Duality of Bloch Spaces and Norm Convergence of Taylor Series

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1. Introduction

Let X be a Banach space of analytic functions on the open unit disk $\mathbf{D}$ in the complex plane. We always assume that the polynomials are dense in X. Given f in X, let

$$f(z)=\sum_{k=0}^{\infty} a_k z^k$$

be the Taylor expansion of f. For any integer $n \geq 1$, let

$$f_n(z)=\sum_{k=0}^{n} a_k z^k$$

be the nth Taylor polynomial of f. It is natural to ask the following question: When does $\{f_n\}$ converge to f in the norm topology of X? We will consider the question for the following spaces in this paper: H^p spaces and VMOA; weighted Bergman spaces; Besov spaces; and the little Bloch space. We give the definitions of these spaces first.

For $1 \leq p < +\infty$, the Hardy space H^p consists of analytic functions f on the open unit disk $\mathbf{D}$ such that

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < +\infty.$$

It is well known that each function in H^p has boundary values almost everywhere on the unit circle. We will not distinguish between functions in H^p and their boundary values. Note that the norm of a function in H^p is precisely the (normalized) Lebesgue L^p norm of its boundary value function on the circle. VMOA is the predual of H^1 under the complex integral pairing

$$\langle f, g\rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\overline{g(\theta)} \, d\theta.$$

Again we will not distinguish between functions in VMOA (of the disk) and their boundary values. A function on the unit circle is in VMOA if and only if it is the Szegö projection of a continuous function (see VI.5 of [3] or Theorem 8.4.7 of [6]). For $1 \leq p < +\infty$ and $\alpha > -1$, the weighted Bergman space

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$L_a^p(dA_\alpha)$ is the subspace of $L^p(\mathbf{D}, dA_\alpha)$ consisting of analytic functions, where

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$$

and $dA(z)$ is the area measure on $\mathbf{D}$ normalized so that the area of $\mathbf{D}$ is equal to 1. $L_a^p(dA_\alpha)$ is a closed subspace of $L^p(\mathbf{D}, dA_\alpha)$, and hence a Banach space itself. It is easy to see that the polynomials are dense in each $L_a^p(dA_\alpha)$. We will write $L_a^p(\mathbf{D})$ for the unweighted Bergman spaces corresponding to $\alpha = 0$. The Bloch space of $\mathbf{D}$, denoted $\mathcal{B}$, is the space of analytic functions $f(z)$ on $\mathbf{D}$ such that $(1-|z|^2)f'(z)$ is bounded on $\mathbf{D}$. $\mathcal{B}$ is a Banach space with the norm

$$\|f\| = |f(0)| + \sup\{(1-|z|^2)|f'(z)| : z \in \mathbf{D}\}.$$

The little Bloch space of $\mathbf{D}$, denoted $\mathcal{B}_0$, is the closure of the polynomials in $\mathcal{B}$. It is well known (see [2] for example) that an analytic function $f(z)$ on $\mathbf{D}$ is in $\mathcal{B}_0$ if and only if $(1-|z|^2)f'(z)$ belongs to $\mathbf{C}_0(\mathbf{D})$, the space of complex continuous functions on the closed unit disk which vanish on the unit circle $\partial\mathbf{D}$. Finally, for $1 \le p < +\infty$ the analytic Besov space B_p consists of analytic functions $f(z)$ on $\mathbf{D}$ such that $(1-|z|^2)^2 f''(z)$ is in $L^p(\mathbf{D}, d\lambda)$, where

$$d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}$$

is the (unique up to constant multiple) Möbius invariant measure on $\mathbf{D}$. See [1] or 5.3 of [6].

The problem of norm convergence of Taylor series for functions in the H^p spaces is classical: The Taylor series for each function in H^p converges in norm if and only if $p > 1$. The result is equivalent to the boundedness of the Szegö projection on L^p of the circle when $p > 1$ and the unboundedness of the Szegö projection on L^1 of the circle (see [3, pp. 108–109]). This result will easily imply that the Taylor series of every function in the weighted Bergman space $L_a^p(dA_\alpha)$ converges in norm if $p > 1$. The corresponding result for analytic Besov spaces (when $p > 1$) follows similarly.

The (Hardy space) proof of the equivalence of the boundedness of the Szegö projection on L^p and the norm convergence of Taylor series does not work in the Bergman space setting. This is mainly because a function on the circle consists of just an analytic part and an anti-analytic part, while a function on the disk is much more complicated. Although we know that the Bergman projection is unbounded on L^1 of the disk (with the area measure), this does not imply that Taylor series of functions in the Bergman space $L_a^1(\mathbf{D})$ do not converge in norm. Furthermore, it is well known (see 4.2 of [6]) that (unlike the Hardy space case) there are many bounded projections from $L^1(\mathbf{D}, dA)$ onto $L_a^1(\mathbf{D})$. Thus one wonders if the results in the Bergman space setting are a little different.

