Roughness Properties of Norms on Non-Asplund Spaces
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I. Introduction

In this note we apply James’ techniques on norm-attaining linear forms to roughness properties of every equivalent norm on a non-Asplund space. We answer in particular a question of Klee [10] which we recall: A Banach space $X$ is said to have the $A$-property [1] if for every norm-compact subset $K$ of $X$, the restriction of the duality mapping $J$ to $K$ has a selector $\sigma_K$ such that $\sigma_K(K)$ is norm-relatively compact. Klee asked whether every space, or separable space, can be renormed to have the $A$-property; it follows from Theorem II.1 that only Asplund spaces may have such a norm. The proof relies heavily on an inequality of Simons [16].

NOTATION. We work in real Banach spaces, and keep notation which is standard in Banach space theory. In particular, $S_1(X)$ and $X_1$ denote the unit sphere and the unit ball of $X$, respectively. More generally, if $r > 0$ and $x \in X$, then $B_r(x)$ denotes the closed ball of radius $r$ centered at $x$. We denote by $J$ the duality map of $X$, that is, the multivalued map of $X$ into $X^*$ defined by

$$J(x) = \{ y \in X^* : y(x) = \| y \|^2 = \| x \|^2 \}.$$

We refer to ([12], [9], [4]) for the construction of rough norms on non-Asplund spaces, and to ([3], [18]) for James’ theorem and its applications.

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II. The Results

The following result asserts that, if $X$ is not an Asplund space, then there are points at which the norm is very “not Frechét differentiable”.

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THEOREM II.1. Let $X$ be a Banach space which is not an Asplund space. Then for every $\epsilon > 0$, there is a norm-convergent sequence $\{x_n\}$ in $S_1(X)$ such that, for every $n \neq k$,

$$\text{dist}(J(x_n), J(x_k)) > 1 - \epsilon.$$ 

Proof. Let us assume first that $X$ is separable. Then, by a result of Mazur [14], the set $G$ of points of $S_1(X)$ at which the norm is Gâteaux-smooth is a norm-dense $G_\delta$ of $S_1(X)$. We recall that $x \in G$ if and only if $J(x)$ is reduced to a point.

We now fix a countable dense subset $D$ of $G$. For every $x \in S_1(X)$, we pick a sequence $\{x_n\}$ in $D$ which is norm-convergent to $x$, and we also pick a $w^*$-cluster point $\sigma(x)$ of $\{J(x_n)\}$ in $(X^*_1, w^*)$; it is clear that $\sigma(x) \in J(x)$. We let

$$B = \{\sigma(x) | x \in S_1(X)\}.$$ 

The following lemma is the crucial point of the proof; it relies on Simons’ inequality [16], which itself stems from James' characterization of weakly compact sets [7].

LEMMA II.2. Given $\epsilon > 0$, there exists $x_0 \in S_1(X)$ such that for every $x \in D$, $\|\sigma(x_0) - J(x)\| > 1 - \epsilon$.

Proof. If the result is false, we have

$$B \subset \bigcup\{B_{1-\epsilon}(J(x)) | x \in D\}.$$ 

Since $X^*$ is not separable and $D$ is countable, there is $z \in X^{**}$ with $\|z\| = 1$ and $z(J(x)) = 0$ for every $x \in D$. We pick $y_0 \in X_1^*$ such that $z(y_0) > 1 - \epsilon/2$.

Since $X_1$ is $w^*$-dense in $X^{**}$, $z$ belongs to the closure of $X_1$ for the topology of pointwise convergence on $J(D) \cup \{y_0\}$, and thus we may find $\{x_n\}$ in $X_1$ such that

$$\forall n, \quad x_n(y_0) > 1 - \epsilon/2$$

and

$$\forall x \in D, \quad \lim_{n \to \infty} x_n(J(x)) = 0.$$ 

Clearly, (1), (3), and $\|x_n\| \leq 1$ imply

$$\forall y \in B, \quad \liminf_{n \to \infty} x_n(y) \leq 1 - \epsilon.$$ 

Observe that every $x \in X$ attains its norm at some point of $B$. Therefore, by using the Simons’ inequality [16, Thm. 3], (4), and (2), it follows that

$$1 - \epsilon \geq \sup_{y \in B} \liminf_{n \to \infty} x_n(y) \geq \liminf_{n \to \infty} x_n(y_0) \geq 1 - \frac{\epsilon}{2}.$$ 

