Interpolating Blaschke Products and Factorization in Douglas Algebras

PAMELA GORKIN[1] & RAYMOND MORTINI[2]

Introduction

The following problem of Guillory, Izuchi, and Sarason is proven: Let B be a Douglas algebra and let u be a unimodular function in B which does not vanish identically on any nontrivial Gleason part in B. If q is a function in B whose zero set contains that of u, then u divides q^N for some $N \in \mathbf{N}$. By using function-theoretic methods we shall also generalize a recent theorem of Tolokonnikov on zero sets of ideals in H^∞.

Let H^∞ be the Banach algebra of all bounded analytic functions in the open unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ and let $M(H^\infty)$ denote its maximal ideal space. For $m, x \in M(H^\infty)$, let $\rho(m,x) = \sup\{|f(x)| : f(m) = 0, \|f\| = 1\}$ denote the pseudohyperbolic distance of the points m and x in $M(H^\infty)$. By Schwarz–Pick's lemma, $\rho(z,w) = |(z-w)/(1-\bar{z}w)|$ if $z, w \in \mathbf{D}$. Let

$$P(m) = \{x \in M(H^\infty) : \rho(m,x) < 1\}$$

be the Gleason part of $m \in M(H^\infty)$. Defining m to be equivalent to x, $m \sim x$, if $\rho(m,x) < 1$ then one can show [4, p. 402] that $\sim$ is an equivalence relation in $M(H^\infty)$. Thus the Gleason parts of two points are either disjoint or equal.

A Gleason part P is called an *analytic disk* if there exists a continuous, bijective map L of $\mathbf{D}$ onto P such that $\hat{f} \circ L$ is analytic in $\mathbf{D}$ for every $f \in H^\infty$, where $\hat{f}$ denotes the Gelfand transform of $f \in H^\infty$.

In his famous paper [8], Hoffman showed that any Gleason part $P(m)$ in $M(H^\infty)$ is either a single point or an analytic disk. Moreover, the latter occurs if and only if $m \in \mathbf{D}$ or lies in the (weak-*-) closure of an interpolating sequence in $\mathbf{D}$, that is, in the closure of a sequence $\{z_n\}$ satisfying

$$\inf_{m \in \mathbf{N}} \prod_{\substack{n \in \mathbf{N} \\ n \neq m}} \rho(z_n, z_m) \geq \delta > 0.$$

This leads us to the following definition.

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Let G denote the set of all points in $M(H^\infty)$ whose Gleason parts are analytic disks. As in Hoffman [8], the elements of G will be called *nontrivial points*.

The set of points in $M(H^\infty)$ whose Gleason part reduces to a single point is denoted by S, and its elements are called *trivial points*. By Hoffman's theorem we have $M(H^\infty)=G\cup S$, where G is an open and S a closed set in the weak-*-topology of $M(H^\infty)$.

If $\{z_n\}$ is an interpolating sequence in $\mathbf{D}$, then the associated Blaschke product b given by

$$b(z)=\prod_{n=1}^{\infty}\frac{\bar{z}_n}{|z_n|}\frac{z_n-z}{1-\bar{z}_n z},\quad z\in\mathbf{D},$$

is called an *interpolating Blaschke product* (where $\bar{z}_n/|z_n|=1$ if $z_n=0$).

Now let (z_n) be any ρ-separated sequence in $\mathbf{D}$, that is, a sequence satisfying $\rho(z_n,z_m)\geq\delta>0$ for $n\neq m$. Then each z_n is a nontrivial point, and it is easy to see from well-known theorems on interpolating sequences that there exists an interpolating Blaschke product which vanishes on a subsequence of (z_n) (see [7, p. 204]). It is now a natural question to ask whether a similar situation holds for sequences of nontrivial points in G. In the first part of this paper we shall give an answer.

Let L^∞ denote the space of essentially bounded Lebesgue measurable functions on the unit circle $\partial\mathbf{D}$. A Douglas algebra is a uniformly closed subalgebra of L^∞ containing H^∞. For a Douglas algebra B, the maximal ideal space $M(B)$ of B can be viewed as a compact subset of $M(H^\infty)$ (see [4, p. 375]). We shall also identify a function in B with its Gelfand transform. In a recent paper of the first author, the following result is proven.

THEOREM 0.1 ([5, Thm. 1]). *Let f be a function in H^∞ which does not vanish on any trivial part in $M(B)$. Then f can be factorized in a product $f=b_1b_2\cdots b_Ng$, where b_j $(j=1,\dots,N)$ are interpolating Blaschke products and where $g\in H^\infty$ is invertible in B.*

In Section 2 we shall present an extension of this result to ideals in H^∞. Incidentally we obtain an extension of a result of Tolokonnikov [14], whose operator-theoretic methods do not seem to work in the present context of Douglas algebras.

Finally, in Section 3 we answer a question of Guillory, Izuchi, and Sarason [6] on the divisibility structure of Douglas algebras. To this end we show, by using results of Section 1, that if b is a Blaschke product which does not vanish identically on any nontrivial Gleason part contained in $M(B)$, then b actually satisfies the assumption of Theorem 0.1. This enables us then to give a complete solution to the problem of Guillory, Izuchi, and Sarason.

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1. Sequences of Nontrivial Points in H^∞

It is well known ([7] and [8]) that if (z_n) is a sequence in **D** then there exists at least one cluster point of (z_n) whose Gleason part is nontrivial. Theorem 1.2 now will generalize this fact to an arbitrary sequence in G. To prove it, we need the following notation and lemma.

