Some Results on BMOH and VMOH on Riemann Surfaces

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1. Introduction

The uniformization theorem for hyperbolic Riemann surfaces states that, given a Riemann surface R, there exist both a Fuchsian group Γ acting on the unit disk Δ and an analytic function $\phi\colon \Delta \to R$ such that ϕ is an automorphic function relative to the group Γ; that is, $\phi(T(z)) = \phi(z)$ for each $z \in \Delta$ and each $T \in \Gamma$ (see, e.g., [4, Theorem, p. 209]). If we start with a Riemann surface R which possesses a Green's function and a function f analytic on R, we say that $f \in \text{BMOA}(R)$ if

$$\sup_{\lambda \in R} \iint_R |f'(w)|^2 G_R(w, \lambda)\, dA(w) < \infty,$$

where $G_R(w, \lambda)$ is the Green's function on R with singularity at λ and $dA(w)$ denotes the element of area on R. We may also define the analytic function $f_* = f \circ \phi$ on Δ. Here, f_* is an automorphic function on Δ. We say that $f_* \in \text{BMOA}(\Delta/\Gamma)$ if

$$\sup_{w \in F} \iint_F |f_*'(z)|^2 G_R(\phi(z), \phi(w))\, dA(z) < \infty,$$

where

$$F = \{z \in \Delta \colon |z| \le |T(z)| \text{ for each } T \in \Gamma\}$$

is the so-called Ford fundamental region for the group Γ and where $dA(z)$ is the element of Euclidean area in Δ. The set F, also known as the Dirichlet polygon, is a fundamental set for the group Γ, together with some additional boundary points for this fundamental set. Although a wide variety of choices for a fundamental region are possible, it will avoid a number of difficulties to deal only with this normalized fundamental region.

For $\lambda \in R$, let a be a point in Δ such that $\phi(a) = \lambda$, and define $G_\Gamma(z, a) = G_R(\phi(z), \lambda)$. By a result of Myrberg,

$$G_\Gamma(z, a) = \sum_{T \in \Gamma} \log\left|\frac{1 - \overline{T(a)}z}{z - T(a)}\right| \quad \text{for } z, a \in F$$

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(see [8, Thm. XI.13, p. 522]). Clearly $f_* \in \mathrm{BMOA}(\Delta/\Gamma)$ whenever $f \in \mathrm{BMOA}(R)$. We can state the condition for $f_* \in \mathrm{BMOA}(\Delta/\Gamma)$ without any reference to the Riemann surface R by noting that $f_* \in \mathrm{BMOA}(\Delta/\Gamma)$ if

$$\sup_{a\in F}\iint_F |f_*'(z)|^2 G_\Gamma(z,a)\,dA(z) < \infty.$$

This removal of all reference to the Riemann surface enables us to remove all notational distinction between the two functions f and f_*, and hereafter we will use the notation f when we mean f_* as given in the context above. In all cases, the domain of the function f should be clear from the context, so that the ambiguity of our notation should cause no problem.

Another characterization for the class $\mathrm{BMOA}(\Delta/\Gamma)$ is as follows. If f is an automorphic function with respect to the Fuchsian group Γ, let $a \in \Delta$ and let $h_a(z)$ denote the least harmonic majorant for $|f(z)-f(a)|^2$. Then $f \in \mathrm{BMOA}(\Delta/\Gamma)$ if there exists a constant M such that $h_a(a) \le M$ for each $a \in F$. The equivalence of this characterization for $\mathrm{BMOA}(\Delta/\Gamma)$ with the integral condition given above is an easy consequence of Green's integral formula.

There is a special case where $\Gamma = \{\text{identity}\}$, so that Δ/Γ is actually Δ. The case $f \in \mathrm{BMOA}(\Delta)$ has been extensively studied with many characterizations of various types given. For our purposes, $f \in \mathrm{BMOA}(\Delta)$ means

$$\sup_{a\in \Delta}\iint_\Delta |f'(z)|^2 \log\left|\frac{1-\bar{a}z}{z-a}\right| dA(z) < \infty.$$

It is known that for each Fuchsian group Γ, the containment $\mathrm{BMOA}(\Delta/\Gamma) \subset \mathrm{BMOA}(\Delta)$ is valid (see [5, Prop. 2, p. 1257]).

