A Density Criterion for Frames of Complex Exponentials

S. JAFFARD

1. Introduction

The notion of a frame has been introduced by Duffin and Schaeffer in [1]. It can be defined in a general Hilbert space $H$ as follows. A sequence $(e_n)$ of vectors of $H$ is a frame if there exist positive constants $C_1$ and $C_2$ such that, for all $f$ in $H$,

$$C_1 |f|^2 \leq \sum |\langle f, e_n \rangle|^2 \leq C_2 |f|^2.$$  

(1)

Frames are important in the study of complex exponentials (cf. [1] and the book of R. M. Young on nonharmonic Fourier series [3]).

The following problem will be studied in this paper. Let $\Lambda = (\lambda_n)_{n \in \mathbb{Z}}$ be a sequence of distinct real numbers. What is the upper bound of all numbers $R$ such that the sequence of functions $(e^{i\lambda_n t})$ is a frame of $L^2([-R, R])$? This number, denoted $R(\Lambda)$, will be called the frame radius of the sequence $\Lambda$. Partial results were found by Duffin and Schaeffer [1] and Landau [2]. They are summarized in Theorems 1 and 2. The goal of the present paper is to give a necessary and sufficient condition for $\Lambda$ to have a strictly positive finite frame radius, and, when it does, to obtain a formula for that radius.

We shall consider only sequences with distinct $\lambda_n$'s since the general case can be dealt with as follows. The frame radius of the sequence $\lambda_n$ is not changed if we repeat some $\lambda_n$'s a finite and uniformly bounded number of times. If the number of repetitions is not bounded, the functions $(e^{i\lambda_n t})$ can never be a frame on any interval. Note also that, if the sequence of functions $(e^{i\lambda_n t})$ is a frame for the interval $I$, it is also a frame for each subinterval of $I$.

The reference space is $L^2(I)$, where $I$ is a finite interval, and the inner product is given by

$$\langle f | g \rangle = \frac{1}{|I|} \int_I f(t) \overline{g(t)} \, dt,$$

where $|I|$ denotes the length of the interval. We denote by $C$, $C_1$, and $C_2$ constants which can change from one line to the next.

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2. Some Definitions and Results

A sequence $\Lambda$ is said to be separated if

$$\inf_{n \neq m} |\lambda_n - \lambda_m| > 0.$$ 

If $\Lambda$ is separated, it has a uniform density $d(\Lambda)$ if there exists a number $L$ such that, for all integers $n$,

$$\left| \lambda_n - \frac{n}{d(\Lambda)} \right| \leq L.$$ 

Duffin and Schaeffer [1] proved the following theorem.

**THEOREM 1.** If $\Lambda$ is a sequence of uniform density $d(\Lambda)$, then the frame radius of $\Lambda$ is at least $\pi d(\Lambda)$.

Let $\Lambda$ be a separated sequence and $n^-(r)$ the smallest number of $\lambda_i$ in any interval of length $r$. The following result has been obtained by Landau [2].

**THEOREM 2.** If $\Lambda$ is a separated sequence, then the lower uniform density of $\Lambda$, defined as

$$D^-(\Lambda) = \lim_{r \to \infty} \frac{n^-(r)}{r},$$

always exists, and the frame radius of $\Lambda$ is at most $\pi D^-(\Lambda)$.

Let $U(\Lambda)$ be the set of all the subsequences of $\Lambda$ with a uniform density. Then the “frame density” of $\Lambda$ is defined by

$$D^f(\Lambda) = \sup_{\Theta \in U(\Lambda)} d(\Theta).$$

(2)

In this paper, the following result will be proved.

**THEOREM 3.** Let $\Lambda$ be a sequence of distinct real numbers. Then: If $U(\Lambda) = \emptyset$, or if the numbers $A_n$ of elements of $\Lambda \cap [n, n+1]$ are not bounded ($n$ taking all integer values), then there exists no interval over which $(e^{i\lambda_n t})$ is a frame. Otherwise, the frame radius of $\Lambda$ is equal to $\pi D^f(\Lambda)$.

The proof of Theorem 3 is divided into two parts. In the first part, we obtain a “qualitative” result on a convenient partitioning of the $\lambda_n$ when the sequence of functions $e^{i\lambda_n t}$ is a frame over a certain interval, thus proving the first part of Theorem 3 and half of the last equality; in the second part, we complete the frame radius equality.
3. A Partitioning of $\Lambda$

The first lemma is a direct consequence of the definition of a frame.

