Quasiconformal Extension and Univalency Criteria

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1. Introduction and Main Results

Following Ahlfors [A2] and Anderson and Hinkkanen [AH], Harmelin [H] recently obtained a univalency criterion for analytic functions $f(z)$ in the upper half-plane U. It says:

Suppose that $f(z)$ is analytic and $f'(z) \neq 0$ in U and satisfies the

(1) $$\left|2y\frac{f''(z)}{f'(z)} - c\right| \le k \quad \textit{for } y = \operatorname{Im} z > 0,$$

where c is some given complex number with $|c| \le k$. If $k < 1$, then $f(z)$ is univalent in U and has a k-quasiconformal extension to the whole plane. If $k = 1$ and $|c| < 1$, then (1) *implies that $f(z)$ is univalent in U.*

By a k-quasiconformal mapping, where $0 \le k < 1$, we mean a quasiconformal homeomorphism whose maximal dilatation does not exceed $(1+k)/(1-k)$, or, equivalently, whose complex dilatation μ satisfies $\|\mu\|_\infty \le k$.

Our initial observation is the following: Simply by applying Lehto's standard argument to Harmelin's criterion, the condition $|c| \le k$ can be dropped from (1). Later we find that the above constant c can be replaced by some analytic functions related to $f(z)$. So it is natural to ask the following question: *Do there exist analytic functions $a(z)$ and $c(z)$ related to $f(z)$ such that the inequality*

(2) $$|(\bar{z} - z)a(z) + c(z)| \le k < 1 \quad \textit{for } \operatorname{Im} z > 0$$

implies that $f(z)$ is univalent in U and has a k-quasiconformal extension to the whole plane?

In this paper, we will give a positive answer. In fact, with the help of singular integrals we have obtained the following results.

THEOREM. *Let $f(z)$ and $a(z)$ be analytic functions in U or $B = \{z : |z| < 1\}$ with $f'(z) \neq 0$ for all z in U or B, and let c be a nonzero complex constant. Suppose that*

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$$(3)\qquad |(\bar{z}-z)a(z)+cf'(z)e^{-\int a(z)\,dz}-1|\le k \quad \textit{for all } z\in U$$

or that

$$(4)\qquad |z(1-|z|^2)a(z)+(cf'(z)e^{-\int a(z)\,dz}-1)|z|^2|\le k \quad \textit{for all } z\in B,$$

as appropriate. If $k<1$, then $f(z)$ is univalent in its domain of definition and has a k-quasiconformal extension to the whole plane. If $k=1$, then $f(z)$ is univalent in its domain.

Choosing some specific $a(z)$ will lead to interesting univalency criteria. For example, from (3) we have ($\operatorname{Im} z>0$)

1. $|f'(z)-1|\le k$ when $a(z)=0$ and $c=1$;
2. $\left|2y\dfrac{f''(z)}{f'(z)}+c-i\right|\le k$ when $a(z)=\dfrac{f''(z)}{f'(z)}$.

From (4), we have ($|z|<1$)

3. $|f'(z)-1|\le \dfrac{k}{|z|^2}$ when $a(z)=0$ and $c=1$;
4. $\left|z(1-|z|^2)\dfrac{f''(z)}{f'(z)}+(c-1)|z|^2\right|\le k$ when $a(z)=\dfrac{f''(z)}{f'(z)}$.

REMARK 1. Criterion 1 is a refinement of the Noshiro-Warschawski univalency criterion for analytic functions in convex domains [D, p. 47]. We know that the function $f(z)=z+\gamma e^{iz}$ ($|\gamma|>1$) is not univalent in U. However $|f'(z)-1|=|\gamma|e^{-y}$. So the constant k in criterion 1 cannot be replaced by any number bigger than 1.

REMARK 2. Criterion 3 was obtained by Krzyz under the additional assumption $f'(0)=1$ in [K]. The function $f(z)=z/(z^2-\gamma)$, where $|\gamma|>1$, which is not univalent in B, shows that in criterion 3, applied to $1/f$ instead of f, we cannot allow k to be larger than 1 if we wish to deduce that f is univalent.

