Boundary Behavior of Positive Solutions of the Helmholtz Equation and Associated Potentials

ROBERT BERMAN & DAVID SINGMAN

1. Main Results

Let $\kappa > 0$. Let n be an integer greater than 1. The Helmholtz equation on $\mathbf{R}^n$ is given by

$$\Delta u = 2\kappa u, \tag{1.1}$$

where Δ denotes the Laplacian, $(\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2)$.

The positive solutions of the Helmholtz equation on all of $\mathbf{R}^n$ are precisely of the form

$$K\mu(x) = \int_{S^{n-1}} e^{\lambda\langle x, b\rangle}\, d\mu(b), \tag{1.2}$$

where $\lambda = \sqrt{2\kappa}$, $\langle \cdot , \cdot \rangle$ denotes the usual inner product on $\mathbf{R}^{n-1}$, $S^{n-1} = \{y \in \mathbf{R}^n : |y| = 1\}$ and μ is a positive Borel measure on S^{n-1}.

The potential theory of the Helmholtz equation is described by means of a Green's function $g(x,y)$ on $\mathbf{R}^n \times \mathbf{R}^n$ which is given by

$$g(x,y) = \int_0^\infty \frac{\exp(-|x-y|^2/(2t) - \kappa t)}{(2\pi t)^{n/2}}\, dt. \tag{1.3}$$

A Helmholtz potential is an extended real-valued function on $\mathbf{R}^n$ of the form

$$G\nu(x) = \int g(x,y)\, d\nu(y), \tag{1.4}$$

where ν is a positive Borel measure on $\mathbf{R}^n$ such that $G\nu \not\equiv \infty$.

Let σ denote unit Lebesgue surface measure on S^{n-1}. In this paper we prove results concerning the behavior of $K\mu(x)/K\sigma(x)$ and $G\nu(x)/K\sigma(x)$ as $|x| \to \infty$.

Let Ω be a subset of $\mathbf{R}^n$ such that, as $|x| \to \infty$ within Ω, $|x/|x| - e| \to 0$, where $e = (1, 0, \ldots, 0)$. Let $O(n)$ denote the set of orthogonal transformations on $\mathbf{R}^n$. Let $\mathfrak{T} = \{T_b : b \in S^{n-1}\}$ be any subset of $O(n)$ such that for each $b \in S^{n-1}$, T_b maps e to b. Let $\Omega_b = T_b(\Omega)$. We think of Ω_b as being an "approach region to b" as $|x| \to \infty$ in the direction of b. An example of such an Ω_b is the parabolic set

Received January 2, 1990. Revision received November 14, 1990.
Michigan Math. J. 38 (1991).

(1.5) $$A(\alpha,b)=\{x\in\mathbf{R}^n\colon |x-|x|\cdot b|\le\alpha\cdot|x|^{1/2}\},$$

where $\alpha>0$.

We say a real-valued function f on $\mathbf{R}^n$ has an $(\Omega,\mathfrak{J})$-limit at $b\in S^{n-1}$ if the limit of $f(x)$ exists as $|x|\to\infty$, $x\in\Omega_b$. In case Ω is invariant with respect to all unitary transformations of $\mathbf{R}^n$ which fix e, we simply use the term Ω-limit. Note that in this case $T(\Omega)$ is the same for all $T\in O(n)$ such that $T(e)=b$, so for such an Ω we define $\Omega_b=T(\Omega)$ for any such T. For $n=2$ we shall always assume that Ω satisfies this condition. The following result was proved in [KT].

THEOREM [KT]. *Let μ be a positive regular Borel measure on S^{n-1}. Let σ denote unit Lebesgue surface measure on S^{n-1}. Then there is a subset E of S^{n-1} having full σ measure such that, for all b in E and all $\alpha>0$, $K\mu/K\sigma$ has $A(\alpha,b)$-limit equal to the Radon–Nikodym derivative $(d\mu/d\sigma)(b)$.*

One of the aims of this paper is to prove a similar almost everywhere limit result with $A(\alpha,b)$ replaced by an approach region, Ω, which is largest possible in some sense. We shall call such a region *admissible.*

Before defining the admissible sets we first define the set $C(\alpha,x)$, where $\alpha>0$ and x is a point in $\mathbf{R}^n$ whose modulus is not 0:

$$C(\alpha,x)=\left\{y\in\mathbf{R}^n\colon 0<|y|\le|x| \text{ and } \left|\frac{y}{|y|}-\frac{x}{|x|}\right|\le\alpha\left(\frac{1}{|y|}-\frac{1}{|x|}\right)^{1/2}\right\}.$$

$C(\alpha,x)$ is a long and thin "balloon"-shaped set whose axis of symmetry extends from the origin to x. In case $|x|=0$ we let $C(\alpha,x)=\{x\}$.