**LEMMA 1.** Let $I$ be a finite interval. If $(e^{i\lambda_n t})$ is a frame of $L^2(I)$, then the number of points $\lambda_n$ inside each interval of length 1 is uniformly bounded.

**Proof.** Suppose that $(e^{i\lambda_n t})$ is a frame of $L^2([-a,a])$, since the position of the interval $I$ is obviously of no importance. Let $\epsilon$ be chosen so that, for all $\eta$ with $|\eta| \leq \epsilon$,

$$\left| \frac{1}{2a} \int_{-a}^{a} e^{i\eta t} \, dt \right|^2 \cong \frac{1}{2}. \quad (3)$$

Suppose now that the number of $\lambda_n$ inside an interval of length 1 is not bounded; then neither is the number of $\lambda_n$ inside an interval of length $\epsilon$. Thus, there exists a sequence $\mu_k$ of real numbers with the following property: the number of $\lambda_n$ inside $[\mu_k - \epsilon, \mu_k + \epsilon]$ is at least $k$. Let $f_k$ be the function $e^{i\mu_k t}$. Then, an immediate consequence of (3) is that

$$\sum_n |\langle f_k | e^{i\lambda_n t} \rangle|^2 \geq \frac{k}{2}. \quad (4)$$

But $\|f_k\| = 1$, so that the second inequality of (1) cannot hold, and the contradiction proves the lemma. \qed

The following lemma gives the structure of all sequences $\Lambda = (\lambda_n)$ such that $(e^{i\lambda_n t})$ is a frame of $L^2(I)$, for a certain interval $I$.

**LEMMA 2.** The following two assertions are equivalent.

(a) There exists $I$ such that $(e^{i\lambda_n t})_{\lambda_n \in \Lambda}$ is a frame of $L^2(I)$.

(b) $\Lambda$ is the disjoint union of a sequence with a uniform density (denoted by $d_1$) and a finite number of separated sequences.

Furthermore, if (b) holds, then $(e^{i\lambda_n t})$ is a frame of $L^2(I)$ for each $I$ such that $|I| < 2\pi d_1$. Hence $R(\Lambda) \geq \pi d_1$.

**Proof.** Let us prove (b) $\Rightarrow$ (a). Let $\Lambda = \Lambda^1 \cup \cdots \cup \Lambda^n$, where $\Lambda^1$ has a positive uniform density $d_1$ and $\Lambda^2, \ldots, \Lambda^n$ are separated. By Theorem 1, $(e^{i\lambda t})_{\lambda \in \Lambda^1}$ is a frame for each interval of length less than $2\pi d_1$; denote one such interval by $I$. Then there exist positive constants $C_1$ and $C_2$ such that, for all $f$ in $L^2(I)$,

$$C_1 \|f\|^2 \leq \sum_{\lambda \in \Lambda^1} |\langle f | e^{i\lambda t} \rangle|^2 \leq C_2 \|f\|^2. \quad (4)$$

A direct computation (performed in [1]) shows that, for each separated sequence $\Lambda'$ and each interval $I$, there exists a constant $C$ such that for all $f$ in $L^2(I)$,
(5) \[ \sum_{\lambda \in \Lambda'} |\langle f | e^{i\lambda t} \rangle|^2 \leq C |f|^2. \]

Hence, there are \( C'_j \) for \( j = 2, \ldots, n \), such that
\[ \sum_{\lambda \in \Lambda_j} |\langle f | e^{i\lambda t} \rangle|^2 \leq C'_j |f|^2. \]

Adding inequalities, \( \Lambda \) satisfies the inequalities of (1). Hence (b) \( \Rightarrow \) (a) and the last statement of Lemma 2 is established.