We prove in Section 3 that there are functions in the (unweighted) Bergman space $L_a^1(\mathbf{D})$ whose Taylor series do not converge in norm. It follows from a well-known duality between $L_a^1(\mathbf{D})$ and the little Bloch space that there are functions in the little Bloch space whose Taylor series do not converge in norm. In Section 4 we prove a duality theorem between the Bloch spaces and the weighted Bergman space $L_a^1(dA_\alpha)$. This will imply that there exist functions in the weighted Bergman space $L_a^1(\mathbf{D}, dA_\alpha)$ whose Taylor series do not converge in norm. The classical result for H^p spaces is included in Section 2 (for completeness), along with some of its consequences.

2. Norm Convergence of Taylor Series in Hardy Spaces

In this section we prove the classical result that norm convergence of Taylor series in H^p is equivalent to the boundedness of the Szegö projection. First we prove a general result about norm convergence of Taylor series.

PROPOSITION 1. *Suppose X is a Banach space of analytic functions in $\mathbf{D}$ with the property that the polynomials are dense in X. Then $\|f_n-f\|\to 0$ $(n\to+\infty)$ for each $f\in X$ if and only if there is a positive constant $C>0$ such that $\|S_n\|\le C$ for all $n\ge 1$, where S_n is the operator $S_n f=f_n$ defined on X.*

Proof. If the Taylor series of each function in X converges in norm then, for each $f\in X$, $S_n f\to f$ in X as $n\to+\infty$. By the uniform boundedness principle, there exists a constant $C>0$ such that $\|S_n\|\le C$ for all $n\ge 1$.

Conversely, if $\|S_n\|\le C$ for some constant $C>0$ and all $n\ge 1$, we show that the Taylor series of each function in X converges in norm. Fix $f\in X$ and $\epsilon>0$; since the polynomials are dense in X, we can find a polynomial $p(z)$ such that $\|f-p\|<\epsilon$. It follows that

$$\begin{aligned}\|S_n f-f\|&\le\|S_n f-S_n p\|+\|S_n p-p\|+\|p-f\|\\&\le C\|f-p\|+\|S_n p-p\|+\|p-f\|\\&<(C+1)\epsilon+\|S_n p-p\|.\end{aligned}$$

Since $S_n p=p$ for n large and ϵ is arbitrary, we see that $\|S_n f-f\|\to 0$ $(n\to+\infty)$, completing the proof of the Proposition. □

Recall that the Szegö projection is the orthogonal projection P from $L^2(\partial\mathbf{D}, d\theta)$ onto the Hardy space H^2. If we think of H^2 as defined on the open unit disk $\mathbf{D}$, then P is an integral operator:

$$Pf(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(\theta)\,d\theta}{1-ze^{-i\theta}},\quad z\in\mathbf{D}.$$

It is clear that the domain of P can be extended to be $L^1(\partial\mathbf{D}, d\theta)$. A classical theorem of Riesz states that P maps $L^p(\partial\mathbf{D}, d\theta)$ boundedly onto H^p if and only if $1<p<+\infty$. Next we show that the problem of norm convergence of Taylor series for functions in the Hardy spaces is equivalent to the Riesz theorem. Recall that S_n is the operator which maps an analytic function to its nth Taylor polynomial.

THEOREM 2. *Let $\|S_n\|_p$ be the norm of S_n on H^p. Then $\{\|S_n\|_p\}$ is a bounded sequence if and only if the Szegö projection P is bounded on $L^p(\partial\mathbf{D}, d\theta)$.*

Proof. Since the trigonometric polynomials are dense in $L^p(\partial\mathbf{D}, d\theta)$ for each $p \geq 1$, we have

$$\|P\|_p = \sup\{\|Pf\|_p : \|f\|_p = 1, f \text{ is a trigonometric polynomial}\}.$$

