Multipliers and Cyclic Vectors in the Bloch Space

LEON BROWN & A. L. SHIELDS*

I. Introduction

In this paper we study the cyclic vectors in $\mathcal{B}$, the Bloch space with the weak* topology, and in $\mathcal{B}_0$, the "little" Bloch space with the norm topology. A result is obtained which implies that every outer function in $\mathcal{B}(\mathcal{B}_0)$ is cyclic. We also obtain a simple characterization of multipliers in $\mathcal{B}$ and $\mathcal{B}_0$.

The Bloch space $\mathcal{B}$ in the open unit disc D in the complex plane is the space of all those analytic functions f such that $(1-|z|^2)f'(z)$ is bounded in D. We norm $\mathcal{B}$ as follows:

$$\|f\| = |f(0)| + \sup\{(1-|z|^2)|f'(z)| : z \in D\}. \tag{1}$$

With this norm $\mathcal{B}$ is a Banach space and $\mathcal{B}_0$ a closed subspace. Here $\mathcal{B}_0$, sometimes called the "little" Bloch space, denotes the set of those f in $\mathcal{B}$ for which $(1-|z|^2)f'(z) \to 0$ as $|z| \uparrow 1$. For information about $\mathcal{B}$ and $\mathcal{B}_0$, see [1] and [2].

The space $\mathcal{B}$ with the norm (1) is isometric to the second dual $\mathcal{B}_0^{**}$ (see [9]). Furthermore, the polynomials are norm dense in $\mathcal{B}_0$ and in $\mathcal{B}_0^*$, and are weak* dense in $\mathcal{B}$. Note that $\mathcal{B}$ is not norm separable.

We have a growth estimate for Bloch functions (see, e.g., [3, Eq. (4)]):

$$|f(z)| \le \left\{1 + \log \frac{1}{1-|z|}\right\} \|f\|. \tag{2}$$

Thus $\mathcal{B}$ is contained in L_a^p (the analytic L^p functions in D) for $p < \infty$. For the Hardy spaces we have $H^\infty \subset \mathcal{B}$, but H^p is not contained in $\mathcal{B}$ for any $p < \infty$; also, $\mathcal{B}$ is not contained in the Nevanlinna class.

The second section of this paper deals with multipliers in $\mathcal{B}$. A complex-valued function ϕ in D is called a *multiplier on* $\mathcal{B}$ if $\phi\mathcal{B} \subset \mathcal{B}$. By M_ϕ we denote the operator of multiplication by ϕ: $M_\phi f = \phi f$ $(f \in \mathcal{B})$. The set of all multipliers will be denoted by $M(\mathcal{B})$. An application of the closed graph theorem shows that M_ϕ is a bounded linear transformation on $\mathcal{B}$. Hence it has a finite norm $\|M_\phi\|$.

Received February 21, 1990. Revision received September 24, 1990.
*Deceased. The research of this author was supported in part by the National Science Foundation.
Michigan Math. J. 38 (1991).

In the third section we study (weak*) cyclic vectors for the space $\mathcal{B}$. These are the function f in $\mathcal{B}$ whose polynomial multiples are weak* dense in $\mathcal{B}$ (i.e., they are cyclic vectors in the weak* topology for the operator of multiplication by z on $\mathcal{B}$). Note that a duality argument yields the fact that if f is in $\mathcal{B}_0$, then f is (norm) cyclic in $\mathcal{B}_0$ if and only if it is weak* cyclic in $\mathcal{B}$. When we refer to cyclic vectors in $\mathcal{B}$, the weak* topology is always understood.

In [3], Anderson, Fernández, and Shields show that if f is outer and bounded, then f is cyclic. Theorem 3 shows that the outer requirement alone is enough to ensure that a Bloch space function is cyclic.

II. Multipliers in the Bloch Spaces

THEOREM 1. *The following are equivalent:*

(a) ϕ *is a multiplier on* $\mathcal{B}$;

(b) ϕ *is a multiplier on* $\mathcal{B}_0$;

(c) $\varphi \in H^\infty$ *and*

$$|\phi'(z)| = O\left(\frac{1}{(1-|z|)\log(1/(1-|z|))}\right).$$

Proof. (a) ⇒ (c) We thank the referee for suggesting the following proof; our original proof was more complicated.

