Compact Contractive Projections in Continuous Function Spaces

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The problem of characterizing subspaces of $\mathcal{C}(K)$ admitting contractive projections has been considered by different authors. From Nachbin, Goodner, and Kelley's theorems we obtain that if K is an extremely disconnected compact Hausdorff space, then a closed subspace of $\mathcal{C}(K)$ is the range of a projection of norm 1 in $\mathcal{C}(K)$ if and only if it is isometric to the continuous functions on an extremely disconnected compact Hausdorff space. Lindenstrauss and Wulbert [4] extended this result when K is a compact Hausdorff space, showing that a Banach space Y is isometric to the range of a contractive projection in some $\mathcal{C}(K)$ if and only if Y is isometric to $\mathcal{C}_\sigma(L) := \{f \in \mathcal{C}(L) : f(x)+f(\sigma(x)) = 0\ \forall x \in L\}$ for some compact Hausdorff space L and some involutive homeomorphism σ of L. Later on, Lindberg [2] gave necessary and sufficient conditions for a closed separating subspace E of $\mathcal{C}(K)$ to be the range of a projection of norm 1, obtaining that each contractive projection onto E can be given in terms of a real-valued continuous function defined on the closure of the single extreme points and the closure of the double extreme points.

In this paper we attempt to discuss the conditions for a separating subspace of $\mathcal{C}(X)$ to be the range of a compact contractive projection in $\mathcal{C}(X)$, X being a Hausdorff completely regular topological space with a fundamental sequence of compact sets.

Throughout this paper X will stand for any Hausdorff completely regular topological space and $\mathcal{C}(X)$ for the space of the continuous real-valued functions on X endowed with the compact-open topology. Given a linear subspace E of $\mathcal{C}(X)$ and a compact subset K of X, we shall set $E_K := \{f \in E : |f(x)| \leq 1\ \forall x \in K\}$ and $C_K := \{f \in \mathcal{C}(X) : |f(x)| \leq 1\ \forall x \in K\}$, and denote by E_K° and C_K° their polar sets in the topological dual spaces of E and $\mathcal{C}(X)$, respectively. E is said to be *separating* if, for each $x, y \in X$, $x \neq y$, there is some $f \in E$ such that $f(x) \neq f(y)$. E separates points and closed sets of X if, for each closed subset A of X and $x \in X \setminus A$, there is some $f \in E$ such that $f(x) \notin \overline{f(A)}$. For each $x \in X$, δ_x will denote the linear form of $(\mathcal{C}(X))'(E')$ such that $\delta_x(f) = f(x)\ \forall f \in \mathcal{C}(X)(E)$. If A is a subset of

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$(\mathcal{C}(X))'$ then $z \in A$ is said to be an *extreme point* of A if $z = \lambda x + (1-\lambda)y$ (with $0<\lambda<1$ and $x, y \in A$) implies that $z = x = y$.

Given $x \in X$, x is a *double point* of X if there is some $y_x \in X$ such that $f(x)+f(y_x)=0$ for every $f \in E$. If x is not a double point then x is called a *single point*. If E is separating and x is a double point then y_x is unique. The set $d(E)$ of all the double points may fail to be closed. So, for instance, 0 is not a double point of $E = \{f \in \mathcal{C}(\mathbf{R}) : f(1/n)+f(n)=0,\ n \in \mathbf{N}\}$ since for each $a \in \mathbf{R}$ there is some $f \in E$ such that $f(0)+f(a) \neq 0$.