Let $Z(b)=\{m\in M(H^\infty): b(m)=0\}$ denote the zero set in $M(H^\infty)$ of a Blaschke product b. Finally, for a set $E\subseteq M(H^\infty)$ and $x\in M(H^\infty)$, let $\rho(x,E)=\inf\{\rho(x,m): m\in E\}$.

LEMMA 1.1. *Let (y_k) be a sequence of nontrivial points in $M(H^\infty)$ and let $0<\eta_0<1$. If there exists a closed set E of nontrivial points such that $\rho(y_k,E)<\eta_0$ for every $k\in\mathbf{N}$, then there exists a cluster point of (y_k) whose Gleason part is nontrivial.*

Proof. By hypothesis, for every n there exists a point $x_n\in E$ such that $\rho(x_n,y_n)\le\eta_0$. Let x be any cluster point of (x_n), say $x_{n(\alpha)}\to x$ for some net $n(\alpha)$. By taking subnets we may assume that $y_{n(\alpha)}$ converges to a point $y\in M(H^\infty)$. Since ρ is lower semi-continuous [8, p. 103], we get $\rho(x,y)\le\eta_0$. But $x\in E$ is a nontrivial point, and since $y\in P(x)$, y is nontrivial, too. □

HOFFMAN'S LEMMA ([8, pp. 86, 106]; see also [4, p. 404]). *Let $0<\delta<1$ and $0<\eta<(1-\sqrt{1-\delta^2})/\delta$; that is, $0<\eta<\rho(\delta,\eta)$ and let*

$$\epsilon=\epsilon(\delta)=\frac{\delta-\eta}{1-\delta\eta}\eta.$$

If b is any interpolating Blaschke product with zeros z_n such that $\delta(b)=\inf_{n\in\mathbf{N}}(1-|z_n|^2)|b'(z_n)|\ge\delta$, then

$$\{m\in M(H^\infty): |b(m)|<\epsilon\}\subseteq\bigcup_{y\in Z(b)}\{m\in M(H^\infty): \rho(m,y)<\eta\}.$$

Later on we will need to use the facts that $0<(1-\sqrt{1-\delta^2})/\delta<\delta$ is a monotone increasing function of $\delta\in(0,1)$ and that $\epsilon<\eta<\delta$. These are easy to check.

We are now ready to prove the following theorem.

THEOREM 1.2. *Let (y_n) be a sequence of nontrivial points in $M(H^\infty)$. Then there exists a cluster point of (y_n) whose Gleason part is nontrivial.*

Proof. If there are infinitely many y_n in **D**, then we are done by the remarks above. So let $y_n\in M(H^\infty)\setminus\mathbf{D}$ for all $n\in\mathbf{N}$. Let $0<\epsilon_n<1$ satisfy $\prod_{n=1}^\infty\epsilon_n>0$. Choose δ_n and η_n as in Hoffman's lemma so that $\epsilon(\delta_n)>\epsilon_n$ and η_n is increasing.

We shall now construct inductively a subsequence $\{y_{k_n}\}$ of $\{y_n\}$ and a sequence of interpolating Blaschke products b_n with $b_n(y_{k_n})=0$ such that b_n is "large" on the zero sets of the preceding b_j and such that the preceding b_j $(j=1,\ldots,n-1)$ are "large" on the zero set of b_n.

Step 1. Choose an interpolating Blaschke product b_1 with zeros $z_{k,1}$ such that

(1) $$b_1(y_1)=0.$$

Using another result of Hoffman [4, p. 407], we can factor b_1 to make $\delta(b_1)$ as large as we wish. Thus we may assume

(2) $$\delta(b_1)>\delta_1$$

and

(3) $$\sum_{k=1}^{\infty}(1-|z_{k,1}|)<\frac{1}{2}.$$

Step 2. Let $y_1=y_{k_1}$. Now every point in $Z(b_1)$ is nontrivial. Thus, by Lemma 1.1 we may assume that there exists a point y_{k_2} such that $\rho(y_{k_2},Z(b_1))>\eta_2$. Since for a compact set $E\subseteq M(H^\infty)$ the function $x\mapsto\rho(x,E)$ is lower semi-continuous on $M(H^\infty)$ (see [8, p. 103]), there exists a neighborhood U_2 of y_{k_2} such that $\rho(U_2,Z(b_1))>\eta_2$.

Now let b_2 be an interpolating Blaschke product with zeros $z_{k,2}$ which satisfies

(1) $$b_2(y_{k_2})=0 \quad\text{and}\quad Z(b_2)\subseteq U_2;$$

(2) $$\delta(b_2)>\delta_2$$

(again by using Hoffman's result [4, p. 407]); and

(3) $$\sum_{k=1}^{\infty}(1-|z_{k,2}|)<\frac{1}{2^2}.$$

Since $\rho(Z(b_2),Z(b_1))>\eta_2>\eta_1$, by Hoffman's lemma we obtain

$$|b_2(m)|>\epsilon_2 \quad\text{for every}\quad m\in Z(b_1)$$

and

$$|b_1(m)|>\epsilon_1 \quad\text{for every}\quad m\in Z(b_2).$$

Step n. Let us assume that $b_1,b_2,\dots,b_{n-1}$ have already been constructed. By Lemma 1.1, we may assume that there exists a point y_{k_n} such that

$$\rho\left(y_{k_n},\bigcup_{j=1}^{n-1}Z(b_j)\right)>\eta_n.$$

By lower semi-continuity, there exists a neighborhood U_n of y_{k_n} such that

$$\rho\left(U_n,\bigcup_{j=1}^{n-1}Z(b_j)\right)>\eta_n.$$

We remark that the fact that $\{\eta_n\}$ is increasing implies $\rho(U_n,Z(b_j))>\eta_j$ for $j=1,\dots,n-1$.