There is also another class we will consider here. We say $f \in \mathrm{VMOA}(\Delta/\Gamma)$ (or equivalently, using the conventions above, that $f \in \mathrm{VMOA}(R)$ where R is the appropriate Riemann surface) if

$$\lim_{\substack{a\to\partial\Delta\\ a\in F}}\iint_F |f'(z)|^2 G_\Gamma(z,a)\,dA(z) = 0.$$

Here we have that $\mathrm{VMOA}(\Delta/\Gamma) \subset \mathrm{BMOA}(\Delta/\Gamma)$ (we will give a new proof of this containment below), but $\mathrm{VMOA}(\Delta/\Gamma)$ is not a subset of $\mathrm{VMOA}(\Delta)$ (see [1]).

There is also a characterization for $\mathrm{VMOA}(\Delta/\Gamma)$ in terms of least harmonic majorant. As before, for f automorphic relative to the Fuchsian group Γ, let $a \in \Delta$ and let $h_a(z)$ be the least harmonic majorant for $|f(z)-f(a)|^2$. Then $f \in \mathrm{VMOA}(\Delta/\Gamma)$ if $h_a(a) \to 0$ as $|a| \to 1$ from within F.

In a manner similar to the above, if we start with a function u harmonic on R then the corresponding $u_* = u \circ \phi$ is also harmonic on Δ, and we define $u \in \mathrm{BMOH}(R)$ to be equivalent to $u_* \in \mathrm{BMOH}(\Delta/\Gamma)$ if and only if

$$\sup_{a\in F}\iint_F |\nabla u_*(z)|^2 G_\Gamma(z,a)\,dA(z) < \infty.$$

As before, we will suppress the distinction between the functions u and u_*, and will use u to denote each of these. The domain of the function u will be clear from the context.

If u is harmonic and automorphic in Δ relative to the Fuchsian group Γ, then u has the harmonic conjugate $\tilde{u}$ on Δ, and the function $f=u+i\tilde{u}$ has the property that f is additive automorphic relative to Γ; that is, for each $T\in\Gamma$ there exists a constant A_T such that $f(T(z))=f(z)+A_T$ for each $z\in\Delta$. This follows immediately by noting that the real part of

$$g_T(z)=f(T(z))-f(z)$$

is zero throughout Δ, and thus A_T is purely imaginary.

If u is harmonic in Δ and automorphic with respect to the Fuchsian group Γ, we define $u\in\mathrm{VMOH}(\Delta/\Gamma)$ if

$$\lim_{\substack{a\to\partial\Delta\\ a\in F}}\iint_F|\nabla u(z)|^2G_\Gamma(z,a)\,dA(z)=0.$$

As $|\nabla u(z)|=|f'(z)|$, the corresponding characterizations for $\mathrm{BMOH}(\Delta/\Gamma)$ and $\mathrm{VMOH}(\Delta/\Gamma)$ in terms of $h_a(z)$, which is the least harmonic majorant of $|f(z)-f(a)|^2$ where $f=u+i\tilde{u}$, are identical with those of $\mathrm{BMOA}(\Delta/\Gamma)$ and $\mathrm{VMOA}(\Delta/\Gamma)$, respectively.

If f is automorphic in Δ relative to the Fuchsian group Γ, we say that $f\in\mathrm{AD}(\Delta/\Gamma)$ if

$$\iint_F|f'(z)|^2\,dA(z)<\infty.$$

Similarly, if u is harmonic in Δ and automorphic relative to the Fuchsian group Γ, we say that $u\in\mathrm{HD}(\Delta/\Gamma)$ if

$$\iint_F|\nabla u(z)|^2\,dA(z)<\infty.$$

Throughout, we will consider only Riemann surfaces (or equivalently, Fuchsian groups) which possess a Green's function. We will say that such a surface R is *regular* if the Green's function $G_R(z,a)$ has the property that $G_R(z,a)\to 0$ as $z\to\partial R$ for each choice of $a\in R$. We say that a Fuchsian group Γ is *regular* if, for each $a\in F$, $G_\Gamma(z,a)\to 0$ as $|z|\to 1$ from within the fundamental region F.