Similarly, since the polynomials are dense in H^p for each $p \geq 1$, we have

$$\|S_n\|_p = \sup\{\|S_n f\|_p : \|f\|_p = 1, f \text{ is a polynomial}\}.$$

Given a trigonometric polynomial

$$q(t) = \sum_{k=-n}^{n} a_k e^{ikt},$$

it is easy to see that the function $h(t) = e^{int} q(t)$ is analytic, $\|h\|_p = \|q\|_p$, and

$$S_n h(t) = \overline{e^{-int} P\bar{q}(t)}.$$

It follows that $\|S_n h\|_p = \|P\bar{q}\|_p$ for any trigonometric polynomial q of degree n. Now if $\|S_n\|_p \leq C$ for some constant $C > 0$ and all $n \geq 1$, then $\|P\bar{q}\|_p \leq C\|q\|_p$ for all trigonometric polynomials q and hence P is bounded on $L^p(\partial\mathbf{D}, d\theta)$. On the other hand, if P is bounded on $L^p(\partial\mathbf{D}, d\theta)$, then

$$\|S_n h\|_p \leq \|P\| \|q\|_p = \|P\| \|h\|_p.$$

When q runs over all trigonometric polynomials, h runs over all (analytic) polynomials; thus we see that $\|S_n\|_p \leq \|P\|$ for all $n \geq 1$. This completes the proof of Theorem 2. □

We derive some corollaries of the above result. First recall that the Riesz theorem states that the Szegö projection is bounded on $L^p(\partial\mathbf{D}, d\theta)$ if $1 < p < +\infty$ (see, e.g., [3, III.3]). On the other hand, it is easy to see that the Szegö projection is unbounded on $L^1(\partial\mathbf{D}, d\theta)$ (see, e.g., [4, p. 150]). Thus we have the following corollary.

COROLLARY 3. *The Taylor series of every function in H^p converges in norm if and only if $1 < p < +\infty$.*

Given $1 < p < +\infty$, Proposition 1 and Corollary 3 imply that there is a constant $C > 0$ such that

$$\int_0^{2\pi} |f_n(e^{it})|^p \, dt \leq C \int_0^{2\pi} |f(e^{it})|^p \, dt$$

for all $f \in H^p$ and $n \geq 1$, where f_n is the nth Taylor polynomial of f. It follows from the use of polar coordinates that for any function f in the weighted Bergman space $L_a^p(dA_\alpha)$,

$$\begin{aligned}\|S_n f\|_{p,\alpha}^p &= \int_{\mathbf{D}} |f_n(z)|^p \, dA_\alpha(z) \\ &= \frac{\alpha+1}{\pi} \int_0^1 r(1-r^2)^\alpha \, dr \int_0^{2\pi} |f_n(re^{it})|^p \, dt \\ &\leq \frac{C(\alpha+1)}{\pi} \int_0^1 r(1-r^2)^\alpha \, dr \int_0^{2\pi} |f(re^{it})|^p \, dt \\ &= C\|f\|_{p,\alpha}^p.\end{aligned}$$

Thus $\{S_n\}$ is a bounded sequence of operators on $L_a^p(dA_\alpha)$ for $1<p<+\infty$. By Proposition 1, we obtain the next corollary.

COROLLARY 4. *If $1<p<+\infty$ and $\alpha>-1$, then the Taylor series of every function in $L_a^p(dA_\alpha)$ converges in norm.*

The question of norm convergence of functions in the Bergman space $L_a^1(dA_\alpha)$ will be answered in Section 4 as a consequence of a duality theorem between the Bloch spaces and $L_a^1(dA_\alpha)$. We first apply this idea to settle the question of norm convergence for functions in the space VMOA. Recall that the dual of VMOA is H^1 under the complex integral pairing

$$\langle f, g\rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta.$$

The rotation invariance of $\partial\mathbf{D}$ and $d\theta$ easily implies that each operator S_n is self-adjoint under the above integral pairing. It follows that the norm of S_n as an operator on VMOA is equivalent to the norm of S_n as an operator on H^1. By Proposition 1 and Corollary 3, $\{S_n\}$ is an unbounded sequence of (bounded) operators on VMOA. This proves the following corollary.

COROLLARY 5. *There exist functions in VMOA whose Taylor series do not converge in norm.*

Using the argument in the proof of Corollary 4 and the fact that an analytic function f on $\mathbf{D}$ is in the Besov space B_p (for $p>1$) if and only if

$$\int_{\mathbf{D}} (1-|z|^2)^p |f'(z)|^p \frac{dA(z)}{(1-|z|^2)^2} < +\infty,$$

we obtain another corollary.

COROLLARY 6. *If $1<p<+\infty$, then the Taylor series of each function in B_p converges in norm.*

We will deal with the Besov space B_1 in the next section after we study the Bergman space $L_a^1(\mathbf{D})$. We conclude this section with the following classical result. A proof can be found in [7, pp. 300–301].