As above, we can find an interpolating Blaschke product b_n with zeros $z_{k,n}$ and satisfying

(1) $$b_n(y_{k_n})=0 \quad\text{and}\quad Z(b_n)\subseteq U_n;$$

(2) $$\delta(b_n) \geq \delta_n;$$

(3) $$\sum_{k=1}^{\infty}(1-|z_{k,n}|) \leq \frac{1}{2^n}.$$

Since $\rho(Z(b_n), Z(b_j)) > \eta_n > \eta_j$, by Hoffman's lemma we have that

$$|b_j(m)| > \epsilon_j \quad \text{for every } m \in Z(b_n) \quad (j=1,2,\ldots,n-1)$$

and

$$|b_n(m)| > \epsilon_n \quad \text{for every } m \in \bigcup_{j=1}^{n-1} Z(b_j).$$

This concludes our construction.

Now let $b = \prod_{n=1}^{\infty} b_n$. Since $\sum_{k=1}^{\infty}(1-|z_{k,n}|) \leq 1/2^n$, the product converges uniformly on compact subsets of $\mathbf{D}$ to the Blaschke product with zeros $\{z_{k,n}\}_{k,n\in\mathbf{N}}$. We shall show that b actually is an interpolating Blaschke product.

Let $z_{k,j}$ be a fixed zero of b_j and let $m > j$. Then

$$(1-|z_{k,j}|^2)\left|\left(\prod_{n=1}^{m} b_n\right)'(z_{k,j})\right| = (1-|z_{k,j}|^2)|b_j'(z_{k,j})| \prod_{\substack{n=1\\ n\neq j}}^{m} |b_n(z_{k,j})|$$

$$\geq \delta_j \prod_{\substack{n=1\\ n\neq j}}^{m} \epsilon_n \geq \prod_{n=1}^{m} \epsilon_n \geq \prod_{n=1}^{\infty} \epsilon_n.$$

(Here we have used the fact that $\delta_j \geq \epsilon_j$.) Since

$$(1-|z_{k,j}|^2)|b'(z_{k,j})| = \lim_{m\to\infty}(1-|z_{k,j}|^2)\left|\left(\prod_{n=1}^{m} b_n\right)'(z_{k,j})\right|$$

we obtain

$$\delta(b) \geq \prod_{n=1}^{\infty} \epsilon_n.$$

Thus b is an interpolating Blaschke product which vanishes at each y_{k_n}. Since $M(H^\infty)$ is compact, there is a cluster point of $\{y_{k_n}\}$. Since b vanishes on any cluster point, by [4, Lemma 3.3, p. 379] and Hoffman's theorem [8, p. 101], any cluster point $\{y_{k_n}\}$ is nontrivial; this concludes the proof. □

2. Zero Sets of Ideals in H^∞

In Tolokonnikov's recent paper [14, p. 94] it is proven that if I is an ideal in H^∞ whose zero set does not contain any trivial point in $M(H^\infty)$, then I contains a Blaschke product b of the form $b = b_1 \cdots b_N$, where b_j $(j=1,2,\ldots,N)$ are interpolating Blaschke products. By using different methods we are able to give an extension of this result to Douglas algebras. This will be done in Theorem 2.3. We begin with some relevant definitions.

Let B be a Douglas algebra and let $f \in B$. Then

$$Z_B(f) = \{m \in M(B) : f(m) = 0\}$$

denotes the zero set of f in B. The hull or zero set of an ideal I in B is the set

$$Z_B(I)=\bigcap_{f\in B} Z_B(f).$$

If $B=H^\infty$, then we omit the subscript.

For an ideal I in H^∞, the set

$$IB=\left\{\sum_{i=1}^{n} f_i q_i : f_i\in I,\ q_i\in B,\ n\in \mathbf{N}\right\}$$

is the ideal generated by I in B. Obviously we have $Z_B(IB)=Z(I)\cap M(B)$, hence it leads to no confusion in writing $Z_B(I)$ for $Z(I)\cap M(B)$.

Let $f\in B$, $m\in M(B)$, and $f(m)=0$. Then

$$\operatorname{ord}(f,m)=\sup\{n\in\mathbf{N}: f=f_1\cdots f_n,\ f_i(m)=0\ (i=1,\dots,n)\}$$

denotes the order of zero of f at the point $m\in M(B)$. If $f(m)\neq 0$, then we set $\operatorname{ord}(f,m)=0$.

Let $P_B(m)=\{x\in M(B): \rho(m,x)<1\}$ be the Gleason part of the point $m\in M(B)$. In the sequel we shall tacitly assume that

$$P_B(m)=P(m)=\{x\in M(H^\infty): \rho(m,x)<1\},$$

a fact proven by Izuchi [9, p. 437].

By a result of Hoffman [8, p. 101] we have $\operatorname{ord}(f,m)=0$ or ∞ for every trivial point $m\in M(H^\infty)$ and $f\in H^\infty$. Let

$$Z_B^\infty(f)=\{m\in M(B): \operatorname{ord}(f,m)=\infty\}\quad (f\in B).$$

For $B=H^\infty$, we let $Z_B^\infty(f)=Z^\infty(f)$.