Metzger [5, Thm. 1, p. 1256] has proved that $\mathrm{AD}(\Delta/\Gamma)\subset\mathrm{BMOA}(\Delta/\Gamma)$. Aulaskari [1, Thm. 1] proved that if Γ is a regular Fuchsian group then $\mathrm{AD}(\Delta/\Gamma)\subset\mathrm{VMOA}(\Delta/\Gamma)$. Aulaskari proved further that if Γ is not a regular Fuchsian group then $\mathrm{VMOA}(\Delta/\Gamma)$ consists of the constant functions, so that $\mathrm{AD}(\Delta/\Gamma)$ is not a subset of $\mathrm{VMOA}(\Delta/\Gamma)$ in that case. Gotoh [2, Thm. 3, p. 335] has shown that, for a Riemann surface R of finite type, the containment $\mathrm{HD}(R)\subset\mathrm{BMOH}(R)$ is valid, but that this containment is not valid for all Riemann surfaces.

In Section 2 we will give some containments relating the classes of functions mentioned above. In Section 3 we will give some examples showing that HD(R) is not necessarily contained in BMOH(R), whether R is a regular surface or not.

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2. Some Basic Containments

We begin by proving a result proved by Aulaskari in [1]. Aulaskari proved that VMOA(Δ/Γ) $\subset$ BMOA(Δ/Γ), but the proof given was rather lengthy. Here we give a shorter, more basic proof, and use the same proof to show that VMOH(Δ/Γ) $\subset$ BMOH(Δ/Γ).

THEOREM 1.

$$\text{VMOA}(\Delta/\Gamma)\subset\text{BMOA}(\Delta/\Gamma) \quad \textit{and} \quad \text{VMOH}(\Delta/\Gamma)\subset\text{BMOH}(\Delta/\Gamma).$$

Proof. Let $f\in$ VMOA(Δ/Γ), let $a\in\Delta$, and let $h_a(z)$ denote the least harmonic majorant of $|f(z)-f(a)|^2$. Recall that $f\in$ VMOA(Δ/Γ) means that $h_a(a)\to 0$ as $|a|\to 1$ from within F. Hence, given $\epsilon>0$, there exists a number r_0, $0<r_0<1$, such that $|h_a(a)|<\epsilon$ whenever $a\in F$ and $|a|>r_0$. Let $a\in F$ with $|a|>r_0$. Then $h_a(z)$ is bounded on $F(r_0)=\bar{F}\cap\{z\in\Delta: |z|\le r_0\}$, since $h_a(z)$ is harmonic on the compact set $F(r_0)$. Hence, there exists a constant M_1 such that $|h_a(z)|<M_1$ for $z\in F(r_0)$.

Let b be fixed in $F(r_0)$ and let $h_b(z)$ be the least harmonic majorant of $|f(z)-f(b)|^2$. Then we have

$$\begin{aligned}|f(z)-f(b)|^2 &= |f(z)-f(a)|^2+|f(a)|^2-|f(b)|^2-2\operatorname{Re}(f(z)(\overline{f(b)}-\overline{f(a)}))\\ &\le |f(z)-f(a)|^2+C_1(a,b)+u_1(z),\end{aligned}$$

where $C_1(a,b) = ||f(a)|^2-|f(b)|^2|$ and $u_1(z) = -2\operatorname{Re}(f(z)(\overline{f(b)}-\overline{f(a)}))$ is a harmonic function (the real part of an analytic function). For b and z in $F(r_0)$ we have that $f(z)$ and $f(b)$ are both bounded, say $|f(z)|\le M_2$ and $|f(b)|\le M_2$, where $M_2=\sup\{|f(z)|: z\in F(r_0)\}$. It follows that $|u_1(z)|\le 2M_2(M_2+|f(a)|)$. Thus, we have

$$|h_b(z)|\le M_1+M_2^2+|f(a)|^2+2M_2(M_2+|f(a)|)=M_3.$$

It follows that $h_b(b)\le M_3$ for $b\in F(r_0)$, and this means that $h_a(a)$ is uniformly bounded for $a\in F$, which is equivalent to $f\in$ BMOA(Δ/Γ). This proves the first containment.

The second containment follows easily if we start with a harmonic function $u\in$ VMOH(Δ/Γ), then define $f=u+i\tilde{u}$, and then repeat the argument above. □

THEOREM 2. *If* $u\in$ BMOH(Δ/Γ) *then* $f=u+i\tilde{u}\in$ BMOA(Δ).