THEOREM 7. *There are functions in the disk algebra $A(\mathbf{D})$ whose Taylor series do not converge in norm.*

3. The Bergman Space $L_a^1(\mathbf{D})$

We write $L_a^p(\mathbf{D}) = L_a^p(dA_\alpha)$ for $\alpha = 0$. Thus $L_a^p(\mathbf{D})$ denotes the usual (unweighted) Bergman spaces. We proved in the previous section that if $1 < p < +\infty$ then the Taylor series of any function in $L_a^p(\mathbf{D})$ converges in norm. In this section we show that this fails in the case $p = 1$.

LEMMA 8. *For $\alpha > -1$, t real, and $z \in \mathbf{D}$, define*

$$I_t(z) = \int_0^{2\pi} \frac{d\theta}{|1 - \bar{z}e^{i\theta}|^{1+t}} \quad \textit{and} \quad J_{t,\alpha}(z) = \int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha \, dA(w)}{|1 - \bar{z}w|^{2+\alpha+t}}.$$

If $t < 0$, then $I_t(z)$ and $J_{t,\alpha}(z)$ are bounded in $z \in \mathbf{D}$. If $t > 0$, then

$$I_t(z) \sim J_{t,\alpha}(z) \sim \frac{1}{(1-|z|^2)^t} \quad \textit{as } |z| \to 1^-.$$

If $t = 0$, then

$$I_0(z) \sim J_{0,\alpha}(z) \sim \log \frac{1}{1-|z|^2} \quad \textit{as } |z| \to 1^-.$$

Proof. See 1.4.10 of [5]. □

THEOREM 9. *There exist functions in $L_a^1(\mathbf{D})$ whose Taylor series do not converge in norm.*

Proof. By Proposition 1, it suffices to show that the operators S_n are not uniformly bounded on $L_a^1(\mathbf{D})$, that is, there is no constant $C > 0$ such that $\|S_n\| \le C$ for all $n \ge 1$. In what follows, all norms will be taken in $L_a^1(\mathbf{D})$.

Fix $a \in \mathbf{D}$ and consider the function

$$f_a(z) = \frac{1-|a|^2}{(1-\bar{a}z)^3}$$

for $z \in \mathbf{D}$. By Lemma 8, there is a constant $C > 0$ such that $\|f_a\| \le C$ for all $a \in \mathbf{D}$. The Taylor expansion of f_a is given by

$$f_a(z) = (1-|a|^2) \sum_{k=0}^{\infty} (k+1)(k+2)\bar{a}^k z^k.$$

For any $n \ge 1$,

$$\begin{aligned} S_n f_a(z) &= (1-|a|^2) \sum_{k=0}^{n} (k+1)(k+2)\bar{a}^k z^k \\ &= (1-|a|^2)\left[-\frac{(n+2)(n+3)(\bar{a}z)^{n+1}}{1-\bar{a}z} + \frac{2(n+3)(\bar{a}z)^{n+2}}{(1-\bar{a}z)^2} + \frac{2(1-(\bar{a}z)^{n+3})}{(1-\bar{a}z)^3}\right]. \end{aligned}$$

Next we estimate the L_a^1-norm of each of the above three functions (in reverse order). First, by Lemma 8, there is a constant $C_1 > 0$ such that

$$A_1 = \int_{\mathbf{D}} \frac{2(1-|a|^2)|1-(\bar{a}z)^{n+3}|}{|1-\bar{a}z|^3}\, dA(z) \le C_1$$

for all $a \in \mathbf{D}$. Second, by elementary calculus, there is a constant $C_2 > 0$ such that

$$2(n+3)(1-|w|^2)|w|^{n+2} \le C_2$$

for all $w \in \mathbf{D}$ and all $n \ge 1$. It follows that, for each $a \in \mathbf{D}$,

$$\begin{aligned} A_2 &= \int_{\mathbf{D}} \frac{2(1-|a|^2)(n+3)|\bar{a}z|^{n+2}}{|1-\bar{a}z|^2}\, dA(z) \\ &= (1-|a|^2)\int_{\mathbf{D}} \frac{2(n+3)(1-|az|^2)|az|^{n+2}}{|1-\bar{a}z|^2(1-|az|^2)}\, dA(z) \\ &\le C_2(1-|a|^2)\int_{\mathbf{D}} \frac{dA(z)}{|1-\bar{a}z|^2(1-|az|^2)} \\ &= C_2. \end{aligned}$$

Finally, for each $a \in \mathbf{D}$,

$$A_3 = (n+2)(n+3)(1-|a|^2)|a|^{n+1}\int_{\mathbf{D}} \frac{|z|^{n+1}\, dA(z)}{|1-\bar{a}z|}.$$

We show that the above integral is unbounded as a function of n and a. This will prove that $\{S_n\}$ is an unbounded sequence of (bounded) operators on $L_a^1(\mathbf{D})$ and hence complete the proof of the theorem.