Now we have the following proposition.

PROPOSITION 2.1. *The function $m\to\operatorname{ord}(f,m)$ is upper semi-continuous on $M(B)$ for every $f\in H^\infty$; that is, $\{m\in M(B): \operatorname{ord}(f,m)\geq n\}$ is a closed set for every $n\in\mathbf{N}$. In particular, the set $Z_B^\infty(f)$ is closed ($f\in H^\infty$).*

Proof. We will show that for each n the set $\{x\in M(B): \operatorname{ord}(f,x)<n\}$ is open in $M(B)$. For $n=1$, the result is clear. Let $\operatorname{ord}(f,m)=p$ and let $f=bg$ be the Riesz factorization of f, where $g\in H^\infty$ has no zeros in $\mathbf{D}$ and where b is a Blaschke product. Because every zero of g in $M(H^\infty)$ has infinite order (see [8, p. 78]), we have $g(m)\neq 0$. By [8, p. 100], m lies in the closure of an interpolating subsequence of the zero sequence of b in $\mathbf{D}$. Repeating this argument p times, we obtain $f=b_1\cdots b_p\cdot h$, where b_i are interpolating Blaschke products with $\operatorname{ord}(b_i,m)=1$ and where $h\in H^\infty$ does not vanish at m. Hence there exists a neighborhood U of m on which $\operatorname{ord}(f,x)\leq p<p+1$. It therefore follows that $\{x\in M(B): \operatorname{ord}(f,x)<n\}$ is open. Hence $Z_B^\infty(f)=\bigcap_{n=1}^\infty\{x\in M(B): \operatorname{ord}(f,x)\geq n\}$ is closed. □

We do not know if Proposition 2.1 holds for every $f\in B$.

PROPOSITION 2.2. *Let $f \in H^\infty$ and let E be a compact subset of $M(H^\infty)$ such that $\sup_{m \in E} \operatorname{ord}(f, m) \leq N$ for some $N \in \mathbf{N}$. Then f can be written in the form $f = cg$, where c is a finite product of interpolating Blaschke products and where $g \in H^\infty$ does not vanish on a neighborhood of E.*

Proof. This works the same way as part of the proof of Theorem 1 in [5], as soon as one notices that there exists a neighborhood U of E on which $\sup_{m \in U} \operatorname{ord}(f, m) < \infty$. But this holds because, by Proposition 2.1, the map $m \to \operatorname{ord}(f, m)$ is upper semi-continuous. □

THEOREM 2.3. *Let B be a Douglas algebra and let I be an ideal in H^∞ whose zero set does not contain any trivial point in $M(B)$. Then the ideal $IB \cap H^\infty$ contains a function of the form $b = b_1 \cdots b_n$, where b_i are interpolating Blaschke products.*

Proof. By compactness there exist finitely many functions $f_1, \ldots, f_N \in I$ with $\bigcap_{i=1}^N Z_B(f_i) \cap S = \emptyset$. Let $J = (f_1, \ldots, f_N)$ be the ideal generated by the functions f_i $(i = 1, \ldots, N)$ in H^∞. We claim that

$$\sup_{m \in Z_B(J)} \min_{j=1,\ldots,N} \operatorname{ord}(f_j, m) = k < \infty. \tag{1}$$

Otherwise, there would exist for each $n \in \mathbf{N}$ a point $m_n \in Z_B(J)$ with

$$\operatorname{ord}(f_j, m_n) \geq n \quad \text{for every } j \in \{1, \ldots, N\}.$$

Let $m \in \overline{\{m_n\}} \setminus m_n$, where $\overline{\{m_n\}}$ denotes the (weak-*-closure) of $\{m_n\}$ in $M(H^\infty)$. By Proposition 2.1, we have $\operatorname{ord}(f_j, m) = \infty$ for every $j \in \{1, \ldots, N\}$. Thus all the functions f_j vanish identically on the closure $\overline{P(m)}$ of the part $P(m)$ (see [8, pp. 79, 101]). But by Budde's result [2, p. 11], $\overline{P(m)}$ contains a trivial point x. Therefore $x \in Z_B(J) \cap S \neq \emptyset$, which contradicts our assumption. Thus (1) holds.

Let $E = \bigcap_{j=1}^N Z^\infty(f_j)$. Then (1) implies that $E \cap Z_B(J) = \emptyset$. Choose open neighborhoods U_j in $M(H^\infty)$ of $Z^\infty(f_j)$ such that

$$\bigcap_{j=1}^{N} \bar{U}_j \cap Z_B(J) = \emptyset. \tag{2}$$

Put $V_j = M(H^\infty) \setminus \bar{U}_j$. Then $\bar{V}_j \cap Z^\infty(f_j) = \emptyset$ $(j = 1, \ldots, N)$ and $Z_B(J) \subseteq \bigcup_{j=1}^N V_j$. This implies that

$$\sup_{m \in \bar{V}_j} \operatorname{ord}(f_j, m) = K_j < \infty \quad (j = 1, \ldots, N). \tag{3}$$

