \quad\text{if and only if}\quad \left|\operatorname{Arg}\left\{\left[\frac{f'(z)}{h'(z)}\right]^{1/\lambda}\right\}\right|<\frac{\pi}{2}.$$

We want to show that there exists a $\delta>0$ such that for all $|\theta|<\delta$,

$$\frac{-\pi}{2}<\operatorname{Arg}\left\{\left[\frac{f'(z)}{h'(z)}\right]^{1/\lambda}\right\}-\theta<\frac{\pi}{2}.$$

But as (ii) holds, this is true for $\delta<(1-\lambda)(\pi/2)$.

A symmetric argument shows that (i) implies (ii). □

Half-lines in $\mathbb{C}\setminus f(\mathbb{D})$ are “opened up” into sectors in the complement of $f_\lambda(\mathbb{D})$. This concept is made precise in Lemma 4, which shows that $\mathbb{C}\setminus f_\lambda(\mathbb{D})$ is the union of sectors of certain angular width. We note that later we will need only part of this result, namely that $\mathbb{C}\setminus f_\lambda(\mathbb{D})$ contains a suitable union of sectors. (Also see Pommerenke [14].)

LEMMA 4. *Let* $\{f_\lambda\}$, $0<\lambda<1$, *be the family of functions approximating an* $f\in\mathbf{L}$. *Then there exists a* $\psi=\psi(\lambda)>0$ *such that* $\mathbb{C}\setminus f_\lambda(\mathbb{D})$ *is the union of sectors of angle at least* ψ.

Proof. Choose an f_λ. By the previous lemma there exists a $\delta=\delta(\lambda)$ such that

$$\text{(i)}\qquad \operatorname{Re}\left\{\frac{f_\lambda'}{e^{i\theta}h'}\right\}>0\quad\text{for all}\quad|\theta|<\delta.$$

Let $t\in[0,\infty)$, $|\theta|<\delta$, and define the auxiliary function

$$F(z,t,\theta)=f_\lambda(z)+te^{i\theta}zh'(z).$$

Since

$$F_z(z,t,\theta)=f_\lambda'(z)+te^{i\theta}[h'(z)+zh''(z)]$$

and

$$F_t(z,t,\theta)=e^{i\theta}zh'(z),$$

we obtain

$$\text{(ii)}\qquad \frac{F_z}{e^{i\theta}h'}=\frac{zF_z}{F_t}=\frac{f_\lambda'}{e^{i\theta}h'}+t\left[1+\frac{zh''}{h'}\right].$$

As h is convex and (i) holds,

$$\text{(iii)} \qquad \operatorname{Re}\left\{\frac{F_z}{e^{i\theta}h'}\right\}>0$$

for all $z\in\mathbb{D}$. In particular, $F\in\mathbf{L}$ and thus is univalent for all $t\in[0,\infty)$ and $|\theta|<\delta$. Also,

$$\text{(iv)} \qquad \operatorname{Re}\left\{\frac{zF_z}{F_t}\right\}>0.$$

Therefore, since F is univalent and satisfies (iv), we claim that

$$\text{(v)} \qquad F(\mathbb{D},t_1,\theta)\subset F(\mathbb{D},t_2,\theta)$$

for $t_1<t_2$; that is, F is an *unnormalized* Löwner chain.

To prove the claim, we first assume that F is smooth on $\mathrm{cl}(\mathbb{D})\times\mathbb{R}^+$. Therefore, inequality (iv) holds for all $(z,t)\in\mathrm{cl}(\mathbb{D})\times\mathbb{R}^+$, and this implies that the flow vector F_t always makes angle ν, $0<\nu<\pi$, with the tangent vector $\tau=ie^{i\omega}F_z(e^{i\omega})$. Because each F is univalent, the flow is always directed outward, and we get (v) in the smooth case. In the general case we consider a sequence

$$\{\rho_n: 0<\rho_n<\rho_{n+1}<1,\ \rho_n\to 1 \text{ as } n\to\infty\},$$

and define

$$F_n(z,t,\theta)=F(\rho_n z,t,\theta).$$

Since

$$F_n(\mathbb{D},t_1,\theta)\subset F_n(\mathbb{D},t_2,\theta)$$

for $t_1<t_2$ and all n, we obtain the general case by letting $n\to\infty$ and applying the Carathéodory kernel theorem.