DEFINITION 1.6. (a) Let Ω be any subset of $\mathbf{R}^n$. Let $t>0$. The *t-section* of Ω is defined to be

$$\Omega(t)=\{b\in S^{n-1}\colon bt\in\Omega\}.$$

(b) Let Ω be any subset of $\mathbf{R}^n$. Let $\alpha>0$. Define the *α-thickening* of Ω to be

$$\Omega_\alpha=\bigcup\{C(\alpha,x)\colon x\in\Omega\}.$$

DEFINITION 1.7. Let $b\in S^{n-1}$ and let Ω be any subset of $\mathbf{R}^n$. We say that Ω *converges to ∞ in the direction of b* provided that Ω is unbounded and that whenever $\{x_k\}$ is a sequence in Ω converging to ∞, we have

$$\left|\frac{x_k}{|x_k|}-b\right|\to 0.$$

REMARK 1.8. A simple computation shows that if $y\in C(\alpha,x)$ then

$$C(\alpha,y)\subset C(2\alpha,x).$$

Thus $(\Omega_\alpha)_\alpha\subset\Omega_{2\alpha}$.

DEFINITION 1.9. Let Ω be a subset of $\mathbf{R}^n$ that converges to ∞ in the direction of e. We say that Ω is *admissible* if there exists $\alpha>0$ and $M<\infty$ (M depending on α) such that

(1.10) $\sigma\Omega_\alpha(t) \le M \cdot t^{-(n-1)/2}$ for all $t > 0$.

REMARK 1.11. (a) We claim that if (1.10) holds for some α then it holds for all $\alpha > 0$ (where M varies with the choice of α). It suffices to show that if (1.10) holds for Ω_α and $\beta > \alpha$, then it also holds for Ω_β. Indeed,

$$\Omega_\beta(t) = \bigcup \left\{ b \in S^{n-1} \colon \left| b - \frac{x}{|x|} \right| \le \beta \left(\frac{|x| - t}{t|x|} \right)^{1/2}, \; x \in \Omega \right\}.$$

This is a union of balls in S^{n-1}. By the covering lemma on page 9 of [S2] we can extract a countable disjoint subfamily $B_1, B_2, \ldots$ of these balls such that

$$\sum_k \sigma B_k \ge c\sigma\Omega_\beta(t),$$

where c depends only on the dimension n. If B'_k denotes the ball on S^{n-1} with the same center as B_k but with radius decreased by a factor of α/β, then $\bigcup_k B'_k \subset \Omega_\alpha(t)$ and so

$$\sigma\Omega_\alpha(t) \ge \sigma \bigcup_k B'_k = \sum \sigma B'_k = c\left(\frac{\alpha}{\beta}\right)^{n-1} \sum_k \sigma B_k \ge c\left(\frac{\alpha}{\beta}\right)^{n-1} \sigma\Omega_\beta(t).$$

Here c varies from step to step. This completes the proof of the claim. Thus there is no need to define the concept of an α-admissible set.

(b) From this and Remark 1.8 it follows that if Ω is admissible then Ω_α is admissible for each $\alpha > 0$.

(c) In the next section we give examples of admissible sets that are not contained in any set of the form $A(\alpha, e)$.

REMARK 1.12. Let Ω be a subset of $\mathbf{R}^n$ that converges to ∞ in the direction of e. We claim that $\Omega_\alpha \supset \{te \colon t > 0\}$. Indeed, let $t > 0$. Let $0 < \epsilon < (\alpha/2)t^{-1/2}$. Since Ω converges to ∞ in the direction of e, there exists s_0 such that if $s > s_0$ and $sb \in \Omega$ for some $b \in S^{n-1}$, then $|b - e| < \epsilon$. Choose $s > \max(s_0, 4t/3)$ such that $sb \in \Omega$ for some $b \in S^{n-1}$. Then $\alpha(1/t - 1/s)^{1/2} > (\alpha/2)t^{-1/2} > \epsilon > |b - e|$, so that $te \in C(\alpha, sb) \subset \Omega_\alpha$. This proves the claim.

We shall prove the following result.

THEOREM A. *Let μ be a positive regular Borel measure on S^{n-1}. Let σ denote unit Lebesgue surface measure on S^{n-1}. Let Ω be admissible. Let $\mathfrak{J} = \{T_b \colon b \in S^{n-1}\}$ be a family of orthogonal transformations of $\mathbf{R}^n$ such that, for each $b \in S^{n-1}$, T_b maps $e = (1, 0, \ldots, 0)$ to b. Then there is a subset E of S^{n-1} having full σ measure such that, for all b in E, $K\mu/K\sigma$ has $(\Omega, \mathfrak{J})$-limit equal to the Radon-Nikodym derivative $(d\mu/d\sigma)(b)$.*

Results of this kind for functions defined by convolutions on the upper half-space $\mathbf{R}^{n+1}_+$ were first considered in [NS]. For generalizations see also [C], [MPS2], [MS], [Su1], [Su2], and [W].

We shall prove the following converse result.