Suppose that φ is a multiplier of $\mathcal{B}$. Then by [5, Lemma 11] $\varphi \in H^\infty$ and $|\varphi(z)| \le \|M_\varphi\|$. Let $z_0 \in D\backslash\{0\}$ be arbitrary. We complete the proof of the implication (a) ⇒ (c) by showing

$$(*) \qquad |\varphi'(z_0)| \le \frac{4\|M_\varphi\|}{(1-|z_0|)\log(1/(1-|z_0|))}.$$

Let $z_0 = re^{i\theta}$. Since $L(z) = \log(1-e^{-i\theta}z)$ is in $\mathcal{B}$ and φ is a multiplier of $\mathcal{B}$, we have

$$\|\varphi L\| \le \|M_\varphi\|\|L\| = 2\|M_\varphi\|.$$

It follows that

$$\begin{aligned}(1-|z|^2)|\varphi'(z)L(z)| &\le 2\|M_\varphi\| + (1-|z|^2)|\varphi(z)L'(z)| \\ &\le 2\|M_\varphi\| + 2\|\varphi\|_\infty \le 4\|M_\varphi\|.\end{aligned}$$

Hence

$$|\varphi'(z)| \le \frac{4\|M_\varphi\|}{(1-|z|^2)|\log(1-e^{-i\theta}z)|}.$$

To see that (*) holds set $z = z_0$ and replace $1-|z_0|^2$ with the smaller value $1-|z_0|$.

(b) ⇒ (c) Given $z_0 = re^{i\theta} \ne 0$ and α in $(0,1)$, let

$$L_\alpha(z) = \left(\frac{1}{z}\log(1-e^{-i\theta}z)\right)^\alpha.$$

A calculation shows that $L_\alpha \in \mathcal{B}_0$ and $\sup_\alpha \|L_\alpha\| = k < +\infty$. In a manner similar to the proof of (a) ⇒ (c), one obtains that if φ is a multiplier of $\mathcal{B}_0$ then, for each α,

$$|\varphi'(z_0)| \leq \frac{2k\|M_\varphi\|}{(1-|z_0|^2)|(\log(1-|z_0|)/z_0)^\alpha|}.$$

Hence,

$$|\varphi'(z_0)| \leq \frac{2k\|M_\varphi\|}{(1-|z_0|)\log(1/(1-|z_0|))}.$$

(c) ⇒ (a) Assume $f \in \mathcal{B}$, $\varphi \in H^\infty$, and

$$|\varphi'(z)| = O\left(\frac{1}{(1-|z|)\log(1/(1-|z|))}\right).$$

Note that $(\varphi f)' = \varphi f' + \varphi' f$ and we have

$$|(\varphi f')(z)| \leq \|\varphi\|_\infty |f'(z)| \leq C\frac{1}{1-|z|}.$$

For $|z| \geq \frac{1}{2}$ we have $|f(z)| \leq C\log(1/(1-|z|))$, which implies $|(\varphi' f)(z)| \leq C/(1-|z|)$. Thus $\varphi f \in \mathcal{B}$.

(c) ⇒ (b) One can easily show that if $f \in \mathcal{B}_0$ then $f(z) = o(\log(1/(1-r)))$. Using this fact, the proof is similar to the proof of (c) ⇒ (a). □

LEMMA 1 (see [3, Lemma 1] or [4, Prop. 2]).

(a) *If* $\{f_n\} \subset B$ *then* $f_n \to 0$ *weak* if and only if* $f_n(z) \to 0$ *for all* z *in* D, *and* $\sup\|f_n\| < \infty$.

(b) *If* $\{f_\alpha\} \subset \mathcal{B}$, $0 \leq \alpha < 1$, *then* $\lim f_\alpha \to 0$ (*as* $\alpha \uparrow 1$) *weak* if and only if* $\lim f_\alpha(z) \to 0$ $(\alpha \uparrow 1)$ *for all* z *in* D, *and* $\limsup \|f_\alpha\| < \infty$.

REMARK. Both (a) and (b) remain valid if $\mathcal{B}$ is replaced by $\mathcal{B}_0$ and weak* is replaced by weakly.

Let $[f]$ denote the weak* (norm) closure in $\mathcal{B}(\mathcal{B}_0)$ of polynomial multiples of f. Set $\|f\|_0 = \sup_z (1-|z|)|f(z)|$ if it is finite.

LEMMA 2. *If* $g \in M(\mathcal{B}_0) = M(\mathcal{B})$ *then*

(a) $f \in \mathcal{B}_0$ *implies* $gf \in [f]$ (*norm topology*);

(b) $f \in \mathcal{B}$ *implies* $gf \in [f]$ (*weak* topology*).