We must now demonstrate the sector property. Again, we first assume that F extends smoothly to the boundary. For θ fixed and $\eta\in[0,2\pi)$, (v) gives us that $f_\lambda(\mathbb{D})=F(\mathbb{D},0,\theta)$ does not contain the ray

$$\rho_\theta(\eta)=\{f_\lambda(e^{i\eta})+te^{i\theta}e^{i\eta}h'(e^{i\eta}): t\geq 0\}.$$

Therefore, for all $\theta\in(-\delta,\delta)$,

$$\bigcup_{\eta\in[0,2\pi)}\sigma(\eta)\subseteq\mathbb{C}\setminus f_\lambda(\mathbb{D}),$$

where

$$\sigma(\eta)=\{f(e^{i\eta})+te^{i\theta}e^{i\eta}h'(e^{i\eta}): t\geq 0,\ |\theta|<\delta\}.$$

Now, choose $z\in\mathbb{C}\setminus f_\lambda(\mathbb{D})$. Since $f_\lambda(\mathbb{D})=F(\mathbb{D},0,\theta)$, for some $\theta\in(-\delta,\delta)$ we have $z\in\mathbb{C}\setminus F(\mathbb{D},0,\theta)$. $F\in\mathbf{L}$ for that θ, and so by a result of Bielecki and Lewandowski [6] z is contained in some ray $\rho_\theta(\eta)$. The ray $\rho_\theta(\eta)$ is contained in the sector $\sigma(\eta)$ for $|\theta|<\delta$. Therefore we get the reverse inclusion, and so

$$\bigcup_{\eta\in[0,2\pi)}\sigma(\eta)=\mathbb{C}\setminus f_\lambda(\mathbb{D}).$$

Thus, we obtain the sector property for $\psi(\lambda)=\delta(\lambda)$ when F is smooth.

In the general case consider $\omega \in \partial F(\mathbb{D})$. There is a sequence of points $z_k \in \mathbb{D}$ with $f(z_k) \to \omega$. Consider the corresponding sequence of sectors σ_k, where

$$\sigma_k \subset \mathbb{C} \setminus f_\lambda(|z| < |z_k|).$$

Each σ_k has vertex $f_\lambda(z_k)$ and angle ψ_k, with $\psi_k \geq \delta$ for all k. Therefore, there is a limit sector σ with vertex ω and angle $\alpha \geq \psi$. Thus a subsequence of the σ_k converges to σ in the sense of Carathéodory. Because $\sigma \subset \mathbb{C} \setminus f_\lambda(\mathbb{D})$, and σ has angle $\alpha \geq \psi(\lambda) > 0$, this completes the proof of Lemma 4. $\square$

Let ρ be the spherical metric on the Riemann sphere $\hat{\mathbb{C}}$, and let $\omega_1, \omega_2 \in f(\mathbb{D}) \setminus f(0)$, where f is a normalized conformal map on $\mathbb{D}$. Define

$$p(\omega_1, \omega_2) = \inf_{C} \sup_{\alpha_j \in C} \rho(\alpha_1, \alpha_2), \quad j = 1, 2,$$

where C is any Carathéodory crosscut of Ω which separates ω_1, ω_2 from $f(0)$. This defines the prime end metric of Mazurkiewicz [12] (also see Pommerenke [15, p. 351]). Let $z_j = f^{-1}(\omega_j)$, $j = 1, 2$. We have the following.

LEMMA 5 (Mazurkiewicz [12]). *There exist constants $K_1, K_2 > 0$ such that*

$$K_1 |z_1 - z_2|^2 \leq p(\omega_1, \omega_2) \leq \frac{K_2}{(\log(3/|z_1 - z_2|))^{1/2}}.$$

This allows us to get a uniform bound on the modulus of continuity of the functions f_λ.

For $f \in \mathbf{L}_0$, we have the following lemma.