REMARK 3. If we choose $a(z)=f''(z)/f'(z)$ and $c(z)=-1$ in the proof of Proposition 1 below, then (16) and (17) in Section 4 below read $h_z=a(h+1)$ and $F''(z)/F'(z)=a(z)$, respectively. Thus criterion 2 remains valid even if $c=0$. The same is true for criterion 4, which was obtained by Ahlfors [A2] with the condition that $|c-1|<1$.

REMARK 4. In order to generalize the criterion of Anderson and Hinkkanen [AH], one would need to consider the following problem which remains open: *Find a relation between analytic functions $a(z)$, $c(z)$, and $f(z)$ such that the inequality*

(5) $$|(\bar{z}-z)^2 a(z)+c(z)|\le k<1 \quad \textit{for } \operatorname{Im} z>0$$

implies that $f(z)$ is univalent in U and has a k-quasiconformal extension to the whole plane.

2. Proof of Theorem

The proof is based on the following propositions.

PROPOSITION 1. *Suppose that $a(z)$ and $c(z)$ are analytic in U and satisfy*

(6) $$|(\bar{z}-z)a(z)+c(z)|\le k<1, \quad z\in U.$$

Set

(7) $$\mu(z)=\begin{cases} 0 & \textit{for } z\in U, \\ (z-\bar{z})a(\bar{z})+c(\bar{z}) & \textit{for } z\in L=\{z: \operatorname{Im} z<0\}. \end{cases}$$

Let $F(z)$ be any k-quasiconformal homeomorphism of $\mathbb{C}$ with dilatation $\mu(z)$. Then

(8) $$\frac{F''(z)}{F'(z)}=a(z)+\frac{c'(z)}{c(z)+1} \quad \textit{for } z\in U.$$

PROPOSITION 2. *Suppose that $a(z)$ and $c(z)$ are analytic in B and satisfy*

$$|z(1-|z|^2)a(z)+c(z)|z|^2|\le k<1, \quad z\in B.$$

Set

$$\mu(z)=\begin{cases} 0 & \textit{for } z\in B, \\ -\dfrac{1}{\bar{z}^2}\left[\left(z-\dfrac{1}{\bar{z}}\right)a\left(\dfrac{1}{\bar{z}}\right)+c\left(\dfrac{1}{\bar{z}}\right)\right] & \textit{for } z\in \Delta=\{z: |z|>1\}. \end{cases}$$

Let $F(z)$ be any k-quasiconformal homeomorphism of $\mathbb{C}$ with dilatation $\mu(z)$. Then

$$\frac{F''(z)}{F'(z)}=a(z)+\frac{c'(z)}{c(z)+1} \quad \textit{for } z\in B.$$

We postpone the proof of the propositions.

Now suppose that $f(z)$ satisfies (3). Set

$$c(z)=cf'(z)e^{-\int a(z)\,dz}-1.$$

First we assume $k<1$. Then $a(z)$ and $c(z)$ satisfy (6). Let $F(z)$ be a k-quasiconformal homeomorphism of $\mathbb{C}$ with dilatation of form (7). Then, from Proposition 1,

$$\frac{F''(z)}{F'(z)}=a(z)+\frac{c'(z)}{c(z)+1}=\frac{f''(z)}{f'(z)} \quad \textit{for } z\in U.$$

So there exist some complex constants α, β with $\alpha\ne 0$ such that

$$f(z)=\alpha F(z)+\beta \quad \text{for } z\in U. \tag{9}$$

It is clear that (9) is a quasiconformal extension formula for $f(z)$.