Proof. (a) We show that if $g \in M(\mathcal{B}_0)$ and $f \in \mathcal{B}_0$ then $g_t(f) \in [f]$, where $g_t(z) = g(tz)$. One easily shows that if P_n is the partial sum of the power series for g_t then $P_n f \to g_t f$ (norm). Thus we have $g_t f$ is in the closure of polynomial multiples of f, which implies $g_t f \in [f]$. For $z \in D$, $g_t(z)f(z) \to g(z)f(z)$. Furthermore,

$$\begin{aligned}\|g_t f\| &= \|g_t f' + f(g_t)'\|_0 + |g(0)f(0)| \\ &\leq \|g\|_\infty \|f\| + \|f(g_t)'\|_0 + |f(0)g(0)|.\end{aligned}$$

If $|z|=r$ and $|z|\geq\frac{1}{2}$, then

$$\begin{aligned}|f(z)(g_t(z))'(1-r^2)| &\leq C\log\left(\frac{1}{1-r}\right)t|g'(tz)|(1-r)\\ &\leq C\log\left(\frac{1}{1-r}\right)\max_{|y|=r}|g'(ty)|(1-r)\\ &\leq C\log\left(\frac{1}{1-r}\right)\max_{|y|=r}|g'(y)|(1-r)\\ &\leq C \quad \text{(by Theorem 1).}\end{aligned}$$

Thus, by Lemma 1, $g_t f\to gf$ weakly and the proof is complete.

(b) The proof is essentially the same for $f\in\mathcal{B}$; we omit the details. □

III. Cyclic Vectors in the Bloch Spaces

LEMMA 3. *If $f, g\in\mathcal{B}$ then*

$$\|(f-f_t)g_t'\|_0\leq\|f'\|_0\|g'\|_0.$$

Here $g_t(z)=g(tz)$ for all $z\in D$.

Proof. Since $f(z)-f(tz)=\int_{tz}^{z}f'$ (integrating along the radius), we have

$$\begin{aligned}|f(z)-f(tz)| &\leq\|f'\|_0\int_{rt}^{r}(1-\tau)^{-1}\,d\tau\\ &=\|f'\|_0\log\frac{1-rt}{1-r}.\end{aligned}$$

Also, if $x=(1-r)/(1-tr)$ then

$$\log\left(\frac{1-rt}{1-r}\right)\left(\frac{1-r}{1-tr}\right)=x\log\frac{1}{x}$$

is bounded by $(1/e)\log e$ for $0\leq t\leq 1$ and $0\leq r<1$.

We proceed with the proof as follows: If $0\leq t\leq 1$ and $0\leq r<1$ then

$$\begin{aligned}|(f(z)-f(tz))tg'(tz)|(1-r) &\leq\|f'\|_0\log\left(\frac{1-rt}{1-r}\right)|g'(tz)|(1-tr)\cdot\frac{1-r}{1-tr}\\ &\leq\|f'\|_0\|g'\|_0\log\left(\frac{1-rt}{1-r}\right)\frac{1-r}{1-tr}\\ &<\|f'\|_0\|g'\|_0.\end{aligned}$$ □

LEMMA 4. *If $f\in H^\infty\subset\mathcal{B}$, $g\in\mathcal{B}$, and $fg\in\mathcal{B}$, then $fg\in[g]$.*

Proof. For $z\in D$, $(f_tg)(z)\to(fg)(z)$. We have $(f_tg)'=f_tg'+g(f_t)'$, and

$$\|(f_tg')\|_0\leq\|f_t\|_\infty\|g'\|_0\leq\|f\|_\infty\|g'\|_0;$$

$$\begin{aligned}\|(f_t)'g\|_0 &\leq\|(g-g_t)f_t'\|_0+\|g_t(f_t)'\|_0\\ &<\|g'\|_0\|f'\|_0+\|gf'\|_0;\end{aligned}$$

$$\begin{aligned}\|gf'\|_0 &= \|(fg)'-(fg')\|_0\\ &\leq \|(fg)'\|_0+\|f\|_\infty\|g'\|_0.\end{aligned}$$

Thus

$$\|(f_t g)\| \leq |f(0)g(0)|+\|f'\|_0\|g'\|_0+\|(fg)'\|_0+2\|f\|_\infty\|g'\|_0$$

and $f_t g \to fg$ weak*, which completes the proof. □

REMARK. If g and fg are in $\mathcal{B}_0$ then $f_t g \to fg$ weakly and we have $fg \in [g]$ (norm).