This contradiction completes the proof of Lemma II.2. \qed
We now come back to the proof of Theorem II.1. If \( x_0 \in S_1(X) \) is the point provided by Lemma II.2, then (by construction of \( \sigma \)) there exists a sequence \( \{x_n\} \) in \( D \) such that \( \lim_{n \to \infty} \|x_0 - x_n\| = 0 \) and \( \sigma(x_0) \) is a \( w^* \)-cluster point of \( J(x_n) \). By taking a subsequence, we may and do assume that

\[
\sigma(x_0) = \lim_{n \to \infty} J(x_n)
\]

in \( (X^*, w^*) \). Now, by Lemma II.2, we have

\[
\|\sigma(x_0) - J(x_n)\| > 1 - \epsilon
\]

for every \( n \); this implies, by (5) and the \( w^* \)-lower semicontinuity of the norm, that

\[
\forall n \geq 1, \quad \lim_{k \to \infty} \|J(x_k) - J(x_n)\| > 1 - \epsilon.
\]

It is then easy to construct, by induction, a subsequence \( \{x'_n\} \) of \( \{x_n\} \) such that \( \|J(x'_n) - J(x'_k)\| > 1 - \epsilon \) for every \( n \neq k \). This concludes the proof if \( X \) is separable.

The general case will now follow from the separable one. If \( Y \) is any non-Asplund space, then \( Y \) contains a separable subspace \( X \) such that \( X^* \) is non-separable. By the above, there is a norm-convergent sequence \( \{x_n\} \subset S_1(X) \) of points of Gâteaux-smoothness of the norm of \( X \) such that

\[
\|J(x_n) - J(x_k)\| > 1 - \epsilon \quad \text{for} \quad n \neq k.
\]

If \( Q: Y^* \to X^* \) is the canonical quotient map and \( \bar{J}: S_1(Y) \to S_1(Y^*) \) is the duality mapping of \( Y \), the restriction of \( (Q \circ \bar{J}) \) to \( X \) is equal to \( J \); because \( \|Q\| = 1 \), it follows that

\[
\text{dist}(\bar{J}(x_n), \bar{J}(x_k)) > 1 - \epsilon
\]

for every \( n \neq k \); this concludes the proof.

\[\square\]

It is not clear to us whether or not it is possible to improve Theorem II.1 by replacing in its statement the norm-convergent sequence by a norm-compact set without isolated points. However, it is possible to do so if we replace the mapping \( J \) by one of its “reasonable” selectors. Indeed, we have the next proposition.

**Proposition II.3.** Let \( X \) be a Banach space that is not an Asplund space. Then, for every \( \epsilon > 0 \), there exists a subset \( K \) of \( S_1(X) \) which is norm homeomorphic to the Cantor set \( [0, 1]^\omega \), and a selector \( \sigma \) of the duality mapping \( J \) such that

\[
\|\sigma(x) - \sigma(x')\| > 1 - \epsilon
\]

for every \( x \neq x' \) in \( K \).

**Proof.** Again, we first assume that \( X \) is separable. It follows from [8, Thm. 3] that then there exists a selector \( \sigma: X \to X^* \) of the duality mapping which
is (norm-$w^*$) of the first Borel class. If we let $B = \sigma(S_1(X))$, then $B$ is a $w^*$-analytic subset of $S_1(X^*)$ (cf., e.g., [11, 38.III.5]) on which every element of $X$ attains its norm. This latter condition allows us to apply the proof of Lemma II.2 which shows that $B$ is not contained in a countable union of balls of radius $(1 - \epsilon)$. Since $B$ is $w^*$-analytic, this implies that there is a subset $K_0$ of $B$, $w^*$-homeomorphic to $\{0, 1\}^\omega$, such that $\|y - y'\| > 1 - \epsilon$ for every $y \neq y'$ in $K_0$. This follows from a more general result ([5, Lemma 2.2]; [13]). For the reader’s convenience, we include a few hints on how to prove it directly.