Next we consider the case $k=1$. For $n=1,2,3,\ldots$, define

$$\mu_n(z)=\begin{cases}0 & \text{for } z\in U,\\ n/(n+1)\{(z-\bar{z})a(\bar{z})+c(\bar{z})\} & \text{for } z\in L;\end{cases}$$

then $|\mu_n(z)|<n/(n+1)$. Let $F_n(z)$ be a k-quasiconformal homeomorphism of $\mathbb{C}$ with dilatation $\mu_n(z)$ that agrees with $f(z)$ at three points of U, where $f(z)$ attains distinct values. Then $\{F_n(z)\}$ is a normal family in U. We may choose a subsequence $\{F_{n_k}(z)\}$ locally uniformly converging to an analytic univalent function $F(z)$ in U. So

$$\frac{F''(z)}{F'(z)}=\lim_{k\to\infty}\frac{F''_{n_k}(z)}{F'_{n_k}(z)} \quad \text{for } z\in U.$$

But

$$\frac{F''_n(z)}{F'_n(z)}=\frac{n}{n+1}a(z)+\frac{\frac{n}{n+1}c'(z)}{\frac{n}{n+1}c(z)+1}\to a(z)+\frac{c'(z)}{c(z)+1} \quad \text{as } n\to\infty \quad \text{for } z\in U.$$

Again, we obtain (9) which implies that $f(z)$ is univalent in U.

Similarly, if $f(z)$ satisfies (4) we can apply Proposition 2 to prove the theorem.

3. Lemmas

To prove the propositions we need some lemmas.

For positive numbers r and ϵ, set

$$z_r=-i\sqrt{r^2+1}, \quad D(r)=\{z: |z-z_r|<r\}, \quad B(t,\epsilon)=\{z: |z-t|<\epsilon\}.$$

Note that when $z\in\partial D(r)$,

$$\bar{z}=\bar{z}_r+\frac{r^2}{z-z_r}, \qquad d\bar{z}=-\frac{r^2}{(z-z_r)^2}dz.$$

Let $h(z)$ be a continuous function in L, and define

$$\iint_L \frac{h(z)}{(z-t)^2}dz\wedge d\bar{z}=\lim_{r\to\infty,\,\epsilon\to 0}\iint_{D(r)\setminus B(t,\epsilon)}\frac{h(z)}{(z-t)^2}dz\wedge d\bar{z}$$

and

$$\iint_L \frac{h(z)}{z-t}dz\wedge d\bar{z}=\lim_{r\to\infty,\,\epsilon\to 0}\iint_{D(r)\setminus B(t,\epsilon)}\frac{h(z)}{z-t}dz\wedge d\bar{z}.$$

LEMMA 1. *Let $a(z)$ be analytic in U. Then*

$$\frac{1}{2\pi i}\iint_L \frac{a(\bar{z})}{(z-t)^2}dz\wedge d\bar{z}=\begin{cases}a(t) & \textit{for } t\in U,\\ 0 & \textit{for } t\in L.\end{cases} \tag{10}$$

Proof. When $t \in U$,

$$\frac{1}{2\pi i}\iint_{D(r)} \frac{a(\bar{z})}{(z-t)^2}\, dz \wedge d\bar{z}$$
$$= -\frac{1}{2\pi i}\int_{\partial D(r)} \frac{a(\bar{z})}{z-t}\, d\bar{z}$$
$$= \frac{1}{2\pi i}\int_{\partial D(r)} \frac{a(\bar{z}_r + r^2/(z-z_r))}{z-t}\cdot\frac{r^2}{(z-z_r)^2}\, dz$$
$$= -a\left(\bar{z}_r + \frac{r^2}{t-z_r}\right)\cdot\frac{r^2}{(t-z_r)^2}$$
$$= -a\left(\frac{it\sqrt{r^2+1}-1}{t+i\sqrt{r^2+1}}\right)\cdot\frac{r^2}{(t+i\sqrt{r^2+1}\,)^2} \to a(t) \quad \text{as } r\to\infty.$$