THEOREM B. *Let Ω be a subset of $\mathbf{R}^n$ that converges to ∞ in the direction of e. Suppose that Ω is invariant under all elements of $O(n)$ which* *preserve the point e. If Ω is not admissible then there exists a positive regular Borel measure μ such that $K\mu/K\sigma$ has* Ω-lim sup *equal to ∞ at every point of S^{n-1}.*

Note that, for $n=2$, Theorems A and B completely characterize the sets Ω converging to ∞ in the direction of e for which the conclusion of Theorem A can be drawn; the sets are precisely those for which the cross-sectional measure condition (1.10) holds. In view of Remark 1.13(a) below, Theorem B provides only a partial converse for Theorem A in case $n \geq 3$.

In the next theorem we give a necessary condition on certain types of curves, γ, in $\mathbf{R}^2$ such that every function of the form $G\nu/K\sigma$ has a finite γ-limit as $|x| \to \infty$ along rotates of γ.

THEOREM C. *Let Ω be an open subset of $\mathbf{R}^2$ which converges to ∞ in the direction of e. Suppose that Ω is bounded by a continuous curve γ. If* $\limsup t^{1/2}\sigma\Omega(t)=\infty$ *as $t \to \infty$, then there exists a Helmholtz potential $G\nu$ such that $G\nu/K\sigma$ has* γ-lim sup *equal to ∞ at σ-almost every point of S^1.*

By a γ-lim sup of ∞ at a point ζ of S^1, we mean a lim sup of ∞ along the rotate of the curve γ by ζ.

REMARK 1.13. (a) Let $n \geq 3$ and let Ω be an admissible set which is invariant under all elements of $O(n)$ that preserve e. We show that Ω is necessarily contained in $A(\beta, e)$ for some $\beta > 0$.

Let $x_0 \in \Omega$, $x_0/|x_0| \neq e$. Then Ω contains $D=\{x: |x|=|x_0|, \langle x, e\rangle = \langle x_0, e\rangle\}$, where $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $\mathbf{R}^n$. Let $t=x_0/2$. Then

$$\Omega_\alpha(t) \supset D_\alpha(t) = \bigcup_{x \in D} \{b \in S^{n-1}: |b-|x|^{-1}x| \leq \alpha |x_0|^{-1/2}\}.$$

The latter union has σ measure at least $c|e-|x_0|^{-1}x_0|^{n-2}|x_0|^{-1/2}$, where c depends only on n and α. Thus, by (1.10),

$$M \geq \left|\frac{x_0}{2}\right|^{(n-1)/2} \sigma\Omega_\alpha(t) \geq c\left|\frac{x_0}{2}\right|^{(n-1)/2} |e-|x_0|^{-1}x_0|^{n-2}|x_0|^{-1/2},$$

which implies that $|x_0-|x_0|e| \leq (M2^{(n-1)/2}/c)^{1/(n-2)}|x_0|^{1/2}$, so that Ω is a subset of $A(\beta, e)$ with $\beta=(M2^{(n-1)/2}/c)^{1/(n-2)}$.

(b) Let Ω be any subset of $\mathbf{R}^n$ which is invariant with respect to all elements of $O(n)$ that fix e. Let $t>0$. Suppose $ta \in \Omega$, where $a \in S^{n-1}$, $a \neq e$. Let $T \in O(n)$. Put $\zeta = T(a)$. Then, by definition, $t\zeta \in \Omega_{T(e)}$. We claim that $T(te) \in \Omega_\zeta$.

To show this, let b be the element of S^{n-1} lying in the 2-dimensional subspace determined by a and e such that $\langle b, e\rangle = \langle a, e\rangle$ (so that $b = 2\langle a, e\rangle e - a$). By our assumption, $tb \in \Omega$. There exists $S \in O(n)$ such that $S(b)=e$ and $S(e)=a$, so $T\circ S(e) = \zeta$ and $T\circ S(tb) = T(te)$. Thus $T(te) \in T\circ S(\Omega) = \Omega_\zeta$. This proves the claim. We shall make use of this remark in the proof of Theorem B.

2. Examples

Here we give examples of admissible sets.

(1) Let $\alpha > 0$. If $x \in A(\alpha, e)$, it is a simple computation to show that $C(\alpha, x) \subset A(2\alpha, e)$. Thus $(A(\alpha, e))_\alpha \subset A(2\alpha, e)$. Since $A(2\alpha, e)$ has the correct cross-sectional measure, $A(\alpha, e)$ is admissible.

(2) Let $\{x_k\}$ be a sequence in $\mathbf{R}^n$. Put $t_k = |x_k|$ and $b_k = x_k/|x_k|$. Choose the sequence so that

(i) $|b_k - e| \to 0$ as $k \to \infty$,
(ii) t_k is increasing with limit ∞,
(iii) $t_k^{1/2} \cdot |b_k - e| \to \infty$ as $k \to \infty$,
(iv) $t_k^{1/2} \cdot |b_{k+1} - e| < c$ for all k and for some $c < \infty$.

Condition (iii) says that $\{x_k\}$ is not contained in $A(\beta, e)$ for any $\beta > 0$, and condition (iv) says that the sequence $\{t_k b_{k+1}\}$ is entirely contained in $A(c, e)$. Since $A(\alpha, e)$ has the property that any line starting from the origin must eventually leave $A(\alpha, e)$, it is easy to construct such a sequence $\{x_k\}$.