By polar coordinates,

$$\int_{\mathbf{D}} \frac{|z|^{n+1}\, dA(z)}{|1-\bar{a}z|} = \frac{1}{\pi}\int_0^1 r^{n+2}\, dr \int_0^{2\pi} \frac{dt}{|1-r|a|e^{it}|}.$$

By Lemma 8, there is a constant $c > 0$ such that

$$\frac{1}{\pi}\int_0^{2\pi} \frac{dt}{|1-r|a|e^{it}|} \ge c \log \frac{1}{1-r|a|}$$

for all $r \in (0,1)$ and $a \in \mathbf{D}$. It follows that

$$\int_{\mathbf{D}} \frac{|z|^{n+1}\, dA(z)}{|1-\bar{a}z|} \ge c\int_0^1 r^{n+2} \log \frac{1}{1-r|a|}\, dr$$

for all $a \in \mathbf{D}$ and hence

$$A_3 \ge c(n+2)(n+3)(1-|a|^2)|a|^{n+1}\int_0^1 r^{n+2} \log \frac{1}{1-r|a|}\, dr$$

for all $a \in \mathbf{D}$. Using integration by parts, we have

$$(n+3)\int_0^1 r^{n+2}\log\frac{1}{1-r|a|}\,dr=\log\frac{1}{1-|a|}-|a|\int_0^1\frac{r^{n+3}\,dr}{1-r|a|}$$
$$\geq\log\frac{1}{1-|a|}-\frac{|a|}{1-|a|}\int_0^1 r^{n+3}\,dr$$
$$=\log\frac{1}{1-|a|}-\frac{|a|}{(n+4)(1-|a|)}.$$

This implies that

$$A_3\geq c(n+2)(1-|a|^2)|a|^{n+1}\left[\log\frac{1}{1-|a|}-\frac{|a|}{(n+4)(1-|a|)}\right]$$
$$=c(n+2)(1-|a|^2)|a|^{n+1}\log\frac{1}{1-|a|}-\frac{c(n+2)(1+|a|)|a|^{n+2}}{n+4}.$$

The second term above is bounded in a and n, but the first term tends to infinity if $a=n/(n+1)$ and $n\to+\infty$. This finishes the proof of Theorem 9. □

Recall that the Besov space B_1 consists of analytic functions $f(z)$ on $\mathbf{D}$ such that

$$\int_{\mathbf{D}}|f''(z)|\,dA(z)<+\infty.$$

It is now clear that the following result is a direct consequence of Theorem 9.

COROLLARY 10. *There are functions in B_1 whose Taylor series do not converge in norm.*

The Bloch space $\mathcal{B}$ consists of analytic functions $f(z)$ on $\mathbf{D}$ such that

$$\|f\|=|f(0)|+\sup\{(1-|z|^2)|f'(z)|: z\in\mathbf{D}\}<+\infty.$$

The closure in $\mathcal{B}$ of the polynomials is called the little Bloch space and is denoted by $\mathcal{B}_0$. It is well known (see [2] for example) that

$$\mathcal{B}_0^*\cong L_a^1(\mathbf{D})\quad\text{and}\quad(L_a^1(\mathbf{D}))^*\cong\mathcal{B}$$

under the complex integral pairing

$$\langle f,g\rangle=\int_{\mathbf{D}}f(z)\overline{g(z)}\,dA(z).$$

The symmetry of $\mathbf{D}$ and the rotation invariance of dA clearly imply that each operator S_n is self-adjoint under the above pairing. It follows that the norm of S_n as an operator on $\mathcal{B}_0$ is comparable to the norm of S_n as an operator on $L_a^1(\mathbf{D})$. By Proposition 1 and Theorem 9, we have the following corollary.

COROLLARY 11. *There exist functions in the little Bloch space $\mathcal{B}_0$ whose Taylor series do not converge in norm.*

4. Duality of Bloch Spaces and $L_a^1(dA_\alpha)$

In order to study the norm convergence of Taylor series for functions in the weighted Bergman spaces $L_a^1(dA_\alpha)$, we first establish a duality theorem between $L_a^1(dA_\alpha)$ and the Bloch spaces. It will be clear that the operators S_n are self-adjoint under this duality. Thus Corollary 11 together with Proposition 1 will show that the Taylor series of a function in the weighted Bergman space $L_a^1(\mathbf{D})$ does not necessarily converge in norm.