When $t \in L$,

$$\frac{1}{2\pi i}\iint_{D(r)\setminus B(t,\epsilon)} \frac{a(\bar{z})}{(z-t)^2}\, dz \wedge d\bar{z}$$
$$= -\frac{1}{2\pi i}\int_{\partial D(r)} \frac{a(\bar{z})}{z-t}\, d\bar{z} + \frac{1}{2\pi i}\int_{|z-t|=\epsilon} \frac{a(\bar{z})}{z-t}\, d\bar{z} = I_1 + I_2.$$

Then

$$I_1 = \frac{1}{2\pi i}\int_{\partial D(r)} \frac{a\left(\bar{z}_r + \dfrac{r^2}{z-z_r}\right)}{z-t}\cdot\frac{r^2}{(z-z_r)^2}\, dz = 0$$

and

$$I_2 = \frac{a(\bar{t})}{2\pi i}\int_{|z-t|=\epsilon} \frac{d\bar{z}}{z-t} + \frac{a'(\bar{t})}{2\pi i}\int_{|z-t|=\epsilon} \frac{\overline{z-t}}{z-t}\, d\bar{z}$$
$$+ o(|z-t|^2) \to 0 \quad \text{as } \epsilon \to 0. \qquad \square$$

In the proof of the next two lemmas, we will use the following fact, which is a consequence of Cauchy's integral theorem: *Let D be a Jordan domain containing ∞. Suppose that $a(z)$ is analytic in $\bar{D}$ and has a zero of at least second order at ∞. Then, for $t \in D$,*

$$\text{(11)} \qquad \int_{\partial D} a(z)\log(z-t)\, dz = 2\pi i \int_{\infty}^{t} a(z)\, dz,$$

where $\int_\infty^t$ means any simple path from ∞ to t and ∂D is negatively oriented with respect to D.

LEMMA 2. *Let $a(z)$ be analytic in U. Then, for $t \in U$,*

$$\text{(12)} \qquad \frac{1}{2\pi i}\iint_L \frac{a(\bar{z})}{z-t}\, dz \wedge d\bar{z} = \int_{+\infty i}^{t} a(z)\, dz,$$

provided that the first integral exists.

Proof. If $\bar{t} \in D(r)$ then

$$\frac{1}{2\pi i}\iint_{D(r)} \frac{a(\bar{z})}{z-t}\,dz\wedge d\bar{z} = \frac{1}{2\pi i}\int_{\partial D(r)} a(\bar{z})\log(z-t)\,d\bar{z}$$

$$= -\frac{1}{2\pi i}\int_{\partial D(r)} a\left(\bar{z}_r + \frac{r^2}{z-z_r}\right)\log(z-t)\cdot\frac{r^2}{(z-z_r)^2}\,dz$$

$$= -\int_{\infty}^{t} a\left(\bar{z}_r + \frac{r^2}{z-z_r}\right)\cdot\frac{r^2}{(z-z_r)^2}\,dz$$

$$= \int_{\bar{z}_r}^{\bar{z}_r + r^2/(t-z_r)} a(\zeta)\,d\zeta \quad \left(\zeta = \bar{z}_r + \frac{r^2}{z-z_r}\right)$$

$$\to \int_{+\infty i}^{t} a(\zeta)\,d\zeta \quad \text{as } r\to\infty. \qquad \square$$

LEMMA 3. *Let $a(z)$ be analytic in U. Then, for $t\in L$,*

$$\frac{1}{2\pi i}\iint_{L} \frac{a(\bar{z})}{z-t}\,dz\wedge d\bar{z} = \int_{+\infty i}^{\bar{t}} a(z)\,dz, \tag{13}$$

provided that the first integral exists.