Proof. Since $|\nabla u(z)|=|f'(z)|$, we see that $u\in$ BMOH(Δ) if and only if $f\in$ BMOA(Δ). Using the same relationship between gradient and derivative to modify Metzger's proof that $\mathrm{BMOA}(\Delta/\Gamma) \subset \mathrm{BMOA}(\Delta)$, we have that $\mathrm{BMOH}(\Delta/\Gamma) \subset \mathrm{BMOH}(\Delta)$. Combining these two statements gives the result. □

The space $\mathcal{B}$ of Bloch functions is the collection of all functions f analytic in Δ such that $\sup\{|f'(z)|(1-|z|^2): z \in \Delta\} < \infty$. The following result combines Theorem 2 with the well-known result that $\mathrm{BMOA}(\Delta) \subset \mathcal{B}$.

COROLLARY 1. $\mathrm{VMOH}(\Delta/\Gamma) \subset \mathrm{BMOH}(\Delta/\Gamma)$, $\mathrm{BMOA}(\Delta) \subset \mathcal{B}$, *and, if* $u \in \mathrm{BMOH}(\Delta/\Gamma)$ *and* $f = u + i\tilde{u}$, *then* $f \in \mathrm{BMOA}(\Delta)$.

The space $\mathcal{B}$ has a subspace $\mathcal{B}_0$ which is of interest in many connections. We say that $f \in \mathcal{B}_0$ if $|f'(z)|(1-|z|^2) \to 0$ as $|z| \to 1$. It is well known that $\mathrm{VMOA}(\Delta) \subset \mathcal{B}_0$. However, for $f \in \mathrm{VMOA}(\Delta/\Gamma)$ and f not identically constant, it is not true that $f \in \mathcal{B}_0$, since for each $z \in F$ and each $\epsilon > 0$ there exists $T \in \Gamma$ such that

$$|T(z)| > 1-\epsilon \quad \text{and} \quad |f'(z)|(1-|z|^2) = |f'(T(z))|(1-|T(z)|^2).$$

However, if f is either an automorphic function or an additive automorphic function relative to the Fuchsian group Γ, we can define $f \in \mathcal{B}_0(\Delta/\Gamma)$ if $|f'(z)|(1-|z|^2) \to 0$ whenever $|z| \to 1$ from within F. We then have the following result.

THEOREM 3. $\mathrm{VMOA}(\Delta/\Gamma) \subset \mathcal{B}_0(\Delta/\Gamma)$. *Further, if* $u \in \mathrm{VMOH}(\Delta/\Gamma)$ *and* $f = u + i\tilde{u}$, *then* $f \in \mathcal{B}_0(\Delta/\Gamma)$.

The first containment has been proved in [1, Thm. 2]. The second statement follows from the same proof with the obvious modifications.

THEOREM 4. *For a regular Riemann surface of finite type,*

$$\mathrm{HD}(R) \subset \mathrm{VMOH}(R).$$

Proof. Gotoh [2, Thm. 3, p. 335] proved that, for a Riemann surface of finite type, $\mathrm{HD}(R) \subset \mathrm{BMOH}(R)$. In his proof, Gotoh showed that for $u \in \mathrm{HD}(R)$, for each $\epsilon > 0$, and for each $\lambda \in R$, the following inequality is valid:

$$\iint_R |\nabla u(w)|^2 G_R(w,\lambda)\, dA(w) \le \epsilon \iint_R |\nabla u(w)|^2\, dA(w) + C \iint_{\Omega_{\lambda,\epsilon}} |\nabla u(w)|^2\, dA(w),$$

where C is a positive constant depending on u and R and $\Omega_{\lambda,\epsilon} = \{w \in R: G_R(w,\lambda) > \epsilon\}$. Since R is a regular Riemann surface, for λ sufficiently close to ∂R and K a fixed compact subset of R we have that $\Omega_{\lambda,\epsilon} \cap K = \emptyset$. Since $|\nabla u|^2$ is summable on R, it follows that $\iint_R |\nabla u(w)|^2 G_R(w,\lambda)\, dA(w) \to 0$ as $\lambda \to \partial R$, and this implies that $u \in \mathrm{VMOH}(R)$. □

We may restate Theorem 4 in terms of Fuchsian groups, as follows.

THEOREM 4′. *If* Γ *is the Fuchsian group associated with the Riemann surface* R, *where* R *is regular and of finite type, then*

$$\mathrm{HD}(\Delta/\Gamma) \subset \mathrm{VMOH}(\Delta/\Gamma).$$

THEOREM 5. *If the Fuchsian group* Γ *is not regular, then* $\mathrm{VMOH}(\Delta/\Gamma)$ *consists only of functions which are identically constant.*

It was proved by Aulaskari [1, Thm. 1b] that, under the same circumstances, $\mathrm{VMOA}(\Delta/\Gamma)$ consists only of constant functions. The same proof, with the obvious modifications, works for the class $\mathrm{VMOH}(\Delta/\Gamma)$ also.