For any $\alpha > -1$, let P_α denote the operator defined by

$$P_\alpha f(z) = (\alpha+1)\int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha f(w)\,dA(w)}{(1-z\bar{w})^{2+\alpha}} = \int_{\mathbf{D}} \frac{f(w)\,dA_\alpha(w)}{(1-z\bar{w})^{2+\alpha}}.$$

It is well known that P_α is a projection onto analytic functions. For example, if $\beta > \alpha > -1$ then P_β is a bounded projection from $L^1(\mathbf{D}, dA_\alpha)$ onto $L_a^1(dA_\alpha)$, by an application of Lemma 8 and Fubini's theorem. We also note that P_α is the orthogonal projection from $L^2(\mathbf{D}, dA_\alpha)$ onto $L_a^2(dA_\alpha)$. In particular, P_α is self-adjoint under the integral pairing given by dA_α.

THEOREM 12. *P_α maps $L^\infty(\mathbf{D})$ boundedly onto $\mathcal{B}$ for any $\alpha > -1$. Moreover, the norm on $\mathcal{B}$ is equivalent to the quotient norm induced by the mapping $P_\alpha: L^\infty(\mathbf{D}) \to \mathcal{B}$.*

Proof. If g is in $L^\infty(\mathbf{D})$ and $f = P_\alpha g$, that is,

$$f(z) = (\alpha+1)\int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha g(w)\,dA(w)}{(1-z\bar{w})^{2+\alpha}} \quad \text{for } z \in \mathbf{D},$$

then differentiating under the integral sign gives

$$f'(z) = (\alpha+1)(\alpha+2)\int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha \bar{w} g(w)\,dA(w)}{(1-z\bar{w})^{3+\alpha}}.$$

By Lemma 8, there is a constant $C > 0$ (depending only on α) such that

$$|f'(z)| \le (\alpha+1)(\alpha+2)\|g\|_\infty \int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha\,dA(w)}{|1-z\bar{w}|^{3+\alpha}} \le \frac{C\|g\|_\infty}{1-|z|^2}$$

for all z in $\mathbf{D}$. Thus f is in the Bloch space, and it is clear that the norm of f is dominated by the norm of g.

On the other hand, if f is in the Bloch space then we wish to show that there is a function g in $L^\infty(\mathbf{D})$ such that $f = P_\alpha g$. It is easy to see that P_α of z^k is a constant multiple of z^k. It follows that if $f(z)$ is a polynomial then f is the image of a bounded function under the mapping P_α. It remains to show that if $f(z)$ is in $\mathcal{B}$ and if $f(0) = f'(0) = f''(0) = 0$, then there exists a function g in $L^\infty(\mathbf{D})$ such that $f = P_\alpha g$. This can be done constructively as follows. Since f is in $\mathcal{B}$ and $f(0) = f'(0) = f''(0) = 0$, the function

$$g(z)=\frac{(1-|z|^2)f'(z)}{(\alpha+1)\bar{z}}$$

is in $L^\infty(\mathbf{D})$. Let $F=P_\alpha g$, that is,

$$F(z)=\int_{\mathbf{D}}\frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^{2+\alpha}}\frac{(1-|w|^2)f'(w)}{\bar{w}}\,dA(w).$$

Differentiating under the integral sign, we get

$$F'(z)=(\alpha+2)\int_{\mathbf{D}}\frac{(1-|w|^2)^{\alpha+1}}{(1-z\bar{w})^{3+\alpha}}f'(w)\,dA(w)=P_{\alpha+1}f'(z)=f'(z).$$

A direct computation shows that $F(0)=0$; thus we have $F(z)=f(z)$ for all $z\in\mathbf{D}$. This proves te identity $\mathcal{B}=P_\alpha L^\infty(\mathbf{D})$. The equivalence of the norm on $\mathcal{B}$ and the quotient norm induced by P_α follows from the open mapping theorem. This can also be seen directly from the above proof. □

THEOREM 13. *If $\alpha>-1$, then P_α maps $\mathbf{C}_0(\mathbf{D})$ onto $\mathcal{B}_0$.*

Proof. It is easy to see that P_α of $(1-|z|^2)z^k$ is a constant multiple of z^k; hence every polynomial belongs to $P_\alpha\mathbf{C}_0(\mathbf{D})$. This, along with the construction in the second part of the proof of Theorem 12, shows that $\mathcal{B}_0\subset P_\alpha\mathbf{C}_0(\mathbf{D})$.