Proof. Assume that $B(t,\epsilon)\subset D(r)$. Then

$$\frac{1}{2\pi i}\iint_{D(r)\setminus B(t,\epsilon)} \frac{a(\bar{z})}{z-t}\,dz\wedge d\bar{z} = \frac{1}{2\pi i}\int_{\partial D(r)} a(\bar{z})\log|z-t|^2\,d\bar{z}$$

$$-\frac{1}{2\pi i}\int_{|z-t|=\epsilon} a(\bar{z})\log|z-t|^2\,d\bar{z} = I_1 - I_2.$$

Write $\tilde{t} = z_r + r^2/(\bar{t}-\bar{z}_r)$. Then for $z\in\partial D(r)$,

$$\log|z-t|^2 = \log\frac{z-t}{z-z_r} + \log(\bar{z}_r - \bar{t}) + \log(z-\tilde{t}).$$

So

$$I_1 = -\frac{1}{2\pi i}\int_{\partial D(r)} a\left(\bar{z}_r + \frac{r^2}{z-z_r}\right)\log(z-\tilde{t})\cdot\frac{r^2}{(z-z_r)^2}\,dz$$

$$= -\int_{\infty}^{\tilde{t}} a\left(\bar{z}_r + \frac{r^2}{z-z_r}\right)\cdot\frac{r^2}{(z-z_r)^2}\,dz$$

$$= \int_{\bar{z}_r}^{\bar{z}_r + r^2/(\tilde{t}-z_r)} a(\zeta)\,d\zeta \quad \left(\zeta = \bar{z}_r + \frac{r^2}{z-z_r}\right)$$

$$\to \int_{+\infty i}^{\bar{t}} a(\zeta)\,d\zeta \quad \text{as } r\to\infty.$$

Obviously, $I_2\to 0$ as $\epsilon\to 0$. $\square$

The same method will yield the following lemma.

LEMMA 4. *Let $a(z)$ be analytic in B. Then*

$$-\frac{1}{2\pi i}\iint_{\Delta}\frac{a(1/\bar{z})}{\bar{z}^2(z-t)^2}\,dz\wedge d\bar{z}=\begin{cases} a(t) & \text{for } t\in B,\\ 0 & \text{for } t\in\Delta,\end{cases}$$

and

$$-\frac{1}{2\pi i}\iint_{\Delta}\frac{a(1/\bar{z})}{\bar{z}^2(z-t)}\,dz\wedge d\bar{z}=\begin{cases} \int_0^t a(z)\,dz & \text{for } t\in B,\\ \int_0^{1/\bar{t}} a(z)\,dz & \text{for } t\in\Delta.\end{cases}$$

4. Proofs of the Propositions

To prove Proposition 1 we first assume that, for a suitable $p>2$ to be specified below,

$$\mu(z)\in L^p(\mathbb{C}).$$

For $g(z)\in L^p(\mathbb{C})$, operators **P**, **T** and $\mathbf{T}\mu$ are defined in [A1]:

$$\mathbf{P}g(t)=\frac{1}{2\pi i}\iint_{\mathbb{C}} g(z)\left(\frac{1}{z-t}-\frac{1}{z}\right)dz\wedge d\bar{z},$$

$$\mathbf{T}g(t)=\frac{1}{2\pi i}\iint_{\mathbb{C}}\frac{g(z)}{(z-t)^2}\,dz\wedge d\bar{z},\quad\text{and}$$

$$\mathbf{T}\mu(g)=\mathbf{T}(\mu g).$$

Let **I** be the identity operator. As we know, $\|T\|_p\to 1$ as $p\to 2$. Let p satisfy $k\|T\|_p<1$. Then $(\mathbf{I}-\mathbf{T}\mu)^{-1}$ exists. Let $h(z)=(\mathbf{I}-\mathbf{T}\mu)^{-1}T\mu$. Since $\mu(h+1)\in L^p$, $F(z)=\mathbf{P}[\mu(h+1)]+z$ is well defined.