If $B$ is a continuous image of a Polish space $P$ and $d$ is the semimetric on $P$ obtained by lifting the norm metric on $B$, then the following statement holds true: Given $\delta > 0$, if $P$ is not $\delta$-separable in $d$, then there are $x_0, x_1 \in P$ such that $d(x_0, x_1) > \delta$ and no neighborhood of $x_0$ or $x_1$ in $P$ is $\delta$-separable in $d$. To see that this statement holds true, put $D = \{x \in P, \text{no neighborhood of } x \text{ in } P \text{ is } \delta\text{-separable in } d\}$. From the Lindelöf property of $P$ it follows that $P \setminus D$ is $\delta$-separable in $d$. Because $P$ is not $\delta$-separable in $d$, it follows that $D$ is not $\delta$-separable in $d$ and the statement follows. From the statement and the lower semicontinuity of $d$ on $P$ it follows that there are points $x_0$ and $x_1$ in $P$ and closed neighborhoods $V_0$ and $V_1$ in $P$ (of $x_0$ and $x_1$, respectively) such that $d(V_0, V_1) > \delta$. Because both $V_0$ and $V_1$ are not $\delta$-separable in $d$, we can apply the above argument to both of them. It is then clear how to complete the construction of a Cantor-like set $K_0$ in $B$.

If we now let $\Omega = \sigma^{-1}(K_0)$, then $\Omega$ is a $G_\delta$ set since $\sigma$ is (norm-$w^*$) of the first Borel class. Since $\Omega$ is obviously uncountable, it contains a Cantor subset $K$ which clearly works since $\sigma(K) \subseteq K_0$ (cf. [17, Thm. 119]).

The general case can be done as before: If $Y$ is not an Asplund space, then it contains a separable subspace $X$ with a nonseparable dual; by the above argument there is a Cantor subset $K$ of $S_1(X)$ and a selector $\sigma$ satisfying the above conditions; if now $j: X^* \to Y^*$ is a map such that, for $x^* \in X^*$, $|j(x^*)| = |x^*|$, $Qj = \text{Id}_{X^*}$, $\bar{\sigma} = j\sigma$, and $\bar{\sigma}$ is a selector: $Y \to Y^*$ which extends $\sigma$, it follows from $|Q| = 1$ that $K$ and $\bar{\sigma}$ work.

REMARKS II.4. (1) When the above statements are compared with Šmulian’s lemma (see [2, p. 29]), they naturally appear as “roughness” assertions. For instance, Proposition II.3 means that any norm on a non-Asplund space is “uniformly rough” when restricted to an appropriate Cantor set; note that conversely, such a norm cannot exist on a separable Asplund space $X$, since the conditions of Proposition II.3 clearly imply that $X^*$ is nonseparable. The same techniques provide several statements which stress the dichotomy between Asplund and non-Asplund spaces.

(2) A connection between these results and James’ constructions of trees in non-superreflexive spaces is provided by the notion of flat Banach space; recall that $X$ is said to be flat if there exist $x \in S_1(X)$ and a $2$-Lipschitz map $g$ from $[0, 1]$ into $S_1(X)$ such that $g(0) = x$ and $g(1) = -x$; for this notion and related ones we refer to [15]. If $X$ is flat then $K = g([0, 1])$ is a norm-compact
subset of \( S_1(X) \) such that \( \text{dist}(J(x'), J(x'')) = 2 \) for every \( x' \neq x'' \) in \( K \) (see [6]). However a non-Asplund Banach space is not necessarily isomorphic to a flat space, since (for example) a space with the Radon–Nikodym property cannot be flat; a proof of this latter fact is provided by the observation that the 2-Lipschitz map \( g : [0, 1] \to S_1(X) \) which joins two antipodal points is nowhere differentiable.

(3) We do not know whether, when \( X \) is separable and \( X^* \) is not, every subset \( B \) of \( S_1(X^*) \) on which every \( x \in X \) attains its norm contains an uncountable biorthogonal system. Note that non-norm-separable and \( w^* \)-analytic subsets of dual spaces contain such systems [19]; under a determinacy axiom, this is also true in the \( w^* \)-projective hierarchy (see [5]).

References


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