PROPOSITION 1. *Let E be a separating linear subspace of $\mathcal{C}(X)$ and K a compact subset of X. If z is an extreme point of E_K°, then there exists some $\alpha \in \mathbf{R}$, $|\alpha|=1$, and some $x \in K$ such that $z = \alpha\delta_x$.*

Proof. If $F := \{\alpha\delta_x : |\alpha|=1,\ x \in K\}$, then its weakly closed convex cover is contained in E_K°. On the other hand, if $\varphi \notin \overline{CF}^{\sigma(E',E)}$ then there will be some $f \in E$ such that $\varphi(f)>1$ and $|f(x)| \leq 1$ $\forall x \in K$; therefore $\varphi \notin E_K^\circ$ and $\overline{CF}^{\sigma(E',E)} = E_K^\circ$. Hence the extreme points of E_K° are contained in F [1, §25.1(6)]. □

It is worth pointing out that under the hypothesis of Proposition 1 there may be some compact subset K of X and some $x \in K$ such that δ_x is not an extreme point of E_K°. For example, taking $E := \{f \in \mathcal{C}(\mathbf{R}) : f(\frac{1}{2}) = \frac{1}{2}(f(0)+f(1))\}$, $\delta_{1/2}$ is not an extreme point of $E_{[0,1]}^\circ$ since $\delta_{1/2} = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. Under additional suppositions a situation like this will not be able to hold.

PROPOSITION 2. *Let K be a compact subset of X and let $x \in K$. Suppose that, for each $f \in E$ with $f(x)=0$, at least one of the following two conditions holds:*

(i) *there exists some $g \in E_K$ and $r>0$ such that $g(x)=1$ and $g+f/r \in E_K$; or*

(ii) *there is a sequence $\{f_n : n \in \mathbf{N}\}$ which is uniformly convergent to f in K, and each f_n verifies* (i).

Then δ_x is an extreme point of E_K°.

Proof. Let $\delta_x = \lambda u^* + (1-\lambda)v^*$, with $0<\lambda<1$ and $u^*, v^* \in E_K^\circ$. First we shall show $\ker \delta_x \subset \ker u^* \cap \ker v^*$. Assume $f(x)=0$. If (i) holds, then $1 = \delta_x(g) = \lambda u^*(g) + (1-\lambda)v^*(g)$ and $1 = \delta_x(g+f/r) = \lambda u^*(g+f/r) + (1-\lambda)v^*(g+f/r)$, which requires $u^*(g) = v^*(g) = 1$ and $u^*(g+f/r) = v^*(g+f/r) = 1$. So $u^*(f) = v^*(f) = 0$. If (ii) holds, given $\epsilon>0$ there is some $p \in \mathbf{N}$ such that if $n \geq p$, $\sup\{|(f-f_n)(x)| : x \in K\} < \epsilon$. Therefore $|u^*(f-f_n)| < \epsilon$ and $|v^*(f-f_n)| < \epsilon$. And from $u^*(f_n) = v^*(f_n) = 0$ it is clear that $u^*(f) = v^*(f) = 0$.

Hence $u^* = \alpha\delta_x$ and $v^* = \beta\delta_x$. On the other hand, as $o \in E$, there exists some $g \in E_K$ such that $g(x)=1$, so $1 \geq |u^*(g)| = |\alpha|$ and $1 \geq |v^*(g)| = |\beta|$. And from $1 = \delta_x(g) = \lambda\alpha\delta_x(g) + (1-\lambda)\beta\delta_x(g)$ it is clear that $\alpha = \beta = 1$. Therefore $u^* = \delta_x$ and $v^* = \delta_x$. □

EXAMPLE. *Let* $E:=\{f\in \mathcal{C}(\mathbf{R}): f(0)+f(1)=0\}$. *Then, for each compact subset* $K\subset \mathbf{R}$, *if* $x\in K$ *then* δ_x *is an extreme point of* E_K°. We shall show that Proposition 2 holds. Assume $f\in E$ and $f(x)=0$. For each $\epsilon>0$ there is some $\delta>0$ such that if $|h|<\delta$ then $|f(x+h)|<\epsilon$. In case $x\neq 0,1$, we shall take $\delta<\max\{|x|,|x-1|\}$. Let $f_\epsilon\in E$ such that

$$f_\epsilon\equiv 0 \quad \text{in} \quad \left]x-\frac{\delta}{2},x+\frac{\delta}{2}\right[,$$

$$f_\epsilon(y)=-\frac{2}{\delta}f(x-\delta)\left(y-\left(x-\frac{\delta}{2}\right)\right) \quad \text{for } y\in\left[x-\delta,x-\frac{\delta}{2}\right],$$

$$f_\epsilon(y)=\frac{2}{\delta}f(x+\delta)\left(y-\left(x+\frac{\delta}{2}\right)\right) \quad \text{for } y\in\left[x+\frac{\delta}{2},x+\delta\right],$$

and

$$f_\epsilon(y)=f(y) \quad \text{for } y\in\mathbf{R}\backslash[x-\delta,x+\delta].$$