The following corollary is immediate.

COROLLARY 2. *If* Γ *is a Fuchsian group which is not regular, then the inclusion* $\mathrm{HD}(\Delta/\Gamma) \subset \mathrm{VMOH}(\Delta/\Gamma)$ *if and only if* $\mathrm{HD}(\Delta/\Gamma)$ *consists only of constant functions.*

3. Some Examples

In this section, we deal with some examples of functions u such that $u \in \mathrm{HD}(R)$ but u is not in $\mathrm{BMOH}(R)$, where R is a Riemann surface. We have mentioned that Gotoh [2, Example, pp. 337–339] has given an example of a function u on a Riemann surface such that $u \in \mathrm{HD}(R)$ but u is not in $\mathrm{BMOH}(R)$. In a sense, Gotoh's example is too strong because, for the function u constructed, the analytic function $f = u + i\tilde{u}$ is not a Bloch function. We note that the Riemann surface in Gotoh's example is a regular surface. Below, we give two examples of functions u and Riemann surfaces R such that $u \in \mathrm{HD}(R)$, u is not in the class $\mathrm{BMOH}(R)$, but $f = u + i\tilde{u}$ is a Bloch function. In one of the examples, R is a regular Riemann surface, and in the other R is a nonregular Riemann surface.

We will make use of a recent result of Stegenga and Stephenson [7, Thm. A, p. 243] which we call Theorem S.

THEOREM S. *Let* f *be analytic in* Δ, *let* R_f *be the Riemann surface corresponding to the image of* f, *let* U *be a disk with radius* $r > 0$ *in the complex plane, let* p *be a point on* R_f *lying over the center of the disk* U *and let* $\Omega(p)$ *be the component containing* p *in* R_f *of the inverse projection of* U, *and let* $\omega_p(r) = \omega(p, \partial\Omega(p) \cap \partial R_f, \Omega(p))$, *the harmonic measure at* p *of* $\partial\Omega(p) \cap \partial R_f$ *in the region* $\Omega(p)$. *Then* $f \in \mathrm{BMOA}(\Delta)$ *if and only if for each* δ, $0 < \delta < 1$, *there exists a number* $r_1 > 0$ *such that* $\omega_p(r_1) > \delta$ *for each* $p \in R_f$.

We note the following corollary to Theorem S.

COROLLARY 3. *If* f *is analytic in* Δ *and if there exist arbitrarily large disks* U *such that some component of the inverse projection of* U *in* R_f *has only a countable number of boundary points which project into the interior of* U, *then* f *is not in* $\mathrm{BMOA}(\Delta)$.

Proof. Fix a disk U such that, if p is a point in R_f lying over the center of U and if $\Omega(p)$ denotes the component of the inverse projection, then the boundary of $\Omega(p)$ projects onto only a countable number of points interior to U. Let f_1 denote the one-to-one mapping from Δ onto R_f and let Π denote the natural projection mapping from R_f into the complex plane such that $f=\Pi\circ f_1$. Then $\Omega'=f_1^{-1}(\Omega(p))$ is a simply connected subset of Δ. Let τ be a conformal mapping from Δ onto Ω'. Then $f\circ\tau$ is a (bounded) analytic function sending Δ into (but not necessarily onto) U, and thus $f\circ\tau$ has angular limits almost everywhere on $\partial\Delta$. However, for any countable subset E in U, the set of points in $\partial\Delta$ at which $f\circ\tau$ has an angular limit in E is of measure zero; the set of points in $\partial\Delta$ at which $f\circ\tau$ fails to have an angular limit is also a set of measure zero. It follows that the pre-image of E has harmonic measure zero at every point of Δ. Let $\partial^*\Omega(p)$ denote that part of the ideal boundary of $\Omega(p)$ which does not project onto ∂U. Then, by conformal mapping, the harmonic measure of $\partial^*\Omega(p)$ in $\Omega(p)$ is zero at each point of Ω'. Now Theorem S implies that f is not in BMOA(Δ). □

THEOREM 6. *There exists a nonregular Riemann surface $R=\Delta/\Gamma$, where Γ is a Fuchsian group on Δ, and a function $f=u+i\tilde{u}$ analytic on Δ, such that f is a Bloch function and $u\in$* HD(R) *but u is not in* BMOH(R).