Let $\Omega = \{x_k : k \geq 1\}$. We show that Ω is admissible. Let $\alpha, t > 0$. Suppose that bt is in Ω_α. Choose K so that $t_{K-1} < t$ and $t_K \geq t$. Thus if bt is in $C(\alpha, x_k)$ then k must be at least K. Suppose first that k is at least $K+1$. Then

$$|b - b_k| \leq \alpha \left(\frac{1}{t} - \frac{1}{t_k} \right)^{1/2},$$

and so

$$\begin{aligned} |b - e| &\leq |b - b_k| + |b_k - e| \\ &\leq \alpha \left(\frac{1}{t} - \frac{1}{t_k} \right)^{1/2} + |b_k - e| \\ &\leq \alpha t^{-1/2} + c t_{k-1}^{-1/2} \\ &\leq (\alpha + c) t^{-1/2}, \end{aligned}$$

since $k \geq K+1$ implies that $t_{k-1} \geq t$. Thus

$$\{b \in S^{n-1} : bt \in \Omega_\alpha\} \subset A((\alpha + c), e) \cup C(\alpha, x_K).$$

This completes the proof that Ω is admissible.

Example (2) thus provides an example of an admissible set that cannot be contained in $A(\beta, e)$ for any $\beta > 0$.

3. Maximal Functions

In this section we fix $\alpha > 0$, Ω an admissible subset of $\mathbf{R}^n$, and $\mathfrak{J} = \{T_b : b \in S^{n-1}\}$ a family of orthogonal transformations of $\mathbf{R}^n$ such that T_b maps $e = (1, 0, \ldots, 0)$ to b. For each $b \in S^{n-1}$ define

$$\Phi_b = \left\{ (\zeta, t) \in S^{n-1} \times (0, \infty) : \frac{\zeta}{t^2} \in (\Omega_b)_\alpha \right\}$$

and

$$\Phi'_b = \left\{(\zeta, t) \in S^{n-1} \times (0, \infty) : \frac{\zeta}{t^2} \in (\Omega_b)_{2\alpha}\right\}$$

(recall Definition 1.6(b)). In case $b = e$ we will not use a subscript in these definitions.

PROPOSITION 3.1. *Fix* $b \in S^{n-1}$.

(a) *Let* $(\zeta_0, t_0) \in \Phi_b$. *Suppose* $\zeta \in S^{n-1}$ *and* $t > t_0$. *If* $|\zeta - \zeta_0| < \alpha(t - t_0)$, *then* $(\zeta, t) \in \Phi'_b$.

(b) *Let* $(\zeta, t) \in \Phi_b$. *Then* $(\zeta, s) \in \Phi_b$ *for all* $s > t$.

(c) *There exists* $M < \infty$ *(depending on* α*) such that, for all* $t > 0$,

$$t^{-(n-1)} \cdot \sigma\{\zeta \in S^{n-1} : |\zeta - \zeta_0| < t \text{ for some } (\zeta_0, t) \in \Phi_b\} \leq M.$$

Proof. (a) $|\zeta - \zeta_0| \leq \alpha(t - t_0) \leq \alpha(t^2 - t_0^2)$. Thus

$$\frac{\zeta}{t^2} \in C\left(\alpha, \frac{\zeta_0}{t_0^2}\right) \subset ((\Omega_b)_\alpha)_\alpha \subset (\Omega_b)_{2\alpha},$$

giving us $(\zeta, t) \in \Phi'_b$.

(b) This follows from the fact that $(\Omega_b)_\alpha$ is starlike with respect to the origin of $\mathbf{R}^n$.

(c) Let (ζ_0, t) be an element of Φ_b and suppose that $|\zeta - \zeta_0| < t$. Then $|\zeta - \zeta_0| < \alpha((1 + \alpha^{-1})t - t)$. By part (a) of this proposition, $(\zeta, (1 + \alpha^{-1})t)$ is an element of Φ'_b. Thus

$$\sigma\{\zeta \in S^{n-1} : |\zeta - \zeta_0| < t \text{ for some } (\zeta_0, t) \in \Phi_b\}$$
$$\leq \sigma\left\{\zeta \in S^{n-1} : \frac{\zeta}{[(1 + \alpha^{-1}) \cdot t]^2} \in (\Omega_b)_{2\alpha}\right\}.$$

The result follows from Definition 1.9 and Remark 1.11. □

We now define a Hardy–Littlewood type maximal function on S^{n-1}.

DEFINITION 3.2. Let ν be a regular Borel measure on S^{n-1}. For $b \in S^{n-1}$ let

$$M\nu(b) = \sup\{t^{-(n-1)} \nu B(\zeta, t) : (\zeta, t) \in \Phi_b\},$$

where $B(\zeta_0, t) = \{\zeta \in S^{n-1} : |\zeta - \zeta_0| < t\}$.

Proposition 3.1 allows us to apply Theorem 1.5 of [Su1] and deduce the following.