Now $\{f_n^*:=f_{1/n},\ n\in\mathbf{N}\}$ is uniformly convergent to f in K, and for each $n\in\mathbf{N}$, if we take $g_n\in E_K$ defined by

$$g_n\equiv 0 \quad \text{in } \mathbf{R}\backslash\left]x-\frac{\delta}{2},x+\frac{\delta}{2}\right[,$$

$$g_n(y)=\frac{2}{\delta}\left(y-\left(x-\frac{\delta}{2}\right)\right) \quad \text{for } y\in\left[x-\frac{\delta}{2},x\right],$$

$$g_n(y)=-\frac{2}{\delta}\left(y-\left(x+\frac{\delta}{2}\right)\right) \quad \text{for } y\in\left[x,x+\frac{\delta}{2}\right],$$

and $r_n>\sup\{|f_n^*(x)|: x\in K\}$, then $g_n+f_n^*/r_n\in E_K$. □

LEMMA 1. *If E separates points and closed sets of X, then the mapping* $\sigma: d(E)\to d(E)$ *such that* $\sigma(x)=y_x$ *is continuous.*

Proof. Let $\{x_\alpha: \alpha\in I\}$ be a net converging to x in $d(E)$. Then for each $f\in E$, $\{f(x_\alpha): \alpha\in I\}$ converges to $f(x)$. So $\{f(y_{x_\alpha}): \alpha\in I\}$ converges to $f(y_x)$ and, since the topology on X is the initial topology defined by E, $\{y_{x_\alpha}: \alpha\in I\}$ converges to y_x; that is, $\{\sigma(x_\alpha): \alpha\in I\}$ converges to $\sigma(x)$. □

From now on we shall assume that $\{K_n: n\in\mathbf{N}\}$ is a fundamental sequence of compact sets of X. Given a linear subspace E of $\mathcal{C}(X)$, we shall say that a projection p of $\mathcal{C}(X)$ onto E is *compact contractive* if, for each $n\in\mathbf{N}$, $\sup\{|pf(x)|: x\in K_n\}\le\sup\{|f(x)|: x\in K_n\}\ \forall f\in\mathcal{C}(X)$.

DEFINITION 1. We shall say that a single (double) point $x\in X$ is an *extreme* single (double) point if δ_x is an extreme point of some $E_{K_n}^\circ$.

We shall denote by S (D) the set of all the extreme single (double) points of X.

LEMMA 2. *The restriction of σ to D is an involutive homeomorphism.*

Proof. Take $x \in D$; then δ_x is an extreme point of some $E^\circ_{K_n}$. Clearly $\delta_{\sigma(x)} \in E^\circ_{K_n}$, and if $\delta_{\sigma(x)} = \lambda u^* + (1-\lambda)v^*$ (with $0 < \lambda < 1$ and $u^*, v^* \in E^\circ_{K_n}$) then $\delta_x = \lambda(-u^*) + (1-\lambda)(-v^*)$, so $\delta_x = -u^* = -v^*$ and $\delta_{\sigma(x)} = u^* = v^*$. Hence $\sigma(x) \in D$. □

PROPOSITION 3. *Let E be a separating subspace of $\mathcal{C}(X)$, p a compact contractive projection of $\mathcal{C}(X)$ onto E, and p^* the transpose linear mapping of p. Then:*

(i) *for each $x \in \bar{S}$, $p^*(\delta_x) = \delta_x$;*

(ii) *for each $x \in D$, $p^*(\delta_x) = t\delta_x - (1-t)\delta_{\sigma(x)}$, $0 \le t \le 1$.*

Moreover, if E separates points and closed sets of X and $d(E)$ is closed, then (ii) *also holds for each $x \in \bar{D}$.*