THEOREM 2. *If $f, g \in \mathcal{B}$, $|f(z)| \geq |g(z)|$ in D, and g is cyclic, then f is cyclic.*

Proof. We have $g/f \in H^\infty$ and $(g/f)f = g \in [f]$, which implies that f is cyclic. □

PROPOSITION 1 (see [3, Cor. to Prop. 1]).

(a) *If f is cyclic in H^∞ (with the weak* topology) then f is cyclic in $\mathcal{B}$. Note that f is cyclic in H^∞ if and only if f is outer* (*see* [8, Thm. 5.5]).

(b) *If f is cyclic in $\mathcal{B}$ then f is cyclic in L^2_a.*

REMARKS. (a) This result also follows from the fact that the identity i: $H^\infty \to \mathcal{B}$ is weak*-weak* continuous and $i: \mathcal{B} \to L^2_a$ is also weak*-weak* continuous. We omit the details.

(b) In [3], Anderson, Fernández, and Shields exhibit singular inner functions that are cyclic in $\mathcal{B}(\mathcal{B}_0)$. One has a complete description of the singular inner functions that are cyclic in L^2_a from Korenblum [6] and Roberts [7] (see also Shapiro's notes [10]). Namely, the singular measure must put no mass on any closed set of ∂D that is "thin" in the sense of Beurling, Carleson, and Hayman (see [4, p. 274] for these references and for the definition of the sets). It is still an open question as to whether this condition is sufficient for a singular inner function to be cyclic in $\mathcal{B}$.

We conclude this paper with the following theorem.

THEOREM 3. *If f is an outer function in $\mathcal{B}$, then f is cyclic in $\mathcal{B}$.*

Proof. Let

$$g(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\log|g^*(e^{it})|\,dt\right\},$$

where $f^*(e^{it}) = f^*(t) = \lim_{r\uparrow 1} f(re^{it})$ a.e. and

$$|g^*(e^{it})| = |g^*(t)| = \begin{cases} 1 & \text{if } |f^*(t)| \geq 1, \\ |f^*(t)| & \text{if } |f^*(t)| \leq 1. \end{cases}$$

We see that $g \in H^\infty \subset \mathcal{B}$ is an outer function and therefore cyclic in $\mathcal{B}$. Furthermore,

$$
\begin{aligned}
|g(z)| &= \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} P_z(t)\log|g^*(t)|\,dt\right\} \\
&\le \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} P_z(t)\log|f^*(t)|\,dt\right\} = |f(z)|, \quad z \in D,
\end{aligned}
$$

where P_z is the Poisson kernel for the point z.

The result follows from Theorem 2. □

References

1. J. M. Anderson, *Bloch functions: the basic theory,* Operators and Function Theory (S. C. Power, ed.), pp. 1–17, Reidel, Dordrecht, 1985.
2. J. M. Anderson, J. Clunie, and Ch. Pommerenke, *On Bloch functions and normal functions,* J. Reine Angew. Math. 270 (1974), 12–37.
3. J. M. Anderson, J. L. Fernández, and A. L. Shields, *Inner functions and cyclic vectors in the Bloch space,* preprint.
4. L. Brown and A. L. Shields, *Cyclic vectors in the Dirichlet space,* Trans. Amer. Math. Soc. 285 (1984), 269–304.
5. P. L. Duren, B. W. Romberg, and A. L. Shields, *Linear functionals on H^p spaces with* $0<p<1$, J. Reine Angew. Math. 238 (1969), 32–60.
6. B. I. Korenblum, *Cyclic elements in some spaces of analytic functions,* Bull. Amer. Math. Soc. (N.S.) 5 (1981), 317–318.
7. J. W. Roberts, *Cyclic inner functions in the Bergman spaces and weak outer functions in H^p,* $0<p<1$, Illinois J. Math. 29 (1985), 25–38.
8. L. A. Rubel and A. L. Shields, *The space of bounded analytic functions on a region,* Ann. Inst. Fourier (Grenoble) 16 (1966), 235–277.
9. ———, *The second duals of certain spaces of analytic functions,* J. Austral. Math. Soc. Ser. A 11 (1970), 276–280.
10. J. Shapiro, *Cyclic inner functions in Bergman spaces,* preprint, 1980.

Leon Brown
Department of Mathematics
Wayne State University
Detroit, MI 48202

Allen L. Shields
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1003