PROPOSITION 3.3. *M is weak-type; that is, there exists $c < \infty$ such that for all regular Borel measures ν and $\lambda > 0$*

$$\sigma\{b \in S^{n-1} : |M\nu(b)| > \lambda\} \leq \left(\frac{c}{\lambda}\right) |\nu|.$$

4. Proof of Theorem A

LEMMA 4.1. *$K\sigma(x) \approx e^{\lambda|x|}|x|^{(1-n)/2}$ as $x \to \infty$, where $\approx$ means that the quotient of the right and left sides is bounded above and below by positive constants.*

Proof. Since σ is rotation invariant,

$$K\sigma(x) \approx \int_0^1 \frac{e^{\lambda|x|(1-r^2)^{1/2}}}{(1-r^2)^{1/2}} r^{n-2}\,dr.$$

Making the substitution $1-r^2=s^2$ and then $1-s=u/(\lambda|x|)$ gives

$$K\sigma(x) \approx e^{\lambda|x|}(\lambda|x|)^{(1-n)/2} \int_0^{\lambda|x|} e^{-u} u^{(n-3)/2}\,du.$$

As the latter integral remains finite as $x \to \infty$ for $n \geq 2$, the proof is complete. □

Proof of Theorem A. In the proof, c denotes a constant which may depend on other constants and which may vary from occurrence to occurrence.

By the lemma we may replace $K\mu/K\sigma$ by

$$u(x) = |x|^{(n-1)/2} e^{-\lambda|x|} \int e^{\lambda\langle x, b\rangle}\,d\mu(b). \tag{4.2}$$

Define the maximal function N by

$$N\mu(b) = \sup\{u(x) : x \in \Omega_b\}.$$

By a standard argument it is enough to show that N is weak-type and for this it is enough (by Proposition 3.3) to show that $N\mu(b) \leq cM\mu(b)$ for all $b \in S^{n-1}$, where c does not depend on μ or b.

Let $b \in S^{n-1}$ and suppose $x = t\xi \in \Omega_b$, where $t=|x|$ and $\xi = x/|x|$. In what follows, $[\sqrt{t}\,]$ denotes the greatest integer less than or equal to $\sqrt{t}$. Then

$$\begin{aligned}
u(x) &= t^{(n-1)/2} e^{-\lambda t} \int e^{\lambda t\langle \xi, \zeta\rangle}\,d\mu(\zeta) \\
&= t^{(n-1)/2} \int e^{-\lambda(t/2)|\zeta-\xi|^2}\,d\mu(\zeta) \\
&\leq t^{(n-1)/2} \Bigg(\sum_{k=0}^{[\sqrt{t}]} \int_{k/\sqrt{t} < |\zeta-\xi| \leq (k+1)/\sqrt{t}} e^{-\lambda(t/2)|\zeta-\xi|^2}\,d\mu(\zeta) \\
&\qquad + \int_{1<|\zeta-\xi|\leq 2} e^{-\lambda(t/2)|\zeta-\xi|^2}\,d\mu(\zeta) \Bigg) \\
&\leq \sum_{k=0}^{[\sqrt{t}]} \frac{e^{-\lambda k^2/2}(k+1)^{n-1}\mu(B(\xi,(k+1)/\sqrt{t}))}{((k+1)/\sqrt{t}\,)^{n-1}} + |\mu| t^{(n-1)/2} e^{-\lambda t/2} \\
&\leq c(M\mu(b) + |\mu|) \\
&\leq cM\mu(b).
\end{aligned}$$

This completes the proof. □

5. Proof of Theorem B

Let σ denote unit Lebesgue surface measure on S^{n-1} and let H denote unit Haar measure on $O(n)$. For E a subset of S^{n-1}, let χ_E denote the function on S^{n-1} which is 1 at a point of E and 0 otherwise.

LEMMA 5.1. *Let E be a Borel subset of S^{n-1}. Then*

$$\sigma(E)=\int \chi_E(Te)\,dH(T).$$

Proof.

$$\begin{aligned}\int d\sigma(y)\int \chi_E(Ty)\,dH(T) &= \int dH(T)\int \chi_E(Ty)\,d\sigma(y)\\ &= \int dH(T)\int \chi_E(y)\,d\sigma(y)\\ &= \int \sigma(E)\,dH(T)\\ &= \sigma(E).\end{aligned} \tag{5.2}$$

However, the right invariance of H shows that the inner integral in the first line of (5.2) is independent of y. The lemma follows. ☐

LEMMA 5.3. *Let E_1 and E_2 be Borel subsets of S^{n-1}. Then*

$$\sigma(E_1)\cdot\sigma(E_2)=\int \sigma(TE_1\cap E_2)\,dH(T).$$

Proof.

$$\begin{aligned}\int \sigma(TE_1\cap E_2)\,dH(T) &= \int dH(T)\int_{E_2} \chi_{TE_1}(y)\,d\sigma(y)\\ &= \int dH(T)\int_{E_2} \chi_{E_1}(T^{-1}y)\,d\sigma(y)\\ &= \int_{E_2} d\sigma(y)\int \chi_{E_1}(T^{-1}y)\,dH(T)\\ &= \int_{E_2} d\sigma(y)\int \chi_{E_1}(Te)\,dH(T)\\ &= \sigma(E_1)\,\sigma(E_2).\end{aligned}$$