Proof. For each $x \in X$ and $n \in \mathbf{N}$ such that $x \in K_n$, let

$$E_x^{K_n} := \{\varphi \in C^\circ_{K_n} : \varphi|_E = \delta_x\},$$

which coincides with the closed convex cover of its extreme points. Now $p^*\delta_x \in E_x^{K_n}$ since, for each $g \in C_{K_n}$,

$$\begin{aligned} |p^*\delta_x(g)| = |\delta_x(pg)| = |pg(x)| &\le \sup\{|pg(y)| : y \in K_n\} \\ &\le \sup\{|g(y)| : y \in K_n\} \le 1 \end{aligned}$$

and since, for each $f \in E$, $p^*\delta_x(f) = \delta_x(pf) = \delta_x(f)$.

On the other hand, if δ_x is an extreme point of $E^\circ_{K_n}$ then each extreme point of $E_x^{K_n}$ is an extreme point of $C^\circ_{K_n}$; for if φ is an extreme point of $E_x^{K_n}$ and $\varphi = \alpha u^* + (1-\alpha)v^*$ (with $0 < \alpha < 1$ and $u^*, v^* \in C^\circ_{K_n}$), then $\delta_x = \varphi|_E = \alpha u^*|_E + (1-\alpha)v^*|_E$, $u^*|_E, v^*|_E \in E^\circ_{K_n}$. So $\delta_x = u^*|_E = v^*|_E$, and $u^*, v^* \in E_x^{K_n}$, coinciding with φ. Moreover, each extreme point φ of $E_x^{K_n}$ is $\varphi = \alpha\delta_z$ with $|\alpha| = 1$ and $z \in K_n$ (i.e., $\varphi = \delta_z$ or $\varphi = -\delta_z$), and coincides with δ_x on E. As E is separating, if $\delta_z|_E = \delta_x$ then $z = x$ and if $-\delta_z|_E = \delta_x$ then $z = \sigma(x)$, which may happen only if x is a double point and $\sigma(x) \in K_n$.

(i) If $x \in S$ then δ_x is the only extreme point of $E_x^{K_n}$ and $p^*\delta_x = \delta_x$. If $x \in \bar{S}$ then $x = \lim_{i \in I} x_i$ with $x_i \in S$. Therefore

$$\lim_{i \in I} \delta_{x_i} = \delta_x \quad \text{and} \quad p^*\delta_x = \lim_{i \in I} p^*\delta_{x_i} = \lim_{i \in I} \delta_{x_i} = \delta_x.$$

(ii) If $x \in D$, $\sigma(x) \in K_n$, and δ_x and $-\delta_{\sigma(x)}$ are the extreme points of $E_x^{K_n}$, then there will be some $0 \le t \le 1$ such that $p^*\delta_x = t\delta_x - (1-t)\delta_{\sigma(x)}$. If $\sigma(x) \notin K_n$, then δ_x is the only extreme point of $E_x^{K_n}$ and (ii) also holds with $t = 1$.

If E separates points and closed sets in X and $d(E)$ is closed, then for each $x \in \bar{D}$, $x = \lim_{i \in I} x_i$ with $x_i \in D$. Now

$$p^*\delta_x = \lim_{i \in I} p^*\delta_{x_i} = \lim_{i \in I} t_i\delta_{x_i} - (1-t_i)\delta_{\sigma(x_i)} = t\delta_x - (1-t)\delta_{\sigma(x)}$$

for some $0 \le t \le 1$. □

An example showing that the conditions given in Proposition 3 are not sufficient may be obtained by considering the projection p of $\mathcal{C}(\mathbf{N})$ in $\mathcal{C}(\mathbf{N})$ defined by $p(a_1, a_2, a_3, a_4, \ldots) = (-a_2, a_2, a_3, a_4, \ldots)$ and $K_n := \{1, 2, \ldots, n\}$, $n \in \mathbf{N}$. Then $E := p(\mathcal{C}(\mathbf{N}))$ is a separating subspace of $\mathcal{C}(\mathbf{N})$, $p^*(\delta_1) = -\delta_2$, and $p^*(\delta_n) = \delta_n$ for each $n \geq 2$. So the double points of $\mathbf{N}$, 1 and 2, satisfy $p^*(\delta_2) = t\delta_2 - (1-t)\delta_{\sigma(2)}$ with $t = 1$ and $p^*(\delta_1) = t\delta_1 - (1-t)\delta_{\sigma(1)}$ with $t = 0$. However, p is not a compact contractive projection since, examining K_1, $|a_2| \leq |a_1|$ is false in general.