We now prove (a) \( \Rightarrow \) (b). It is sufficient to prove that there exists an \( N > 0 \) and a \( C_N \) larger than 1 such that, for each integer \( k \), the number \( A^k_N \) of \( \lambda_i \) in each interval \([kN, (k+1)N)\) lies between 1 and \( C_N \). For, if it is so, we can define a subsequence \( \mu_k \) of \( \lambda_i \) by picking one of the \( \lambda_i \) in each interval \([2kN, (2k+1)N)\). The \( \mu_k \) satisfy \( |\mu_{k+1} - \mu_k| > N \) and \( |\mu_k - 2kN| < N \); thus the \( \mu_k \) will form a sequence having a uniform density. The remaining \( \lambda_i \) can then be divided into at most \( 2C_N - 1 \) separated sequences by picking at most one \( \lambda_i \) in each interval of the form \([2kN, (2k+1)N)\) (for \( C_N - 1 \) sequences) or of the form \([(2k+1)N, (2k+2)N)\) for the remaining sequences.

We now proceed to show the existence of such an \( N \). Because of Lemma 1, for each \( N \), the number of \( \lambda_i \) in each interval \([kN, (k+1)N)\) is uniformly bounded. So it is sufficient to prove that each \( A^k_N \) is at least 1 for some \( N \). If this were not the case, then for each \( N \) we could pick a half-open interval of length \( N \) such that no \( \lambda_k \) lies in this interval. Let \( \mu_N \) be the center of this interval, and let \( f_N(t) = e^{i\mu_N t} \). Then
\[ |\langle f_N | e^{i\lambda k t} \rangle|^2 = \left| \frac{2 \sin((\lambda_k - \mu_N)|I|/2)}{|I|/(\lambda_k - \mu_N)} \right|^2 \leq \frac{4}{|I|^2 |\lambda_k - \mu_N|^2}. \]

By Lemma 1, there are at most \( C_1 \) numbers \( \lambda_k \) in the interval \([n, n+1)\), and there are none if \( |n - \mu_N| < N/4 \) (for \( N > 4 \)). Thus
\[ \sum_k |\langle f_N | e^{i\lambda k t} \rangle|^2 = \sum_{n \in Z} \sum_{\lambda_k \in [n, n+1)} |\langle f_n | e^{i\lambda k t} \rangle|^2 \]
\[ = \sum_{n, |n - \mu_N| > N/4} \frac{4C_1}{|I|^2(|\mu_N - n| - 1)^2} \]
\[ \leq \frac{C'}{N} \quad \text{(for } N > 4 \text{)}. \]

Since \( |f_N| = 1 \), if \( N \) is chosen large enough then a contradiction with the first inequality of (1) is obtained, and the first part of Lemma 2 follows.

Because of Lemma 2, we shall assume from now on that all the sequences we consider are finite disjoint unions of separated sequences.

Some of the conclusions of Theorem 3 follow immediately from Lemmas 1 and 2. When \( U(A) \) is empty, \( (e^{i\lambda t}) \) cannot be a frame by Lemma 2. When
the cardinality of \( \Lambda \cap \{ n, n+1 \} \) is unbounded, \((e^{i\lambda_n t})\) cannot be a frame by Lemma 1. If \((e^{i\lambda_n t})\) is a frame over some interval, by Lemma 2, \( \Lambda \) is a disjoint union of separated sequences. Consequently, each subsequence of \( \Lambda \) is such a union. If \( \Theta \) is in \( U(\Lambda) \) and has density \( d(\Theta) \), then \( \Lambda = \Theta \cup (\Lambda \setminus \Theta) \), where \( \Lambda \setminus \Theta \) is a disjoint union of separated sequences. Again, using Lemma 2, we see that \( R(\Lambda) \geq \pi d(\Theta) \). Hence

\[
R(\Lambda) \geq \pi \sup_{\Theta \in U(\Lambda)} d(\Theta) = \pi D_f(\Lambda).
\]

The purpose of the next section is to complete the proof of Theorem 3 by showing that \( R(\Lambda) = \pi D_f(\Lambda) \).

It is perhaps worth noting that when \( \Theta \) is a separated sequence of uniform density \( d(\Theta) \), then \( D_f(\Theta) = d(\Theta) \); this together with Theorem 2 establishes the conclusion of Theorem 3 in this case. Similarly, a slight improvement of this argument leads to the same conclusion if \( \Theta \) is only separated. The main difficulty we shall have to deal with in the next part will come from the fact that \( \Theta \) may not be separated.

4. A Determination of the Frame Radius

The key ingredient in this determination is given by the following proposition.