LEMMA 6. *If $\{f_\lambda\}$ $(0 < \lambda < 1)$ is the approximating family for $f \in \mathbf{L}_0$, then $f_\lambda(\mathbb{D})$ is bounded.*

Proof. The convex domain $U = h(\mathbb{D})$ has a boundary with a continuously turning tangent, and therefore must be bounded. The function $f_\lambda : \mathbb{D} \to \Omega_\lambda$ may be written as

$$f_\lambda = g_\lambda \circ h,$$

where $g_\lambda' = q^\lambda$, for q a function with positive real part on U. By Harnack's inequality,

$$|g_\lambda'(\zeta)| \leq c\{\mathrm{dist}(\zeta, \partial U)\}^{-\lambda}.$$

Let l be a half-line in U starting from $\zeta_0 = h(0)$ and ending at $\omega \in l \cap \partial U$. Since U is a convex domain and ∂U is smooth, there exists a constant K such that

$$\mathrm{dist}(\zeta, \partial U) \geq K|\zeta - \omega|$$

for all $\zeta \in l$. Therefore,

$$\begin{aligned} |g_\lambda(\zeta)| &\leq c \int_{\zeta_0}^{\zeta} \frac{|d\xi|}{[\mathrm{dist}(\xi, \partial U)]^\lambda} \\ &\leq [c \cdot K^{-\lambda}] \int_{\zeta_0}^{\zeta} \frac{|d\xi|}{(|\xi - \omega|)^\lambda} \leq \end{aligned}$$

$$\leq \frac{2[c \cdot K^{-\lambda}]}{1-\lambda} |\zeta_0 - \omega|^{1-\lambda}$$
$$< \infty. \qquad \square$$

3. Quasidisks

A conformal map $f: \mathbb{D} \to \Omega$ has a quasiconformal extension to the Riemann sphere $\hat{\mathbb{C}}$ if and only if Ω is a quasidisk (Ahlfors [1]). For $f \in \mathbf{L}_0$, we wish to show that $\{f''_\lambda / f'_\lambda\}$ is in $\mathbf{T}$. Equivalently, we need to prove the following.

THEOREM 2. *If* $\{f_\lambda\}$ $(0 < \lambda < 1)$ *is the approximating family for* $f \in \mathbf{L}_0$, *then* $f_\lambda(\mathbb{D})$ *is a quasidisk.*

We use a criterion due to Gehring [8]. A set $E \subset \hat{\mathbb{C}}$ is *K-locally connected* if for all $z \in \mathbb{C}$ and all r, $0 < r < \infty$,

(i) points in $E \cap \mathrm{cl}(B(z, r))$ can be joined in $E \cap \mathrm{cl}(B(z, Kr))$, and
(ii) points in $E \setminus B(z, r)$ can be joined in $E \setminus B(z, r/K)$.

Here, "joined" means joined by an arc lying in the specified set. Gehring [8] proves that a domain Ω is a quasidisk if and only if Ω is K-locally connected for a finite K.

Proof of Theorem 2. Choose an f_λ for $0 < \lambda < 1$. We assume that $\Omega_\lambda = f_\lambda(\mathbb{D})$ does not satisfy either of Gehring's two conditions, and arrive at a contradiction in both cases.

Case (i): Assume that for any constant $K > 1$ there exists a $z_0 \in \Omega_\lambda$, an $r > 0$, and points $z_1, z_2 \in \Omega_\lambda \cap \mathrm{cl}(B(z_0, r))$ which cannot be joined in $\Omega_\lambda \cap \mathrm{cl}(B(z_0, Kr))$.

Let L be the line segment joining z_1 to z_2. By hypothesis, L must intersect $\mathbb{C} \setminus \Omega_\lambda$. Therefore we may assume that $z_1, z_2 \in \partial\Omega_\lambda$. There exists a point $z_3 \in \partial B(z_0, Kr)$ and a sector σ of angle at least $\psi(\lambda) > 0$ and vertex z_3 such that σ separates z_1, z_2 in $B(z_0, r)$, but that $\sigma \cap \Omega_\lambda = \emptyset$. By elementary geometry,

$$2r \geq |z_1 - z_2| \geq \psi(\lambda)(K-1)r.$$

Thus

$$\psi(\lambda) \leq \frac{2}{K-1},$$

which may be made arbitrarily small, contradicting Lemma 4. Therefore, Ω_λ satisfies condition (i) of Gehring's criterion.