DEFINITION 2. Given a closed subspace X' of X, we shall say that the subspace E of $\mathcal{C}(X)$ is *compact isometric* to the subspace F of $\mathcal{C}(X')$ if the linear mapping $I_{X'}$ of E in $\mathcal{C}(X')$ such that the image of each $f \in E$ is its restriction to X' has range F, and if, for each $n \in \mathbf{N}$ such that $K_n \cap X' \neq \emptyset$, $\sup\{|f(x)| : x \in K_n\} = \sup\{|I_{X'} f(x)| : x \in K_n\}$.

PROPOSITION 4. *If E separates points and closed sets of X, $d(E)$ is closed, and E is the range of a compact contractive projection of $\mathcal{C}(X)$ onto E, then E is compact isometric to $F = \{f \in \mathcal{C}(\bar{S} \cup \bar{D}) : f(x) + f(\sigma(x)) = 0\ \forall x \in \bar{D}\}$.*

Proof. Let us show that the range of $I_{\bar{S} \cup \bar{D}}$ of E in $\mathcal{C}(\bar{S} \cup \bar{D})$ is F. Set $A_n := K_n \cap (\bar{S} \cup \bar{D})$ for each $n \in \mathbf{N}$. Given $f \in F$, if $f_1 := f|_{A_1}$ then by the Tietze extension theorem there is some $\hat{f}_1$ in $\mathcal{C}(K_1)$ such that $\hat{f}_1|_{A_1} = f_1$. For each $i \geq 2$, let $f_i \in \mathcal{C}(K_{i-1} \cup A_i)$ be defined by $f_i := \hat{f}_{i-1}$ on K_{i-1} and by $f_i := f|_{A_i}$ on A_i. Then there is some $\hat{f}_i$ in $\mathcal{C}(K_i)$ such that $\hat{f}_i|_{K_{i-1} \cup A_i} = f_i$. Let us define $\hat{f}$ in $\mathcal{C}(X)$ so that $\hat{f}|_{K_n} = \hat{f}_n$, for which $I_{\bar{S} \cup \bar{D}}(\hat{f}) = f$ clearly holds.

Now $p\hat{f} \in E$ and $I_{\bar{S} \cup \bar{D}}(p\hat{f}) = f$, since $p\hat{f}(x) = \delta_x(p\hat{f}) = p^*\delta_x(\hat{f}) = \delta_x(\hat{f}) = f(x)$ for each $x \in \bar{S}$ and since $p\hat{f}(x) = p^*\delta_x(\hat{f}) = (t\delta_x - (1-t)\delta_{\sigma(x)}(\hat{f}) = t(f(x) + f(\sigma(x)) - f(\sigma(x)) = f(x)$ for each $x \in \bar{D}$.

Finally, if $K_n \cap (\bar{S} \cup \bar{D}) \neq \emptyset$, then

$$\begin{aligned}
\sup\{|f(x)| : x \in K_n \cap (\bar{S} \cup \bar{D})\} &\leq \sup\{|f(x)| : x \in K_n\} \\
&= \sup\{|\varphi(f)| : \varphi \in E^\circ_{K_n}\} \\
&= \sup\{|\varphi(f)| : \varphi \in \operatorname{ext} E^\circ_{K_n}\} \\
&\leq \sup\{|\delta_x(f)| : \delta_x \in \operatorname{ext} E^\circ_{K_n}, x \in K_n\} \\
&\leq \sup\{|f(x)| : x \in K_n \cap (\bar{S} \cup \bar{D})\}
\end{aligned}$$

for every $f \in E$. □

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