On the other hand, if g is in $\mathbf{C}_0(\mathbf{D})$ and $f=P_\alpha g$, then differentiating under the integral sign gives

$$f'(z)=\int_{\mathbf{D}}\frac{(1-|w|^2)^\alpha\varphi(w)}{(1-z\bar{w})^{3+\alpha}}\,dA(w),\quad z\in\mathbf{D},$$

where $\varphi(z)=(\alpha+1)(\alpha+2)\bar{z}g(z)$ is still in $\mathbf{C}_0(\mathbf{D})$. Let

$$\varphi_z(w)=\frac{z-w}{1-\bar{z}w}$$

and make the change of variable $w\to\varphi_z(w)$ in the above integral. We then obtain

$$(1-|z|^2)f'(z)=\int_{\mathbf{D}}\varphi\circ\varphi_z(w)\frac{(1-|w|^2)^\alpha}{|1-z\bar{w}|^\alpha}\frac{1-z\bar{w}}{(1-\bar{z}w)^2}\,dA(w).$$

Since $1-|w|^2\le 2|1-z\bar{w}|$ and

$$\int_{\mathbf{D}}\frac{dA(w)}{|1-\bar{z}w|}$$

converges uniformly in z, and since $\varphi\circ\varphi_z(w)\to\varphi(z_0)=0$ for each $w\in\mathbf{D}$ as $z\to z_0\in\partial\mathbf{D}$, dominated convergence gives

$$\lim_{|z|\to1^-}(1-|z|^2)|f'(z)|=0.$$

This completes the proof of Theorem 13. □

LEMMA 14. *Let T_α be the operator defined by*

$$T_\alpha f(z) = (\alpha+2)(1-|z|^2)\int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^{3+\alpha}} f(w)\,dA(w).$$

Then T_α is an embedding from $\mathcal{B}$ into $L^\infty(\mathbf{D})$. Moreover, T_α maps $\mathcal{B}_0$ into $\mathbf{C}_0(\mathbf{D})$.

Proof. If $f \in \mathcal{B}$, then Theorem 12 implies that $f = P_\alpha g$ for some $g \in L^\infty(\mathbf{D})$. It follows easily from Fubini's theorem and the reproducing property of P_α that $T_\alpha f = T_\alpha P_\alpha g = T_\alpha g$. Thus

$$T_\alpha f(z) = (\alpha+2)(1-|z|^2)\int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^{3+\alpha}} g(w)\,dA(w) \quad \text{for } z \in \mathbf{D}.$$

By Lemma 8, there is a constant $C > 0$ such that $\|T_\alpha f\|_\infty \le C\|g\|_\infty$. Since the norm on $\mathcal{B}$ is equivalent to the quotient norm induced by $P_\alpha : L^\infty(\mathbf{D}) \to \mathcal{B}$, taking the infimum of the above inequality over g we see that T_α maps $\mathcal{B}$ boundedly into $L^\infty(\mathbf{D})$.

On the other hand, Fubini's theorem and the reproducing property of $P_{\alpha+1}$ imply that $P_\alpha T_\alpha f = P_\alpha f = f$ for any $f \in \mathcal{B}$. By using the quotient norm on $\mathcal{B}$ induced by $P_\alpha : L^\infty(\mathbf{D}) \to \mathcal{B}$ we see that the norm of f in $\mathcal{B}$ is dominated by $\|T_\alpha f\|_\infty$.

Since T_α maps each polynomial to the product of another polynomial and $(1-|z|^2)$, T_α maps polynomials into $\mathbf{C}_0(\mathbf{D})$. Since T_α is bounded from $\mathcal{B}$ to $L^\infty(\mathbf{D})$, $\mathbf{C}_0(\mathbf{D})$ is closed in $L^\infty(\mathbf{D})$, and $\mathcal{B}_0$ is the closure of the polynomials, we see that T_α maps $\mathcal{B}_0$ into $\mathbf{C}_0(\mathbf{D})$. □

THEOREM 15. *Under the integral pairing*

$$\langle f, g\rangle_\alpha = \int_{\mathbf{D}} f(z)\overline{g(z)}\,dA_\alpha(z)$$

we have the following dualities (with equivalent norms):

$$\mathcal{B}_0^* \cong L_a^1(dA_\alpha) \quad \text{and} \quad (L_a^1(dA_\alpha))^* \cong \mathcal{B}.$$

Proof. First assume that g is in $\mathcal{B}$ and that $f \in L_a^1(dA_\alpha)$ is a polynomial. (Polynomials are dense in $L_a^1(dA_\alpha)$.) By Theorem 12, there exists a function φ in $L^\infty(\mathbf{D})$ such that $g = P_\alpha \varphi$. Since P_α is self-adjoint under the pairing $\langle\cdot,\cdot\rangle_\alpha$, we have

$$\begin{aligned}\langle f, g\rangle_\alpha &= \int_{\mathbf{D}} f(z)\overline{P_\alpha\varphi(z)}\,dA_\alpha(z)\\ &= \int_{\mathbf{D}} P_\alpha f(z)\overline{\varphi(z)}\,dA_\alpha(z)\\ &= \int_{\mathbf{D}} f(z)\overline{\varphi(z)}\,dA_\alpha(z).\end{aligned}$$

It follows that $|\langle f, g\rangle_\alpha| \le \|\varphi\|_\infty \|f\|_{1,\alpha}$ for all polynomials f in $L^1_a(dA_\alpha)$. Thus each function g in $\mathcal{B}$ induces a bounded linear functional on $L^1_a(dA_\alpha)$.