**Proposition 1.** Let \( \Lambda^1 = (\lambda^1_n) \) and \( \Lambda^2 = (\lambda^2_n) \) be two disjoint sequences of distinct real numbers such that

\[
|\lambda^1_n - \lambda^2_n| \to 0 \quad \text{when} \quad |n| \to \infty;
\]

let us also suppose that \( R(\Lambda^1) \) exists. Then

\[
R(\Lambda^1) = R(\Lambda^1 \cup \Lambda^2).
\]

The proof of Proposition 1 will use the two auxiliary lemmas that follow.

**Lemma 3.** If a sequence of vectors \( e_n \) is a frame of a Hilbert space \( H \), then the mapping \( T: l^2 \to H \) defined by

\[
T((a_n)) = \sum a_n e_n
\]

is continuous and onto.

**Proof.** Let \( g \in H \). Then

\[
|\langle T((a_n)), g \rangle| = |\sum a_n \langle e_n, g \rangle| \\
\leq \|a_n\| \left( \sum |\langle e_n, g \rangle|^2 \right)^{1/2} \\
\leq C \|a_n\| \|g\|.
\]

Hence \( T \) is continuous. Thus, to prove that \( T \) is onto it is sufficient to prove that, for any \( f \) in \( H \), if \( f \) is orthogonal to all the \( e_n \) then \( f = 0 \). But this is a consequence of the first inequality of (I).
LEMMA 4. Suppose that a sequence of functions \((e_n)_{n \in \mathbb{Z}}\) is a frame of \(L^2(I)\). Then \((e_n)_{n \neq 0}\) is a frame on each interval \(I' \subset I\) such that \(|I'| < |I|\).

Proof. The \((e_n)_{n \in \mathbb{Z}}\) are a frame of \(L^2(I')\). Then, either \((e_n)_{n \neq 0}\) is a frame of \(L^2(I')\), and we have nothing to prove, or the \((e_n)_{n \in \mathbb{Z}}\) are a Riesz basis of \(L^2(I')\) (cf. [3, p. 186]). We now make this assumption.

Let \(f\) be a square integrable function defined on \(I\), and vanishing on \(I'\) but not on \(I\). By Lemma 3, \(f = \sum a_n e_n\), with \((a_n)\) in \(l^2\). Since the \((e_n)_{n \in \mathbb{Z}}\) are a Riesz basis of \(L^2(I')\), and \(f\) vanishes on \(I'\), we obtain that \(a_n = 0\) for all \(n\). Hence, \(f\) vanishes on \(I\), and a contradiction is obtained.

\(\square\)

Proof of Proposition 1. The proposition will be proved in two steps. The first step is to prove it under the stronger assumption that

\[|\lambda_n^1 - \lambda_n^2| \leq \frac{1}{n^2}.\]

If this assumption holds, then

\[
|\langle f \mid e^{i\lambda_n^1 \hat{t}} \rangle - \langle f \mid e^{i\lambda_n^2 \hat{t}} \rangle| \leq \|f\| |e^{i\lambda_n^1 \hat{t}} - e^{i\lambda_n^2 \hat{t}}| \\
\leq C\|f\| |\lambda_n^1 - \lambda_n^2| \\
\leq C\|f\| \frac{1}{n^2},
\]

so that

\[||\langle f \mid e^{i\lambda_n^1 \hat{t}} \rangle - \langle f \mid e^{i\lambda_n^2 \hat{t}} \rangle||^2 \leq C'\frac{\|f\|^2}{n^2}.\]

We saw that \(R(\Lambda^1 \cup \Lambda^2) \supseteq R(\Lambda^1)\). Let \(I\) be an interval over which \((e^{i\lambda \hat{t}})_{\lambda \in \Lambda_1 \cup \Lambda_2}\) is a frame. There exist \(C_1\) and \(C_2\) such that

\[C_1\|f\|^2 \leq \sum_{\lambda \in \Lambda_1} |\langle f \mid e^{i\lambda \hat{t}} \rangle|^2 + \sum_{\lambda \in \Lambda_2} |\langle f \mid e^{i\lambda \hat{t}} \rangle|^2 \leq C_2\|f\|^2.\]

Let \(N\) be such that

\[C' \sum_{|n| \geq N} \frac{1}{n^2} \leq \frac{C_1}{2}.\]