By Proposition 2.2 we have $f_j = c_j g_j$, where $Z(g_j) \cap \bar{V}_j = \emptyset$ and where the c_j are finite products of interpolating Blaschke products. Because

$$Z_B(g_j) \subseteq M(H^\infty) \setminus V_j = \bar{U}_j \quad \text{and} \quad \bigcap_{j=1}^{N} Z_B(g_j) \subseteq Z_B(J),$$

relation (2) yields that $\bigcap_{j=1}^N Z_B(g_j) = \emptyset$. Thus there exist functions $q_j \in B$ such that $1 = \sum_{j=1}^N q_j g_j$. Hence

$$c_1\cdots c_N=\sum_{j=1}^{N}\Bigg(q_j\prod_{\substack{k=1\\k\neq j}}^{N}c_k\Bigg)c_jg_j\in JB\cap H^\infty\subseteq IB\cap H^\infty. \qquad \square$$

We note that if $B=H^\infty$ then we obtain Tolokonnikov's result [14, p. 94]. If $I=(b)$, where b is a Blaschke product, we obtain Gorkin's result (see Theorem 0.1).

Of course we cannot expect that under the assumptions of Theorem 2.3 the ideal I itself contains a function of the form $b=b_1\cdots b_N$, where the b_i are interpolating Blaschke products. For example, take $B=L^\infty$ and $I=(u)$, where u is a singular inner function. Such counterexamples occur if and only if there exists a trivial point which does not belong to the maximal ideal space of B. This can be shown in the following way.

Let $x\in S\backslash M(B)$. Then the ideal $I=\{f\in H^\infty: f(x)=0\}$ does not contain any interpolating Blaschke product; nevertheless, $Z_B(I)\cap S=\emptyset$. On the other hand, if $S\subseteq M(B)$ then $Z_B(I)\cap S=Z(I)\cap S$. Therefore, if $Z_B(I)\cap S=\emptyset$, Tolokonnikov's result [14, p. 94] shows that I contains a Blaschke product of the desired type.

Using Theorem 6.1 in [11, p. 47], we obtain the following corollary.

COROLLARY 2.4. *Let I be an ideal in H^∞ such that $Z_B(I)\cap S=\emptyset$. Then I contains a function of the form bu, where b is a finite product of interpolating Blaschke products and where u is an inner function invertible in B.*

Proof. Let $J'=(g_1,\dots,g_N)$, where the g_j are the functions constructed above with the property that $\bigcap_{j=1}^{N}Z_B(g_j)=\emptyset$. By [11, p. 47], the ideal J' contains an inner function u invertible in B. Thus $c_1\cdots c_N\cdot u\in(c_1g_1,\dots,c_Ng_N)\subseteq I$, where c_i are the functions constructed in the proof of Theorem 2.3. Let $b=c_1\cdots c_N$. Then bu is the desired function. $\square$

In [11, p. 47], the second author has proven that if I is an ideal in H^∞ whose zero set does not meet $M(B)$, then I is generated algebraically by inner functions invertible in B. We do not know, under the weaker assumption of Theorem 2.3 (or Corollary 2.4), if I is generated by functions of the form bu, where u are inner functions invertible in B and where b are finite products of interpolating Blaschke products.

3. Divisibility in Douglas Algebras

In [6], Guillory, Izuchi, and Sarason proved the following result.

THEOREM [6, p. 3]. *Let f be a function in $H^\infty+C$ which does not vanish identically on any nontrivial Gleason part of $H^\infty+C$. If g is a function in $H^\infty+C$ such that every zero of f is a zero of g of at least as high multiplicity, then f divides g in $H^\infty+C$.*

In the same paper they asked if this result could be extended to arbitrary Douglas algebras. In this section we shall give a positive answer to their question. Further results on the divisibility in Douglas algebras are then deduced.

Let $m \in M(H^\infty)$. Then we denote the support set of the unique representing measure of m on the Shilov boundary of H^∞ by supp m. Since supp m is a weak peak set for H^∞ (see [7, p. 207]), the restriction algebra $H^\infty|_{\operatorname{supp} m}$ is a uniformly closed subalgebra of $C(\operatorname{supp} m)$ (see [3, p. 57ff.]). Its maximal ideal space is the set

$$M(H^\infty|_{\operatorname{supp} m}) = \{x \in M(H^\infty) : \operatorname{supp} x \subseteq \operatorname{supp} m\}$$

(see [3, p. 39]). The main tool to achieve our goal will be the following version of Marshall's theorem.

THEOREM (Marshall [10]). *Let $\delta > 0$. Given finitely many inner functions $u_1, \ldots, u_n$ and an α with $0 < \alpha < 1$, there exists β, $0 < \beta < 1$ (depending only on α but not on $u_1, \ldots, u_n$) and an interpolating Blaschke product b_n such that the following hold:*

(a) *If $b_n(z) = 0$, then $\max_{1 \le j \le n} |u_j(z)| \le \beta$.*
(b) *If $\max_{1 \le j \le n} |u_j(z)| \le \alpha$, then $|b_n(z)| \le \delta$.*

For a proof of Marshall's result see Garnett [4, p. 336]. This version can be obtained by applying the proof in Garnett's book to each function separately.