We claim that $F(z)$ is a k-quasiconformal homeomorphism of $\mathbb{C}$ with dilatation $\mu(z)$. To see this, for $n=1,2,3,\ldots$ define

$$\mu_n(z)=\begin{cases}\mu(z) & \text{for } |z|\le n,\\ 0 & \text{for } |z|>n;\end{cases}$$

$$h_n(z)=(\mathbf{I}-\mathbf{T}\mu_n)^{-1}T\mu_n;\quad\text{and}$$

$$F_n(z)=\mathbf{P}[\mu_n(h_n+1)]+z.$$

From [A1], $F_n(z)$ is a k-quasiconformal homeomorphism of $\mathbb{C}$ with dilatation $\mu_n(z)$ and satisfies $F_n(0)=0$ and $F_n(z)=z+O(1)$ as $z\to\infty$. Note that as $n\to\infty$, we have $\|h_n-h\|_p\to 0$. Thus $F_n(z)\to F(z)$ in L^p. However $\{F_n(z)\}$ is a normal family in $\mathbb{C}$. We may choose a subsequence locally uniformly converging to a k-quasiconformal homeomorphism of $\mathbb{C}$ with dilatation $\mu(z)$. Obviously the limit function must be $F(z)$.

Next we want to prove that $F(z)$ satisfies (8). From [A1] the distributional derivative

$$\{F(z)\}_z = h(z)+1 = 1+\mathbf{T}\mu(z)+\mathbf{T}\mu\mathbf{T}\mu(z)+\mathbf{T}\mu\mathbf{T}\mu\mathbf{T}\mu(z)+\cdots$$
$$= \sum_{n=0}^{\infty}(\mathbf{T}\mu)^n(z).$$

For $z\in U$ and $n=1,2,3,\ldots$, we formally define

$$g_n(z)=\left(\int_{+\infty i}^{z} a(z_n)\int_{+\infty i}^{z_n} a(z_{n-1})\int_{+\infty i}^{z_{n-1}}\cdots\int_{+\infty i}^{z_2} a(z_1)\right)dz_1\cdots dz_{n-1}dz_n.$$

Then

$$g_n(t)=\int_{+\infty i}^{t} a(z)g_{n-1}(z)\,dz.$$

When $t\notin\mathbb{R}$, we conclude with the aid of Lemmas 1–3 that

$$\mathbf{T}\mu(t)=\frac{1}{2\pi i}\iint_L \frac{(z-\bar z)a(\bar z)+c(\bar z)}{(z-t)^2}\,dz\wedge d\bar z$$

$$(14)\qquad =\frac{1}{2\pi i}\iint_L \frac{a(\bar z)}{z-t}\,dz\wedge d\bar z+\frac{1}{2\pi i}\iint_L \frac{(t-\bar z)a(\bar z)+c(\bar z)}{(z-t)^2}\,dz\wedge d\bar z$$

$$=\begin{cases} g_1(t)+c(t) & \text{for } t\in U,\\ g_1(\bar t) & \text{for } t\in L.\end{cases}$$

The first integral in (14) exists since $\mu\in L^p(\mathbb{C})$ so that $\mathbf{T}\mu\in L^p(\mathbb{C})$. The second integral on the second line in (14) exists by Lemma 1. Hence the first integral on the second line in (14) also exists, so that Lemmas 2 and 3 can be applied. This also shows that the integral used in the formal definition of $g_1(z)$ converges.

Similarly,

$$\mathbf{T}\mu\mathbf{T}\mu(t)=\frac{1}{2\pi i}\iint_L \frac{a(\bar z)g_1(\bar z)}{z-t}\,dz\wedge d\bar z$$
$$+\frac{1}{2\pi i}\iint_L \frac{(t-\bar z)a(\bar z)g_1(\bar z)+c(\bar z)g_1(\bar z)}{(z-t)^2}\,dz\wedge d\bar z$$
$$=\begin{cases} g_2(t)+c(t)g_1(t) & \text{for } t\in U,\\ g_2(\bar t) & \text{for } t\in L.\end{cases}$$

By induction,

$$(15)\qquad (\mathbf{T}\mu)^n(t)=\begin{cases} g_n(t)+c(t)g_{n-1}(t) & \text{for } t\in U,\\ g_n(\bar t) & \text{for } t\in L.\end{cases}$$