REMARK. Condition (i) is satisfied for the approximating family $\{f_\lambda\}$ of any f in the full class $\mathbf{L}$. We say in this case that $f_\lambda(\mathbb{D})$ is an "inner quasidisk." The smoothness conditions satisfied by $f \in \mathbf{L}_0$ are needed to show that Ω_λ satisfies condition (ii).

Case (ii): Assume that for any constant $K > 1$ there exist a $z_0 \in \mathbb{C}$, an $r > 0$, and points $z_1, z_2 \in \Omega_\lambda \setminus B(z_0, r)$ which cannot be joined in $\Omega_\lambda \setminus B(z_0, r/K)$.

The two points z_1, z_2 can be joined to arcs in the set $\partial B(z_0, r) \cap \Omega_\lambda$, and so we may assume $z_1, z_2 \in \partial B(z_0, r)$. Also, there must exist an arc τ of the set $\partial B(z_0, r/K) \cap \Omega_\lambda$ which separates z_1 and z_2 in Ω_λ. Let $c_\rho = \{\rho e^{i\theta}: 0 \le \theta \le 2\pi\}$ for $0 < \rho < 1$, and let $\gamma_\rho = f_\lambda(c_\rho)$ with counterclockwise orientation. For ρ sufficiently close to 1, the arc γ_ρ will intersect τ in at least two points ξ_1 and ξ_2. Let $\theta_j = \operatorname{Arg} f_\lambda^{-1}(\xi_j)$, $j = 1, 2$, with $\theta_2 > \theta_1$ and $\delta = \theta_2 - \theta_1$. We denote the maximum change in tangent on γ_ρ between θ_1 and θ_2 by

$$|\Delta_\delta \operatorname{Arg}\{f_\lambda'(\rho e^{i\theta})\}|.$$

LEMMA 7. *For all $K > 0$, there exists an $\eta > 0$ and a $\rho_0 < 1$ such that*

$$|\Delta_\delta \operatorname{Arg}\{f_\lambda'(\rho e^{i\theta})\}| \ge \pi(1-\eta)$$

for $\rho > \rho_0$ and for some pair of points ξ_1 and $\xi_2 \in \Omega_\lambda$. Moreover, $\eta \to 0$ as $K \to \infty$.

Proof. Let c_ρ, γ_ρ, and τ be as above. We may choose ρ_0 sufficiently close to 1, so that for $\rho > \rho_0$ there exist two points $\xi_1, \xi_2 \in \tau \cap \gamma_\rho$, with a point $\xi_3 \in \gamma_\rho$ in between ξ_1 and ξ_2 such that $\xi_3 \notin B(z_0, r)$.

Let l_1, l_2 denote the lines passing through ξ_3 and tangent to $\partial B(z_0, r/K)$, touching $\partial B(z_0, r/K)$ at α_1, α_2. Let $\pi\eta$ be the interior angle of the intersection of l_1 and l_2 (see Figure 1).