THEOREM 3.1. *Let m be a trivial point and let B be a Blaschke product vanishing at m. Then there exists a nontrivial point $x \in M(H^\infty|_{\operatorname{supp} m})$ such that B vanishes identically on the Gleason part $P(x)$ of x.*

Proof. Since m is a trivial point, we can, according to Hoffman's theorem (see [4, p. 412]), inductively factor B as $B = B_1C_1 = B_1B_2C_2 = B_1B_2 \cdots B_nC_n$ with $B_j(m) = C_j(m) = 0$ $(j = 1, 2, \ldots, n)$ and $C_{j-1} = B_jC_j$ $(j = 2, \ldots, n)$. For the same reason, we can factor B_j as $B_j = B_{1j}B_{2j} \cdots B_{nj}C_{nj}$ with $B_{kj}(m) = C_{kj}(m) = 0$, $k, j \in \{1, 2, \ldots, n\}$. For each n, let

$$S_n(z) = \max\{|B_{kj}(z)| : 1 \le k \le n, 1 \le j \le n\}.$$

Since there are only finitely many B_{kj}, we can find an open set U about m such that $|B_{kj}(y)| < \frac{1}{2}$ for every $y \in U$ and all $k, j \in \{1, 2, \ldots, n\}$. By Marshall's theorem there exists β, $0 < \beta < 1$, and interpolating Blaschke products b_n such that the following hold for $z \in \mathbf{D}$:

(a) if $b_n(z) = 0$ then $|S_n(z)| \le \beta$; and
(b) if $|S_n(z)| \le \frac{1}{2}$ then $|b_n(z)| \le \frac{1}{2}$.

Therefore, for $z \in U \cap \mathbf{D}$ we have $|b_n(z)| \le \frac{1}{2}$. The corona theorem now implies that $|b_n(m)| \le \frac{1}{2}$. If $\bar{b}_n|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$, we would have $1 = m(1) = m(b_n)m(\bar{b}_n) = |m(b_n)|^2 \le \frac{1}{4}$. Thus $\bar{b}_n|_{\operatorname{supp} m} \notin H^\infty|_{\operatorname{supp} m}$.

So b_n is not invertible in $H^\infty|_{\operatorname{supp} m}$. Thus there exists $p_n \in M(H^\infty|_{\operatorname{supp} m})$ with $b_n(p_n) = 0$. Since p_n is in the closure of the zeros of b_n in $\mathbf{D}$ (see [4,

p. 379]), from (a) above we have $|B_{kj}(p_n)| \le \beta$ for $1 \le k \le n$ and $1 \le j \le n$. Thus

$$|B_j(p_n)| \le |B_{1j}(p_n)| \cdots |B_{nj}(p_n)| \le (\beta)^n \quad (j=1,2,\dots,n).$$

Since $\{p_n\}$ is a sequence of nontrivial points, by Theorem 1.2 there exists a cluster point p of p_n whose Gleason part is nontrivial, too. Since $p_n \in M(H^\infty|_{\operatorname{supp} m})$ for every n, we have $p \in M(H^\infty|_{\operatorname{supp} m})$.

Given $\epsilon > 0$, there exist infinitely many p_n such that $|B_j(p_n) - B_j(p)| < \epsilon$. Hence $|B_j(p)| \le \epsilon + |B_j(p_n)| \le \epsilon + (\beta)^n$. Letting $n \to \infty$, we see that $(\beta)^n \to 0$ and $B_j(p) = 0$ for every $j = 1, 2, \dots$. Thus B has a zero of infinite order at p and hence B vanishes identically on $P(p)$ (see [8, p. 79]). □

COROLLARY 3.2. *Let m be a trivial point in $M(H^\infty) \setminus M(L^\infty)$. Then m lies in the closure of the set*

$$V = \{x \in M(H^\infty) : x \textit{ nontrivial and } \operatorname{supp} x \subseteq \operatorname{supp} m\};$$

in other words, m belongs to the closure of $M(H^\infty|_{\operatorname{supp} m}) \cap G$.

Proof. If this were not true, there would exist a neighborhood U of m in $M(H^\infty)$ such that the closure of V does not meet U. By Marshall's theorem [10, p. 20], we may take U to be of the form

$$U = \bigcap_{i=1}^{N} \{x \in M(H^\infty) : |b_i(x)| < \epsilon, \, b_i(m) = 0\}$$

for some Blaschke products b_i $(i = 1, \dots, N)$.

The proof of Theorem 3.1, however, shows that if $b_i(m) = 0$ $(i = 1, \dots, N)$, then there exists a nontrivial point x with $\operatorname{supp} x \subseteq \operatorname{supp} m$ such that all the b_i vanish at x. Thus $x \in U$, which contradicts the choice of U. □

REMARK. The proof shows that every trivial point $x \in M(H^\infty) \setminus M(L^\infty)$ with $\operatorname{supp} x \subseteq \operatorname{supp} m$ belongs to the closure of $M(H^\infty|_{\operatorname{supp} m}) \cap G$. However, we were unable to answer the following question.

QUESTION. Let m be a trivial point in $M(H^\infty)$. Is the set of nontrivial points in $M(H^\infty|_{\operatorname{supp} m})$ dense in $M(H^\infty|_{\operatorname{supp} m})$?

In the final section we shall see that at least the union of the support sets of nontrivial points in $M(H^\infty|_{\operatorname{supp} m})$ is dense in $\operatorname{supp} m$, where m is a trivial point.

Combining the results of Sections 1 and 2 with Theorem 3.1, we now obtain the extension of Guillory, Izuchi, and Sarason's result to arbitrary Douglas algebras.