Clearly the existence of $g_n(z)$ is guaranteed by the condition $\mu\in L^p$. Define $g(z)=\sum_{n=1}^{\infty} g_n(z)$ for $z\in U$. Since $g(\bar z)=h(z)\in L^p(L)$, it follows that $g(z)\in L^p(U)$. Then the distributional derivative of g in U is

$$g_z=\sum_{n=1}^{\infty} g_n'(z)=a(z)\sum_{n=2}^{\infty} g_{n-1}(z)+a(z)=a(g+1).$$

Now, for $z\in U$, $h=g+gc+c$. So the distributional derivative of h in U is

(16) $$h_z = g_z + g_z c + g c_z + c_z = (1+c)(1+g)\left(a + \frac{c_z}{1+c}\right) = (h+1)\left(a + \frac{c_z}{1+c}\right).$$

Note that the last expression is an analytic function in U. Therefore

(17) $$\frac{F''(z)}{F'(z)} = \frac{h'(z)}{h(z)+1} = a(z) + \frac{c'(z)}{c(z)+1} \quad \text{for } z \in U.$$

Because any two k-quasiconformal homeomorphisms of $\mathbb{C}$ with same dilatation only differ by an integral linear transformation, we have finished the proof in the case $\mu(z) \in L^p(\mathbb{C})$.

In general, for $n = 1, 2, 3, \ldots$ set

$$\mu_n(z) = \begin{cases} 0 & \text{for } z \in U, \\ \dfrac{n}{n + (-i\bar{z})^{1+\delta}} \{(z-\bar{z})a(\bar{z}) + c(\bar{z})\} & \text{for } z \in L, \end{cases}$$

where δ is a small positive number satisfying $k < \cos(\pi\delta/2)$. Then, for any $p > 2$,

$$\mu_n(z) \in L^p(\mathbb{C}) \quad \text{and} \quad |\mu_n(z)| \le \frac{k}{\cos(\pi\delta/2)} < 1.$$

Let $F_n(z)$ be a quasiconformal homeomorphism of $\mathbb{C}$ with dilatation $\mu_n(z)$ that fixes 0, 1, and ∞. Again $\{F_n(z)\}$ is a normal family. Thus there exists a subsequence $\{F_{n_k}(z)\}$ locally uniformly converging to a k-quasiconformal homeomorphism $F(z)$ with dilatation $\mu(z)$. Denote $\rho_n(z) = n/(n + (-iz)^{1+\delta})$ for $z \in U$. Now, for $z \in U$,

$$\begin{aligned} \frac{F''(z)}{F'(z)} &= \lim_{k\to\infty} \frac{F''_{n_k}(z)}{F'_{n_k}(z)} \\ &= \lim_{k\to\infty} \left\{ \rho_{n_k}(z)a(z) + \frac{\rho'_{n_k}(z)c(z) + \rho_{n_k}(z)c'(z)}{\rho_{n_k}(z)c(z)+1} \right\} = a(z) + \frac{c'(z)}{c(z)+1}. \end{aligned}$$

This completes the proof of the Proposition 1.

It is not difficult to prove Proposition 2 with the aid of Lemma 4. Because every step of the proof is exactly the same as before, we omit the details.

References

[A1] L. V. Ahlfors, *Lectures on quasiconformal mappings,* Van Nostrand Math. Studies, 10, Van Nostrand, New York, 1966.

[A2] ———, *Sufficient condition for quasiconformal extension,* Ann. of Math. Stud., 79, pp. 23–29, Princeton Univ. Press, Princeton, NJ, 1974.

[AH] J. M. Anderson and A. Hinkkanen, *A univalency criterion,* Michigan Math. J. 32 (1985), 33–40.

[D] P. L. Duren, *Univalent functions,* Grundlehren Math. Wiss. 259, Springer, Berlin, 1983.

[H] R. Harmelin, *A univalency criterion,* Complex Variables Theory Appl. 10 (1988), 327–331.

[K] J. Krzyz, *Convolution and quasiconformal extension,* Comment. Math. Helv. 51 (1976), 99–104.

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