Then

\[
\sum_{|n| < N} |\langle f \mid e^{i\lambda_n^1 \hat{t}} \rangle|^2 + \sum_{|n| < N} |\langle f \mid e^{i\lambda_n^2 \hat{t}} \rangle|^2 \\
\leq \sum_{|n| < N} |\langle f \mid e^{i\lambda_n^1 \hat{t}} \rangle|^2 + \sum_{|n| < N} |\langle f \mid e^{i\lambda_n^2 \hat{t}} \rangle|^2 + 2 \sum_{|n| \geq N} |\langle f \mid e^{i\lambda_n^1 \hat{t}} \rangle|^2 + \frac{C_1}{2} \|f\|^2,
\]

so that

\[\frac{C_1}{4} \|f\|^2 \leq \sum_{|n| < N} |\langle f \mid e^{i\lambda_n^1 \hat{t}} \rangle|^2 + \sum_{|n| < N} |\langle f \mid e^{i\lambda_n^2 \hat{t}} \rangle|^2 \leq C_2\|f\|^2.\]

Lemma 4 means that the frame radius is not changed by deleting one element of a sequence (hence also by deleting a finite number), so that Proposition 1 holds under the assumption \(|\lambda_n^1 - \lambda_n^2| \leq 1/n^2\). The general case will be a consequence of the following lemma (proved in [1]).
LEMMA 5. Let \( (e^{i\lambda_nt}) \) be a frame over \( I \). There exists \( \delta_1 > 0 \) such that \( (e^{i\mu_nt}) \) is a frame over the same interval whenever \( (\mu_n) \) is a real sequence such that \( |\mu_n - \lambda_n| \leq \delta_1 \).

We can now complete the proof of Proposition 1.

Suppose that \( |\lambda_n^2 - \lambda_n^2| \to 0 \), and let \( R \) be less than \( R(\Lambda^1 \cup \Lambda^2) \). Then \( (e^{i\lambda_n^1}) \cup (e^{i\lambda_n^2}) \) is a frame over \([-R, R]\). Let \( \delta_1 \) be as in Lemma 5. Change \( \lambda_n^2 \) into \( \mu_n = \lambda_n^1 + 1/n^2 \), if \( n \) is such that \( 1/n^2 \leq \delta_1/2 \) and \( |\lambda_n^1 - \lambda_n^2| \leq \delta_1/2 \). By Lemma 5, the set of functions \( (e^{i\lambda_n^1}) \cup (e^{i\mu_n}) \) is a frame over \([-R, R]\). But, since \( |\lambda_n^1 - \mu_n| = 1/n^2 \) for \( n \) large enough, \( R \leq R(\Lambda^1) \). Thus

\[ R(\Lambda^1) \leq R(\Lambda^1 \cup \Lambda^2), \]

and hence Proposition 1 is proved. \( \square \)

Let us call \( V(\Lambda) \) the set of all the subsequences of \( \Lambda \) that are separated. Then the following lemma holds.

LEMMA 6. For any sequence \( \Lambda \) such that the cardinality of \( \Lambda \cap [n, n+1] \) is bounded, the following equality holds:

\[ \sup_{\Theta \in V(\Lambda)} D^- (\Theta) = \sup_{\Theta \in U(\Lambda)} d(\Theta) = D^f(\Lambda). \]

Proof. A sequence of uniform density is separated, so that

\[ \sup_{\Theta \in V(\Lambda)} D^- (\Theta) \geq \sup_{\Theta \in U(\Lambda)} d(\Theta) \]

because, for a sequence \( \Theta \) with a uniform density, \( d(\Theta) = D^- (\Theta) \). Suppose that \( \mu_n \) is a separated subsequence of \( \Lambda \), with a lower uniform density \( D^- \).

Let \( \epsilon > 0 \). Choose \( R \) large enough so that \( R(D^- - \epsilon) \) is an integer and

\[ \frac{n^-(R)}{R} \geq D^- - \epsilon. \]

In each interval \([kR, (k+1)R]\) there are at least \( R(D^- - \epsilon) \) numbers \( \mu_n \). Extract a subsequence \((\gamma_n)\) of \((\mu_n)\) that has exactly \( R(D^- - \epsilon) \) elements in each of these intervals. The sequence \((\gamma_n)\) is separated and

\[ \left| \gamma_n - \frac{n}{D^- - \epsilon} \right| \leq R \]

so that \((\gamma_n)\) has a uniform density \( D^- - \epsilon \), and

\[ \sup_{\Theta \in U(\Lambda)} d(\Theta) \geq D^- - \epsilon. \]

This proves Lemma 6. \( \square \)

The two following lemmas are a slight generalization of Theorem 2.