THEOREM 3.3. *Let B be a Douglas algebra and let u be an inner function which does not vanish identically on any nontrivial Gleason part of $M(B)$. Then u can be factorized in the form $u = bv$, where b is a finite product of* *interpolating Blaschke products and where v is an (inner) function invertible in B.*

Proof. We claim that the function u satisfies the assumptions of Theorem 0.1, that is, that u does not vanish at any trivial point in $M(B)$. To see this, let $u = b_1 v_1$ be the Riesz–Smirnov factorization of u, where b_1 is a Blaschke product and v_1 is a singular inner function. Assume there exists a trivial point $m \in M(B)$ at which u vanishes. Since $\{x \in M(H^\infty) : \operatorname{supp} x \subseteq \operatorname{supp} m\}$ is a subset of $M(B)$, we conclude from Theorem 3.1 that $b_1(m) \neq 0$. Thus $v_1(m) = 0$. By Corollary 3.2, m lies in the closure of the set $E = M(H^\infty|_{\operatorname{supp} m}) \cap G$. Thus there exists a net $x_\alpha \in E$ with $v_1(x_\alpha) \to 0$. Choose for every $n \in \mathbf{N}$ points $x_{\alpha(n)}$ of the net such that $|v_1(x_{\alpha(n)})| \leq 1/n$. By Theorem 1.2 there exists a cluster point x of $x_{\alpha(n)}$ whose Gleason part is nontrivial. Because $M(H^\infty|_{\operatorname{supp} m})$ is closed, $x \in M(H^\infty|_{\operatorname{supp} m}) \cap G = E$. Because v_1 has no zeros in $\mathbf{D}$, v_1 (and hence $u = b_1 v_1$) vanishes identically on the Gleason part $P(x)$. But $P(x) \subseteq M(H^\infty|_{\operatorname{supp} m}) \subseteq M(B)$, which contradicts the hypothesis on u. Thus v_1 does not vanish anywhere on $M(B)$. Hence u does not vanish on a trivial point in $M(B)$. Theorem 0.1 now yields the assertion. □

REMARKS. Theorem 3.3 shows in particular that the (apparently) weaker condition "u does not vanish identically on any nontrivial Gleason part in $M(B)$" is in fact equivalent to the hypothesis "u does not vanish on any trivial point in $M(B)$," provided that u is an inner function. This answers a question in [5]. It is worth noting that this does not hold for outer functions, as the following example shows.

EXAMPLE. For a Douglas algebra B, let $M_1(B) = \{m \in M(B) : m(z) = 1\}$ be the fiber of $M(B)$ over the point $z = 1$. It is easy to see that $A = \{f \in L^\infty : f|_{M_1(L^\infty)} \in H^\infty|_{M_1(L^\infty)}\}$ is a Douglas algebra with maximal ideal space $M(A) = M(L^\infty) \cup M_1(H^\infty)$ (see [4, §9]). The outer function $f(z) = 1 + z$ does not vanish on any nontrivial Gleason part in $M(A)$, but of course f vanishes on many trivial parts in $M(A)$.

We are now able to investigate the divisibility structure of Douglas algebras, thus answering questions of Guillory, Izuchi, and Sarason [6].

THEOREM 3.4. *Let u be a unimodular function in the Douglas algebra B. Assume that u does not vanish identically on any nontrivial Gleason part in $M(B)$. Let g be a function in B satisfying one of the following conditions:*

(a) *Every zero of u is a zero of g of at least as high multiplicity.*
(b) $|g| \leq |u|$ *on $M(B)$.*

Then g is divisible by u in B.

Proof. We shall show that the hypothesis implies that u does not vanish on any trivial part in $M(B)$. The assertion then follows from Gorkin's result [4].

Of course u does not vanish on the Shilov boundary $M(L^\infty)$ of B, because u is invertible in L^∞. So let $m \in M(B)\setminus M(L^\infty)$. By the Chang-Marshall theorem (see [4, §9]) there exists a function $f \in H^\infty$ such that $f|_{\operatorname{supp} m} = u|_{\operatorname{supp} m}$. In particular, $f(x) = u(x)$ for every x with $\operatorname{supp} x \subseteq \operatorname{supp} m$. Let $f = vF$ be the inner-outer factorization of f. Note that F does not vanish on $\operatorname{supp} m$. The proof of Theorem 3.3 now shows that f cannot vanish on any trivial point x with $\operatorname{supp} x \subseteq \operatorname{supp} m$. Thus u does not vanish on any trivial point in $M(B)$. □

REMARK. If u is a finite product of interpolating Blaschke products, we obtain [1, Lemma 3, p. 91].

COROLLARY 3.5. *Let $u \in B$ be as in Theorem 3.4. If g is a function in B with $Z_B(u) \subseteq Z_B(g)$, then $g^N \bar{u} \in B$ for some $N \in \mathbf{N}$.*

Proof. We claim that

(1) $$\sup_{m \in M(B) \cap G} \operatorname{ord}(u, m) = N < \infty.$$

Otherwise there would exist for every $n \in \mathbf{N}$ a point $x_n \in M(B) \cap G$ with $\operatorname{ord}(u, x_n) \geq n$. Then we can write $u = u_1^{(n)} \cdots u_n^{(n)}$ with $u_j^{(n)}(x_n) = 0$ for $j = 1, \ldots, n$. let L_{x_n} denote the Hoffman map from $\mathbf{D}$ onto $P(x_n)$. Using the Chang-Marshall theorem and the fact that $x_n \in M(B) \cap G$, we see that $u_j^{(n)} \circ L_{x_n}$ is an analytic function on $\mathbf{D}$ for all j and n. Now each $u_j^{(n)} \circ L_{x_n}$ has norm at most 1 and vanishes at the origin, so Schwarz's lemma implies that $|u_j^{(n)} \circ L_{x_n}(z)| \leq |z|$. Thus $|u \circ L_{x_n}(z)| \leq |z|^n$ for all n. By Theorem 1.2 there exists a nontrivial cluster point x of x_n. Since by Hoffman's theorem [8, p. 93] u has a continuous extension to $M(H^\infty)$, a subnet of $u \circ L_{x_n}$ converges therefore to $u \circ L_x$. Hence $u \circ L_x$ is identically zero on $\mathbf{D}$; in other words, u vanishes identically on $P(x)$. Note that $P(x) \subseteq M(B)$. This contradicts the assumption on u, so we have (1).