Proof. We will need to construct an image surface for the function f, and we begin with a construction due to Pommerenke [6]. Let $T=\bigcup_{n=1}^{\infty}(0,2^n)\times[-2^{-3n},2^{-3n}]$. Clearly, T has finite area. For each positive integer n we take a copy of T to be the translate of T given by $T+2^{-3n+1}i$, and we join T to $T+2^{-3n+1}i$ across the segment $(2^{n-1},2^n)\times\{2^{-3n}\}$ where this is the only joining of T to $T+2^{-3n+1}i$. We continue that process in a similar manner until each copy of T is joined to exactly one other copy of T across a horizontal segment of its boundary, and no horizontal boundary segments of any copy of T are left free. Denote by S the resulting Riemann surface, and let Π be the natural projection from S onto the complex plane. Then S is invariant under a group Σ of motions generated by those motions sending a copy of T onto an adjacent copy of T, and if σ is one of these motions then $\Pi\circ\sigma$ is a translation of the plane by $2^{-3n+1}i$. We will use the surface S both in this proof and also in the proof of Theorem 7.

We now modify the surface S. Returning to the original planar region T, let T' be the result when T is punctured at each positive integer. Now let S' be the surface resulting when we remove from S all the points congruent to $T-T'$ under the group Σ. Finally, let R_f denote the universal covering surface of S'. Let U be a disk totally contained in the strip $W_n=\{z: 2^n<\operatorname{Re} z<2^{n+1}\}$, let p be a point on S' which projects onto the center of U, and let $\Omega(p)$ be the component of $\Pi^{-1}(U)$ containing p. We note that p is on a copy of T' and that each copy of T' meeting $\Omega(p)$ is attached to another copy of $\Omega(p)$ across a segment contained in W_n, and so $\Omega(p)$ is a covering surface for the disk U minus a finite number of punctures.

Let f_* be a conformal mapping from Δ onto R_f, let Π_* be the projection mapping from R_f to S', and let $f = \Pi \circ \Pi_* \circ f_*$. Then f satisfies the hypotheses of Corollary 3, so f is not in BMOA(Δ). However, f is a Bloch function, since S' contains no disks of radius larger than 1. Further, S' is invariant under the group Σ, and elements of Σ correspond to "translations by imaginary numbers"; thus it follows that f is additive automorphic with respect to a group Γ on Δ which corresponds under f_* to Σ, and the periods of f are all imaginary. Letting u be the real part of f, we have that u is automorphic relative to the Fuchsian group Γ; it follows from Theorem 2 that u is not in BMOH(Δ/Γ). However, if F denotes the Ford fundamental region of Γ then we have that the planar area of $f(F)$, counting multiplicity, is equal to the planar area of T'. (In fact, we can choose an appropriate "fundamental region" F', where each point of F is congruent under Γ to exactly one point of F—except for a set of area measure zero—so that f maps F' onto T' in a one-to-one manner.) Thus, f is an additive automorphic function for which $f(F)$ has finite planar area and $u = \text{Re}(f) \in \text{HD}(\Delta/\Gamma)$.

Finally, we note that the Riemann surface corresponding to Δ/Γ is simply the region T', with each horizontal boundary segment identified with the horizontal segment lying directly above or below it. It is easily seen that this surface is not regular, since it has punctures. This completes the proof. □

THEOREM 7. *There exists a regular Riemann surface* $R = \Delta/\Gamma$, *where* Γ *is a Fuchsian group on* Δ, *and a function* $f = u + i\tilde{u}$ *analytic on* Δ, *such that* f *is a Bloch function and* $u \in \text{HD}(R)$ *but* u *is not in* BMOH(R).

Proof. Let S be the Riemann surface constructed in the proof of Theorem 6. We wish to delete some disks from the basic region T and then, if $\{U_n\}$ is the collection of disks deleted from T, we will delete $\bigcup_{\sigma \in \Sigma} \bigcup_{n=1}^{\infty} \sigma(U_n)$ from S. We will choose these disks according to the following scheme.