LEMMA 7. If \( \Lambda \) is a finite union of separated sequences, then the limit

\[ D^-(\Lambda) = \lim_{r \to \infty} \frac{n^-(r)}{r} \]

exists; \( D^-(\Lambda) \) is said to be the lower density of \( \Lambda \).
Proof. Define, for \( r \geq 1 \), the function \( Q(r) = n^{-}(r)/r \). Let us first establish three simple properties of the function \( Q \). Because of Lemma 1, there exists a constant \( C \) such that \( n^{-}(r) \leq Cr \). Thus \( Q \) is a bounded function.

Let \( p \) be an integer and \( I \) an interval of length \( pr \). Let us write \( I \) as a disjoint union of \( p \) intervals \( I_1, \ldots, I_p \), each of length \( r \). For each \( k \), the cardinality of \( (\Lambda \cap I_k) \) is at least \( n^{-}(r) \), so that the cardinality of \( (\Lambda \cap I) \) is at least \( pn^{-}(r) \); thus

\[
Q(pr) \geq Q(r).
\]

Let \( \alpha > 1 \). Obviously \( n^{-}(\alpha r) \geq n^{-}(r) \), so that

\[
Q(\alpha r) \geq \frac{1}{\alpha} Q(r).
\]

Let

\[
\bar{Q} = \sup_{r \geq 1} Q(r).
\]

Given \( \epsilon \) in \( (0, 1/2) \), choose a positive and a positive integer \( n \) such that \( Q(a) \geq \bar{Q} - \epsilon \) and

\[
\frac{n+1}{n} \leq \frac{1}{1-\epsilon}.
\]

Consider \( x \geq na \). There exists an integer \( p \) at least equal to \( n \) such that

\[
pa \leq x < (p+1)a.
\]

Then

\[
Q(x) = Q\left(\frac{x}{pa} pa\right) \geq \frac{pa}{x} Q(pa) \geq \frac{pa}{x} Q(a) \geq \frac{pa}{x} (\bar{Q} - \epsilon)
\]

\[
\geq \frac{pa}{(p+1)a} (\bar{Q} - \epsilon) \geq \frac{n}{n+1} (\bar{Q} - \epsilon) \geq (1-\epsilon)(\bar{Q} - \epsilon).
\]

Hence Lemma 7 is established.

LEMMA 8. If \( \Lambda \) is a finite union of separated sequences, the frame radius of \( \Lambda \) is at most \( \pi D^{-}(\Lambda) \).

Proof. Let \( \Lambda \) be a finite union of disjoint separated sequences. Then, for each \( \delta \), one can find a single separated sequence \( \Lambda' \) such that

\[
|\lambda_n - \lambda'_n| < \delta.
\]

From Lemma 5, if \( \delta \) is small enough then the frame radius of \( \Lambda' \) will be at least the frame radius of \( \Lambda \). But

\[
D^{-}(\Lambda) = D^{-}(\Lambda').
\]

Because of Theorem 2,

\[
R(\Lambda') \leq \pi D^{-}(\Lambda'),
\]

so that
\[ R(\Lambda) \leq \pi D^{-}(\Lambda) \]

and Lemma 8 follows. \qed

We can now complete the proof of Theorem 3. It remains only to show that

\[ R(\Lambda) \leq \pi D^{f}(\Lambda) \]

for a sequence \( \Lambda \) which is the union of \( k \) separated sequences.

The idea of the proof is to split \( \Lambda \) into a separated sequence \( \Omega \) of uniform density at least \( D^{f}(\Lambda) - \epsilon \), a finite union of sequences \( \Gamma^{i} \) each of which tends to \( \Omega \) (or a subsequence of \( \Omega \)), and a remaining sequence \( \Theta \) of lower uniform density at most \( 3k\epsilon \).

Suppose that such a splitting is achieved. Then, by Proposition 1, we can disregard the \( \Gamma^{i} \) in the calculation of the frame radius, and by Lemma 8, the frame radius of \( \Omega \cup \Theta \) is at most \( \pi (D^{f}(\Lambda) + 3k\epsilon) \). Thus Theorem 3 will be proved once this splitting of \( \Lambda \) is constructed.