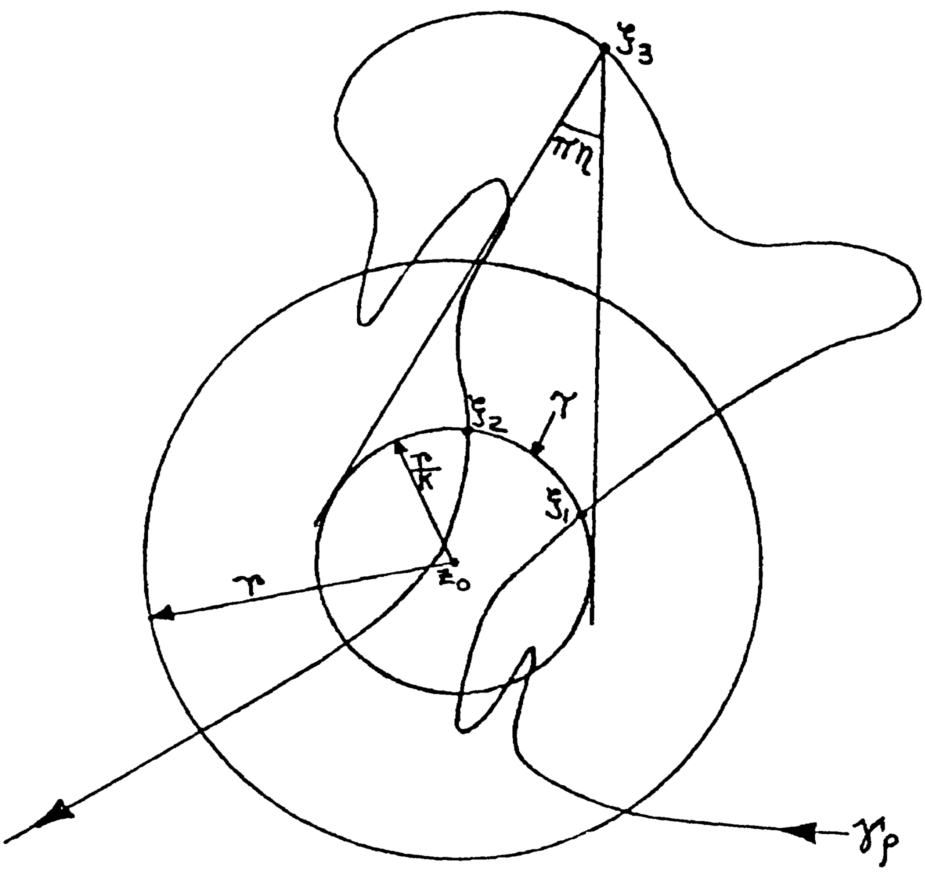

Figure 1 Elementary reductions for the second case

Since $\xi_3 \notin B(z_0, r)$,

$$\pi\eta \le 2\sin^{-1}\left(\frac{1}{K}\right).$$

The arc consisting of the line segments connecting α_1, ξ_3, and α_2 has the minimal change in tangent of any continuous arc passing through those points. This arc has total change in tangent of $\pi(1-\eta)$.

The curve γ_ρ has a continuously turning tangent. Since this curve must pass through ξ_3 and intersect $\partial B(z_0, r/K)$ at ξ_1, ξ_2, there must exist two points on γ_ρ between ξ_1 and ξ_2 such that the change in tangent at those two points is at least $\pi(1-\eta)$. Therefore,

$$|\Delta_\delta \operatorname{Arg}\{f_\lambda'(\rho e^{i\theta})\}| \ge \pi(1-\eta).$$

Clearly, $\eta \to 0$ as $K \to \infty$. □

LEMMA 8. *Given any $\epsilon > 0$, there exists a $\nu = \nu(\epsilon)$ independent of θ such that for any $\rho \le 1$,*

$$|\Delta_{\theta_2-\theta_1} \operatorname{Arg}\{f_\lambda'(\rho e^{i\theta})\}| < \pi\lambda + \epsilon$$

whenever $|\theta_2 - \theta_1| < \nu$. Furthermore, $\nu \to 0$ as $\epsilon \to 0$.

Proof. Now $f_\lambda = g_\lambda \circ h$, with $g_\lambda'(\omega) = p^\lambda(\omega)$ for $\omega \in h(\mathbb{D})$ with h convex. Therefore,

(i) $$|\Delta \operatorname{Arg}\{f_\lambda'(\rho e^{i\theta})\}| \le \lambda |\Delta \operatorname{Arg}\{p(h(\rho e^{i\theta}))\}| + |\Delta \operatorname{Arg}\{h'(\rho e^{i\theta})\}|.$$

Since $\operatorname{Re}\{p(\omega)\} > 0$,

(ii) $$|\Delta \operatorname{Arg}\{p(h(\rho e^{i\theta}))\}| \le \pi$$

for all θ and ρ.

Now $h(\mathbb{D})$ is bounded, and $\partial h(\mathbb{D})$ has a continuously turning tangent in $\mathbb{C}$. The curve c_ρ is smooth and compact for all ρ, $0 < \rho < 1$. Therefore $h'(\rho e^{i\theta})$ is uniformly continuous on c_ρ, and so given any $\epsilon > 0$ there exists a $\nu = \nu(\epsilon)$ independent of θ where $\nu \to 0$ as $\epsilon \to 0$, such that

(iii) $$|\Delta_{\theta_2-\theta_1} \operatorname{Arg}\{h'(\rho e^{i\theta})\}| < \epsilon$$

whenever $|\theta_2 - \theta_1| < \nu$. Moreover, as $\partial h(\mathbb{D})$ has a continuously turning tangent, this estimate holds as $\rho \to 1$.