The second-to-last equality follows by the right invariance of H, and the last inequality follows by Lemma 5.1. ☐

LEMMA 5.4. *Let E be a Borel subset of S^{n-1} having positive σ measure. Put $\alpha=\sigma(E)$ and $m=[1/\alpha]-1$, where $[1/\alpha]$ denotes the greatest integer less than or equal to $1/\alpha$. Then there exist $T_1,\dots,T_m\in O(n)$ such that*

$$\sigma(E \cup T_1 E \cup \cdots \cup T_m E) > \tfrac{1}{2}.$$

Proof. We begin by applying Lemma 5.3 to the sets E and E^c, where E^c refers to the set of points of S^{n-1} not in E. We deduce that there exists $T_1 \in O(n)$ such that

$$\sigma(T_1 E \cap E^c) \geq \sigma(E)\sigma(E^c).$$

Thus

$$\begin{aligned}\sigma(E^c \cap (T_1(E))^c) &= \sigma(E^c) - \sigma(E^c \cap T_1 E)\\ &\leq \sigma(E^c) - \sigma(E^c)\cdot\sigma(E)\\ &= (1-\alpha)^2,\end{aligned}$$

so

$$\sigma(E \cup T_1 E) \geq 1 - (1-\alpha)^2.$$

Similarly, by applying Lemma 5.3 to E and $(E \cup T_1 E)^c$, we find $T_2 \in O(n)$ such that

$$\sigma(E \cup T_1 E \cup T_2 E) \geq 1 - (1-\alpha)^3.$$

Continuing in this way m times, we get $T_1, \ldots, T_m$ in $O(n)$ such that

$$\begin{aligned}\sigma(E \cup T_1 E \cup \cdots \cup T_m E) &\geq 1 - (1-\alpha)^{m+1}\\ &\geq 1 - (1-\alpha)^{1/\alpha}\\ &\geq 1 - \exp(-1)\\ &\geq \tfrac{1}{2}.\end{aligned}$$ □

The following lemma is a special case of Lemma 1 in [S1].

LEMMA 5.5. *Let $\{F_k\}$ be a sequence of Borel subsets of S^{n-1} such that*

$$\sum_{k=1}^{\infty} \sigma(F_k) = \infty.$$

Then there exists a sequence $\{T_k\}$ in $O(n)$ such that

$$\sigma\{y \in S^{n-1} : y \in T_k(F_k) \text{ for infinitely many } k\} = 1.$$

Proof of Theorem B. Let $\epsilon, \alpha > 0$. Since Ω is not admissible, there exist increasing sequences $\{t_k\}$ and $\{C_k\}$ with limits ∞ such that

$$\sum_{k=1}^{\infty} C_k t_k^{-(n-1)/2} (\sigma\Omega_\alpha(t_k))^{-1} < \epsilon. \tag{5.6}$$

Let $E_k = \Omega_\alpha(t_k)$. Let $m_k + 1$ be the integer part of $(\sigma\Omega_\alpha(t_k))^{-1}$. By Lemma 5.4 there exist $T_{k,0}, \ldots, T_{k,m_k} \in O(n)$ such that

$$\sigma(T_{k,0}(E_k) \cup T_{k,1}(E_k) \cup \cdots \cup T_{k,m_k}(E_k)) > \tfrac{1}{2}.$$

By applying Lemma 5.5, we may assume without loss of generality that

$$\sigma\{\zeta \in S^{n-1} : \zeta \in T_{k,0}(E_k) \cup T_{k,1}(E_k) \cup \cdots \cup T_{k,m_k}(E_k) \text{ for infinitely many } k\} = 1. \tag{5.7}$$

For each k consider the m_k+1 points $T_{k,0}(et_k), T_{k,1}(et_k), \ldots, T_{k,m_k}(et_k)$ on the sphere $|x|=t_k$ (recall $e=(1,0,\ldots,0)$). This gives us an infinite sequence of points. Due to (5.7), Remark 1.12, and Remark 1.13(b), we have shown that for σ-almost every $\zeta \in S^{n-1}$ there is a subsequence $\{x_j\}$ of these points with $x_j \in (\Omega_\alpha)_\zeta$, $|x_j| \to \infty$, and $|x_j/|x_j| - \zeta| \to 0$. (It is here we use the fact that Ω, hence Ω_α, is invariant under elements of $O(n)$ that fix e.) For each one of these m_k+1 points on $|x|=t_k$ (call a typical one bt_k), we associate a measure which is a multiple of the restriction of σ to the cap on S^{n-1} centered at b of radius $2\alpha t_k^{-1/2}$, where the multiple is C_k. (By the term *cap* we mean the set of points of S^{n-1} at most a distance to b of $2\alpha t_k^{-1/2}$.) This gives us a measure μ of total magnitude about $\sum_k C_k t_k^{-(n-1)/2} m_k$. This magnitude can be made as small as we wish, depending upon ϵ.