Next assume that ξ is a bounded linear functional on $L^1_a(dA_\alpha)$. Then, by the Hahn–Banach extension theorem, ξ extends to a bounded linear functional on $L^1(\mathbf{D}, dA_\alpha)$. Thus there exists a function φ in $L^\infty(\mathbf{D})$ such that

$$\xi(f) = \int_{\mathbf{D}} f(z)\overline{\varphi(z)}\, dA_\alpha(z)$$

for all f in $L^1_a(\mathbf{D})$. Since $f = P_\alpha f$ and P_α is self-adjoint under the integral pairing associated with the measure dA_α, we see that $\xi(f) = \langle f, P_\alpha \varphi\rangle_\alpha$ and $g = P_\alpha \varphi$ is in $\mathcal{B}$ by Theorem 12.

Finally, assume that ξ is a bounded linear functional on $\mathcal{B}_0$. We prove that there exists a function g in $L^1_a(dA_\alpha)$ such that $\xi(f) = \langle f, g\rangle_\alpha$ for all polynomials f in $\mathcal{B}_0$. (Polynomials are dense in $\mathcal{B}_0$.) Let Y be the image of $\mathcal{B}_0$ under the mapping T_α. By Lemma 14, Y is a closed subspace of $\mathbf{C}_0(\mathbf{D})$ and $\xi \circ T_\alpha^{-1}$ is a bounded linear functional on Y. Applying the Hahn–Banach extension theorem and the Riesz representation theorem, we can find a finite complex Borel measure μ on $\mathbf{D}$ such that

$$\xi \circ T_\alpha^{-1}(f) = \int_{\mathbf{D}} f(z)\, d\bar{\mu}(z)$$

for all $f \in Y$. It follows that

$$\begin{aligned}\xi(f) &= \int_{\mathbf{D}} T_\alpha f(z)\, d\bar{\mu}(z) \\ &= (\alpha+2)\int_{\mathbf{D}} (1-|z|^2)\, d\bar{\mu}(z) \int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha f(w)}{(1-z\bar{w})^{3+\alpha}}\, dA(w)\end{aligned}$$

for all f in the little Bloch space $\mathcal{B}_0$. Let

$$g(z) = \frac{\alpha+2}{\alpha+1}\int_{\mathbf{D}} \frac{(1-|w|^2)\, d\mu(w)}{(1-z\bar{w})^{3+\alpha}}, \quad z \in \mathbf{D}.$$

Fubini's theorem then implies that $\xi(f) = \langle f, g\rangle_\alpha$ for all polynomials f in $\mathcal{B}_0$. It remains to show that g is in $L^1_a(dA_\alpha)$. This can be checked easily using Fubini's theorem and Lemma 8:

$$\begin{aligned}\int_{\mathbf{D}} |g(z)|\, dA_\alpha(z) &\le (\alpha+2)\int_{\mathbf{D}} (1-|z|^2)^\alpha dA(z) \int_{\mathbf{D}} \frac{(1-|w|^2) d|\mu|(w)}{|1-z\bar{w}|^{3+\alpha}} \\ &= (\alpha+2)\int_{\mathbf{D}} (1-|w|^2)\, d|\mu|(w) \int_{\mathbf{D}} \frac{(1-|z|^2)^\alpha dA(z)}{|1-z\bar{w}|^{3+\alpha}} \\ &\le C\int_{\mathbf{D}} \frac{(1-|w|^2)\, d|\mu|(w)}{1-|w|^2} = C\|\mu\|.\end{aligned}$$

This completes the proof of Theorem 15. □

THEOREM 16. *There exist functions in $L_a^1(dA_\alpha)$ whose Taylor series do not converge in norm.*

Proof. Since dA_α is rotation invariant, each S_n is self-adjoint under the pairing

$$\langle f, g\rangle_\alpha = \int_{\mathbf{D}} f(z)\overline{g(z)}\, dA_\alpha(z).$$

By the duality $\mathcal{B}_0^* \cong L_a^1(dA_\alpha)$, the norm of S_n as an operator on $\mathcal{B}_0$ is equvalent to the norm of S_n as an operator on $L_a^1(dA_\alpha)$. The desired result now follows from Proposition 1 and Corollary 11. □

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