Let \( \epsilon \) be fixed. We can extract from \( \Lambda \) a sequence \( \Omega = (\omega_{n}) \) which has a uniform density at least \( D^{f}(\Lambda) - \epsilon \) and is separated. We now construct the sequence \( \Theta = (\theta_{n}) \) by induction. We do this construction only for positive values of \( \lambda_{n} \); it is the same for the negative values. Let

\[ E_{i} = \bigcup_{\omega_{n} \geq 0} [\omega_{n} - 1, \omega_{n} + 1]. \]

Let \( \Theta^{1} = (\theta_{n}^{1}) \) be the subsequence of \( \Lambda \) composed of all the \( \lambda_{n} > 0 \) which are not in \( E_{1} \). The sequence \( \Theta^{1} \) is the union of at most \( k \) separated sequences, none of which has a density larger than \( 2\epsilon \); for, if such a subsequence \( \Sigma \) had a density larger than \( 2\epsilon \), then the union of \( \Sigma \) and \( \Omega \) would be a separated subsequence of \( \Lambda \) with a density at least \( D^{f}(\Lambda) + \epsilon \), which is impossible. Thus \( D^{-}(\Theta^{1}) \leq 2k\epsilon \), and there exists an interval \( I_{1} \) large enough such that the number of \( \theta_{n}^{1} \) in \( I_{1} \) is less than \( 3k\epsilon |I_{1}| \).

Let \( A_{1} = \sup I_{1} \) if \( I_{1} \) does not intersect \( E_{1} \); otherwise, let \( p \) be the largest integer such that

\[ I_{1} \cap [\omega_{p} - 1, \omega_{p} + 1] \neq 0; \]

then \( A_{1} = \omega_{p} + 1 \). The beginning of the construction of \( \Theta \) is as follows: \( \theta_{n} = \theta_{n}^{1} \) if \( \theta_{n}^{1} \leq A_{1} \).

The induction now works as follows. We suppose that \( \Theta \) is constructed for \( \theta_{n} \leq A_{m-1} \). We now define the set

\[ E_{m} = \bigcup_{\omega_{n} \geq A_{m-1}} \left[ \omega_{n} - \frac{1}{m}, \omega_{n} + \frac{1}{m} \right], \]

and the sequence \( \Theta^{m} \) which is composed of the \( \lambda_{n} \) larger than \( A_{m-1} \) that are not in \( E_{m} \). We can find by the same argument as above an interval \( I_{m} \) included in \([A_{m-1}, +\infty)\), of length at least \( m \) and such that the number of elements of the sequence \( \Theta^{m} \) in \( I_{m} \) is less than \( 3k\epsilon |I_{m}| \). Let \( A_{m} = \sup I_{m} \) if \( I_{m} \) does not intersect \( E_{m} \); otherwise, let \( p \) be the largest integer such that
\[ I_m \cap \left[ \omega_p - \frac{1}{m}, \omega_p + \frac{1}{m} \right] \neq 0; \]

then \( A_1 = \omega_p + 1/m \). The sequence \( \Theta \) for \( A_m < \theta_n < A_{m+1} \) is composed of the elements of \( \Theta^m \) in the same interval.

Once the construction of \( \Theta \) is achieved, we have finally split \( \Lambda \) into a sequence \( \Omega \) of uniform density \( A - \epsilon \), a sequence \( \Theta \) of lower density less than \( 3k\epsilon \) (because the number of elements of \( \Theta \) in \( I_N \) is at most \( 3k\epsilon|I_N| \)), and a remaining sequence included in a set

\[ E = \bigcup [\mu_n - \alpha_n, \mu_n + \alpha_n], \]

where the \( \alpha_n \) are certain \( 1/m \), and are such that \( \alpha_n \to 0 \); this sequence can obviously be written as a finite union of sequences \( \Gamma_i \), each of which tends to \( \Omega \) or a subsequence of \( \Omega \). The requested splitting is thus performed and Theorem 3 is proved. \( \square \)

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References


L.A.M.M. (C.E.R.M.A.)
Ecole Nationale des Ponts et Chaussées
La Courtine, 93167
Noisy-le-grand
France