Let bt_k be one of our m_k+1 points and let $s \geq t_k$. Let I be the cap on S^{n-1} centered at b of radius $s^{-1/2}$. Since $s \geq t$, I is contained in a cap centered at b of radius $t_k^{-1/2}$. Thus

$$
\begin{aligned}
\left(\frac{1}{C_k}\right)\frac{K\mu(bs)}{K\sigma(bs)} &\geq \int_I c s^{(n-1)/2} e^{-\lambda s} e^{\lambda s \langle b, \zeta\rangle}\, d\sigma(\zeta) \\
&= \int_I c s^{(n-1)/2} e^{-(\lambda s/2)|b-\zeta|^2}\, d\sigma(\zeta) \qquad (5.8)\\
&\geq c,
\end{aligned}
$$

since the last integral is independent of s. This computation allows us to conclude that for σ-a.e. $\zeta \in S^{n-1}$, the Ω_α-lim sup of $K\mu/K\sigma$ is ∞. We claim that the Ω-lim sup is also equal to ∞ σ-a.e. To see this, let ξs be a point of Ω such that $bt_k \in C(\alpha, \xi s)$, with bt_k as in the first part of this paragraph. Then $|b-\xi| \leq \alpha t_k^{-1/2}$. Thus the cap of radius $2\alpha t_k^{-1/2}$ centered at b in the construction of μ contains a cap centered at ξ of radius about $\alpha t_k^{-1/2}$. Together with the computation in (5.8) this proves the claim.

Finally, consider the exceptional subset E of S^{n-1} of σ-measure 0, where the Ω-lim sup of $K\mu/K\sigma$ is not ∞. There exists a decreasing sequence $\{U_k\}$ of open subsets of S^{n-1} whose intersection contains E for which

$$\sum_{k=1}^{\infty} \sigma(U_k) < \infty. \tag{5.9}$$

Let ω_k be the restriction of σ to an open subset of S^{n-1} which contains the closure of U_k and has at most double the σ measure. Let ω be the sum of these measures. By (5.9), ω is a finite measure. Since $K\omega_k/K\sigma$ has Ω-limit 1 at every point of U_k, $K\omega/K\sigma$ has Ω-lim sup equal to ∞ at every point of E. The measure $\mu = \nu + \omega$ satisfies the requirements of the theorem. □

6. Proof of Theorem C

We first estimate the Green's function. For future reference we prove the estimates for all $n \geq 2$.

LEMMA 6.1. *Let $g(r)=g(x,y)$, where $r=|x-y|$. Then*

(a) $$g(r)\approx\begin{cases}\log(1/r) \text{ as } r\to 0^+ & \text{if } n=2,\\ r^{2-n} \text{ as } r\to 0^+ & \text{if } n\geq 3;\end{cases}$$

(b) $$g(r)\approx e^{-\lambda r}r^{(1-n)/2} \text{ as } r\to\infty.$$

Proof. (a) Write $g(r)$ as I+II, where I and II are obtained from (1.3) by integrating respectively over the intervals $[0,1]$ and $[1,\infty)$.

We first estimate I. Since $e^{-\kappa}\leq e^{-\kappa t}\leq 1$,

$$\text{I}\approx r^{2-n}\int_{r^2/2}^{\infty} e^{-s}s^{(n/2)-2}\,ds. \tag{6.2}$$

If $n\geq 3$, it follows immediately that $\text{I}\approx r^{2-n}$ as $r\to 0^+$. If $n=2$, an application of L'Hôpital's rule shows that the integral in (6.2) $\approx\log(1/r)$ as $r\to 0^+$. Thus $\text{I}\approx\log(1/r)$ if $n=2$.

Consider now II. If $r<1<t$, then $e^{-1/2}<e^{-r^2/(2t)}<1$. Thus

$$\text{II}\approx\int_1^{\infty} e^{-\kappa t}t^{-n/2}\,dt,$$

which is finite and independent of r. This completes the proof of (a).

(b) From the definition of $g(r)$, we have

$$g(r)r^{(n-1)/2}e^{\lambda r}=r^{(n-1)/2}\int_0^{\infty}\exp\left(\frac{-(r-\lambda t)^2}{2t}\right)(2\pi t)^{-n/2}\,dt. \tag{6.3}$$

(Recall that $\lambda^2=2\kappa$.) Writing this as I+II+III+IV, where we integrate respectively over the intervals $(0,r/(2\lambda)]$, $(r/(2\lambda),r/\lambda]$, $(r/\lambda,2r/\lambda]$, and $(2r/\lambda,\infty)$, it is a simple exercise to show that each of these integrals remains bounded as $r\to\infty$. This completes the proof. □

Proof of Theorem C. For any set E in $\mathbf{R}^2$ and $\theta\in S^1$ define $E_\theta=\{z\theta : z\in E\}$. Let $\epsilon>0$. By assumption there exist increasing sequences $\{t_k\}$ and $\{C_k\}$ with limits ∞ such that

$$\sum_{k=1}^{\infty} C_k t_k^{-1/2}(\sigma\Omega(t_k))^{-1}<\epsilon. \tag{6.4}$$