Now it is easy to see that every zero of u is a zero of g^N with at least as high multiplicity. Theorem 3.4 finally yields the assertion of the theorem. □

4. Support Sets in $M(H^\infty)$

In [2, p. 35], Budde noticed that there exist trivial points in $M(H^\infty)$ whose support sets contain the support set of a nontrivial point. In the following we show that the support set of any point which does not belong to the Shilov boundary of H^∞ strictly contains the support set of a nontrivial point. Convention: The symbol "$\subset$" denotes strict inclusion.

PROPOSITION 4.1. *The maximal ideal space of a Douglas algebra is uniquely determined by its set of nontrivial points.*

Proof. Suppose that B_1 and B_2 are two different Douglas algebras, with $M(B_1) \cap G = M(B_2) \cap G$. By the Chang-Marshall theorem, there is an interpolating Blaschke product b which is invertible in one of these algebras but not in the other, say $\bar{b}\in B_1\backslash B_2$. Since $\bar{b}\notin B_2$, there exists $m\in M(B_2)$ with $m(b)=0$. By Hoffman's theorem [8, p. 88] and our assumption, $m\in M(B_2)\cap G=M(B_1)\cap G$. But $\bar{b}\in B_1$, so $|m(b)|=1$, a contradiction. □

THEOREM 4.2. *Let $m\in M(H^\infty)\backslash M(L^\infty)$. Then there exists a nontrivial point x with* $\operatorname{supp} x\subset \operatorname{supp} m$.

Proof. Let B be the Douglas algebra

$$B=H^\infty_{\operatorname{supp} m}=\{f\in L^\infty : f|_{\operatorname{supp} m}\in H^\infty|_{\operatorname{supp} m}\}.$$

Using Newman's result [7, p. 179], choose an inner function u with $u(m)=0$. Then $\bar{u}\notin B$. A theorem of Sundberg [13] tells us that the Douglas algebra $B_1=[B,\bar{u}]$ is strictly contained in L^∞. Thus, by Proposition 4.1, there exists a nontrivial point $x\in M(B_1)$. Using the Chang–Marshall theorem we get an interpolating Blaschke product b with $\bar{b}\in B_1\backslash B$. Hence $|b(x)|=1$. Since the representing measure for x is a probability measure, $|b(x)|=1$ implies that b is constant on $\operatorname{supp} x$. On the other hand $\operatorname{supp} x\subseteq \operatorname{supp} m$. But $|m(b)|<1$, so $\operatorname{supp} x\subset \operatorname{supp} m$. □

COROLLARY 4.3. *There exist no minimal support sets containing more than one point.*

COROLLARY 4.4. *The support set of a trivial point $m\in M(H^\infty)\backslash M(L^\infty)$ strictly contains the support set of another trivial point of $M(H^\infty)\backslash M(L^\infty)$.*

Proof. Choose, according to Theorem 4.2, a nontrivial point x with $\operatorname{supp} x\subset \operatorname{supp} m$. By Budde [2, p. 11] there exists a trivial point $y\in \overline{P(x)}$, and thus $\operatorname{supp} y\subseteq \operatorname{supp} x$ (see [2, p. 25]).

THEOREM 4.5. *Let $m\in M(H^\infty)\backslash M(L^\infty)$ be a trivial point. Then the set*

$$E=\bigcup\{\operatorname{supp} x : x \textit{ nontrivial}, \operatorname{supp} x\subseteq \operatorname{supp} m\}$$

is dense in $\operatorname{supp} m$.

Proof. Assume that $\bar{E}$ is a proper subset of $\operatorname{supp} m$. Consider the Douglas algebras $B_1=H^\infty_{\bar{E}}=\{f\in L^\infty : f|_{\bar{E}}\in H^\infty|_{\bar{E}}\}$ and $B_2=H^\infty_{\operatorname{supp} m}$. Since

$$m\notin M(B_1)=\{x\in M(H^\infty): \operatorname{supp} x\subseteq \bar{E}\}\cup M(L^\infty)$$

[3, p. 39], we have $B_2\subset B_1$ and hence $M(B_1)\subset M(B_2)$. But if $x\in M(B_2)$ is a nontrivial point, then $\operatorname{supp} x\subseteq E$. Thus $x\in M(B_1)$, so we conclude that $M(B_1)\cap G=M(B_2)\cap G$. By Proposition 4.1, $B_1=B_2$, which is a contradiction. □

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Pamela Gorkin
Department of Mathematics
Bucknell University
Lewisburg, PA 17837

Raymond Mortini
Mathematisches Institut I
Universität Karlsruhe
Englestr. 2
D-7500 Karlsruhe 1
Germany