The proof of the lemma follows by combining (i), (ii), and (iii). □

Choose an $f \in \mathbf{L}_0$, and consider the family $\{f_\lambda\}$ as defined in Section 2. The arc τ discussed above is a Carathéodory crosscut. Let ω_1, ω_2 be two points in Ω_λ such that τ separates ω_1, ω_2 from $f_\lambda(0)$, and such that

$$\operatorname{Arg}\{f_\lambda^{-1}(\omega_1)\} = \theta_1 < \theta_2 = \operatorname{Arg}\{f_\lambda^{-1}(\omega_2)\}.$$

Therefore, by the Mazurkiewicz estimate (Lemma 5), for $z_j = f_\lambda^{-1}(\omega_j)$, $j = 1, 2$,

$$K_1 |z_1 - z_2|^2 \le p(\omega_1 - \omega_2).$$

Since

$$p(\omega_1-\omega_2)\le \operatorname{diam}\tau<2\pi\left(\frac{r}{K}\right),$$

we have

$$K_1\delta^2\le 2\pi\left(\frac{r}{K}\right).$$

Clearly,

$$2\pi\left(\frac{r}{K}\right)\le 2\pi\left(\frac{\operatorname{diam}(f_\lambda(\mathbb{D}))}{K}\right).$$

By Lemma 6, $\operatorname{diam}(f_\lambda(\mathbb{D}))<\infty$, and so $\delta\to 0$ as $K\to\infty$. Now Lemma 8 allows us to choose an ϵ sufficiently small so that

$$|\Delta_\delta \operatorname{Arg}\{f_\lambda'(\rho e^{i\theta})\}|<\pi\lambda+\epsilon$$

for all ρ. However, for K sufficiently large, by Lemma 7 there exists an $\eta>0$ such that

$$\pi(1-\eta)>\pi\lambda+\epsilon,$$

and such that, for all $\rho>\rho_0$,

$$|\Delta_\delta \operatorname{Arg}\{f_\lambda'(\rho e^{i\theta})\}|\ge\pi(1-\eta).$$

This contradicts Lemma 8. Therefore, Ω_λ must satisfy condition (ii) of Gehring's criterion, and hence $f_\lambda(\mathbb{D})$ is a quasidisk. □

4. Proof of Theorem 1

Let f be an element of $\mathbf{L}_0$, and let $\{f_\lambda\}$ denote the approximating family for $0<\lambda<1$. Lemma 2 proves that

$$\left\|\frac{f_\lambda''}{f_\lambda'}-\frac{f''}{f'}\right\|\to 0$$

as $\lambda\to 1$. By Theorem 2, $f_\lambda(\mathbb{D})$ is a quasidisk, and therefore f_λ''/f_λ' is an element of $\mathbf{T}$. Thus,

$$\frac{f''}{f'}\in \operatorname{cl}(\mathbf{T}).$$

5. Subclasses of $\mathbf{L}_0$

In this section we prove Corollary 1. First, let f satisfy $\operatorname{Re}\{f'\}>0$. Then f is in the class $\mathbf{L}_0$ with associated convex function $h(z)=z$.

Next, let f be a bounded starlike function. Since f is starlike, there exists a convex function h such that $f(z)=zh'(z)$. And, since $\operatorname{Re}\{zf'/f\}>0$,

$$0<\operatorname{Re}\left\{\frac{zf'}{f}\right\}=\operatorname{Re}\left\{\frac{f'}{h'}\right\}.$$

Thus $f\in\mathbf{L}$, with associated convex function

$$h(z) = \int_0^z \frac{f(\zeta)}{\zeta}\, d\zeta.$$

Because f is a bounded starlike function, $f(\mathbb{D})$ does not contain any sectors. Thus, for the appropriate branch of the logarithm, $\operatorname{Arg}\{f(z)/z\}$ is continuous in $\mathrm{cl}(\mathbb{D})$. But as $\operatorname{Arg}\{f(z)/z\} = \operatorname{Arg}\{h'(z)\}$, $\partial h(\mathbb{D})$ has a continuously turning tangent and therefore $f \in \mathbf{L}_0$. □

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Department of Mathematics and Statistics
The American University
Washington, DC 20016-8050