For each k consider the "gate"

$$g_k=\{te : t\geq t_k\}.$$

(We note that constructions with gates were employed in [BC, §4] in relation to Blaschke products on the unit disc in $\mathbf{C}$ under the hypothesis that γ is a tangential curve converging to 1.) Define the "projection" P_k by

$$P_k=\{\zeta\in S^1 : (g_k)_\zeta\cap\gamma\neq\emptyset\}.$$

Our assumption concerning the behavior of Ω at ∞ implies that $P_k\supset\Omega(t_k)\backslash\{e\}$. Thus (6.4) remains true if we replace $\Omega(t_k)$ by P_k. Let m_k be the integer part of $(\sigma(P_k))^{-1}$. Arguing as we did in the proof of Theorem B, we can construct a set G_k which is a union of m_k rotates of g_k such that for σ-almost every $\zeta \in S^1$, γ_ζ intersects G_k for an infinite number of k.

We now show how to construct a measure ν such that $G\nu \not\equiv \infty$ and $G\nu/K\sigma \geq C_k$ on G_k. We shall choose ν so that its support does not contain the origin 0. Thus, by Lemma 6.1, the condition that $G\nu \not\equiv \infty$ is that the measure $d\mu = e^{-\lambda|y|}|y|^{-1/2}\,d\nu$ be totally finite.

Consider the measure μ_k, which on the gate g_k is given by

$$d\mu_k = C_k t_k^{-1/4} s^{-5/4}\,ds.$$

The total mass of μ_k is $4C_k t_k^{-1/2}$. Let $d\nu_k(s) = e^{\lambda s} s^{1/2}\,d\mu_k(s)$. Using Lemma 4.1 and Lemma 6.1, we deduce that if $t \geq t_k$ then

$$\begin{aligned}\frac{G\nu_k(te)}{K\sigma(te)} &\geq \int_{2t}^{\infty} C_k t_k^{-1/4} s^{-5/4} e^{\lambda s} s^{1/2} (e^{-\lambda(s-t)}(s-t)^{-1/2}) e^{-\lambda t} t^{1/2}\,ds \\ &= \int_{2t}^{\infty} C_k t_k^{-1/4} t^{1/2} s^{-3/4} (s-t)^{-1/2}\,ds \\ &\geq \int_{2t}^{\infty} C_k t_k^{-1/4} t^{1/2} s^{-5/4}\,ds \\ &\geq C_k \left(\frac{t}{t_k}\right)^{1/4} \\ &\geq C_k.\end{aligned}$$

Let ν_k' be the sum of the m_k rotates of the measure ν_k so that ν_k' has its support on G_k. Let $\nu = \sum \nu_k'$. By (6.4), ν satisfies the growth condition that $e^{-\lambda|y|}|y|^{-1/2}\,d\nu$ is totally finite. We have thus shown that $G\nu/K\sigma$ has γ-lim sup equal to ∞ at σ-almost every point of S^1. □

References

[BC] R. Berman and W. S. Cohn, *Tangential limits of Blaschke products and functions of bounded mean oscillation,* Illinois J. Math. 31 (1987), 218–239.

[C] J. Chabrowski, *Representation theorems for parabolic systems,* J. Austral. Math. Soc. Ser. A 32 (1982), 246–288.

[KT] A. Korányi and J. C. Taylor, *Fine convergence and parabolic convergence for the Helmholtz equation and the heat equation,* Illinois J. Math. 27 (1983), 77–93.

[MPS1] B. A. Mair, S. Philipp, and D. Singman, *A converse Fatou theorem on homogeneous spaces,* Illinois J. Math. 33 (1989), 643–656.

[MPS2] ———, *Generalized local Fatou theorems and area integrals,* Trans. Amer. Math. Soc. 321 (1990), 401–413.

[MPS3] ———, *A converse Fatou theorem,* Michigan Math. J. 36 (1989), 3–9.

[MS] B. A. Mair and D. Singman, *A generalized Fatou theorem,* Trans. Amer. Math. Soc. 300 (1987), 705–719.

[NS] A. Nagel and E. M. Stein, *On certain maximal functions and approach regions,* Adv. in Math. 54 (1984), 83–106.

[S1] E. M. Stein, *On limits of sequences of operators,* Ann. of Math. (2) 74 (1961), 140–170.

[S2] ———, *Singular integrals and differentiability properties of functions,* Princeton University Press, Princeton, N.J., 1970.

[Su1] J. Sueiro, *On maximal functions and Poisson-Szegö integrals,* Trans. Amer. Math. Soc. 298 (1986), 653–669.

[Su2] ———, *A note on maximal operators of Hardy-Littlewood type,* Math. Proc. Cambridge Philos. Soc. 102 (1987), 131–134.

[W] N. A. Watson, *Boundary behaviour of harmonic functions and solutions of parabolic systems,* Proc. Edinburgh Math. Soc. (2) 31 (1988), 267–270.

Robert Berman
Department of Mathematics
Wayne State University
Detroit, MI 48202

David Singman
Department of Mathematical Sciences
George Mason University
Fairfax, VA 22030