First, look at $K = \{z : r < |z| < 1\}$. The function $h_r(z) = \log|z|/\log r$ is harmonic in the region K, $h_r(z) = 1$ on the circle $|z| = r$, and $h_r(z) = 0$ on $\partial\Delta$. If $\kappa(z)$ is a conformal mapping of Δ onto itself, then the κ takes the disk $\{z : |z| \leq r\}$ onto a closed disk D with hyperbolic radius $\tanh^{-1} r$. Further, the function $h_{r,\kappa}(z) = \log|\kappa^{-1}(z)|/\log r$ is harmonic in $\Delta - D$, $h_{r,\kappa}(z) = 1$ on ∂D, and $h_{r,\kappa}(z) = 0$ on $\partial\Delta$. If $0 < s < 1$ and $\kappa(z) = (z+s)/(1+sz)$, then

$$\kappa^{-1}(0) = -s \quad \text{and} \quad h_{r,\kappa}(0) = \frac{\log s}{\log r}.$$

If $r < \frac{1}{2}$ and $s \to 1$, then $h_{r,\kappa}(0) \to 0$.

Now suppose $D_1, D_2, \ldots, D_n, \ldots$ are countably many mutually disjoint closed disks, all contained in Δ and all having centers on the positive real axis. If s_n is the hyperbolic center of D_n (where s_n is a positive real number), if $\tanh^{-1}(r_n)$ is the hyperbolic radius of D_n, and if $\kappa_n(z) = (z - s_n)/(1 - s_n z)$, then the harmonic measure $\omega(z)$ of $\bigcup_n \partial D_n$ at z relative to $\Delta - \bigcup_n D_n$ is not greater than $\sum_n \log|\kappa_n(z)|/\log r_n$. In particular, $\omega(0) \leq \sum_n \log s_n/\log r_n$.

We wish to consider a situation where n is fixed and $s_{n,k}=6k/2^n$, where $-2^n<6k<2^n$, $k\neq 0$. Since $\log r\to-\infty$ as $r\to 0$, we may choose numbers $r_{n,k}$, $0<r_{n,k}<1$, such that

$$\sum_{\substack{k=-q\\k\neq 0}}^{q} 2^{4n+7}\frac{\log s_{n,k}}{\log r_{n,k}}<2^{-3n},$$

where q is the greatest integer less than $2^n/6$. For each n and $-2^n<6k<2^n$, $k\neq 0$, let $D'_{n,k}$ denote the disk in Δ with center $s_{n,k}$ and hyperbolic radius $\tanh^{-1}r_{n,k}$, and let

$$D_{n,k}=3\cdot 2^n+2^n D'_{n,k}=\{w: w=3\cdot 2^n+2^n z,\ z\in D'_{n,k}\}.$$

Now, consider the surface S as constructed in the proof of Theorem 6 and delete the closed disks $D_{n,k}$ from T, the basic region from which the surface S was constructed. We note that we can rechoose the $r_{n,k}$, making them smaller if necessary, so that each closed disk $D_{n,k}$ is a subset of T. Finally, we delete from S all points on S which are congruent to points in $D_{n,k}$ under elements of the group Σ. Let S'' denote the resulting surface, and let R_f denote its universal covering surface.

Let f_* denote a conformal mapping from Δ onto R_f, let Π_* denote the natural projection mapping from R_f to S'', and let Π denote the natural projection mapping from S'' into the complex plane. Then $f=\Pi\circ\Pi_*\circ f_*$ is the function we desire. For if p is any point on R_f such that $\Pi\circ\Pi_*(p)=3\cdot 2^n$, and if $\Omega(p)$ is the component in R_f of the inverse projection of $U=\{z: |z-3\cdot 2^n|<2^n\}$, then, for each appropriate value of k, at most 2^{4n+7} copies of $D_{n,k}$ are missing from $\Pi^{-1}(U)$ on S'', and so the harmonic measure $\omega_p(p)$ of $\partial\Omega(p)\cap\Pi_*^{-1}\circ\Pi^{-1}(U)$ at p relative to the region $\Omega(p)$ is less than 2^{-3n}. It follows from Theorem S that f is not in BMOA(Δ). Let Γ be the Fuchsian group on Δ which corresponds to the action of Σ on S''. Then the function $\Pi_*\circ f_*$ is additive automorphic with respect to the group Γ when the projection Π is applied, so f is additive automorphic with respect to Γ. Letting u be the real part of f, it follows from Theorem 2 that u is not in BMOH(Δ/Γ). However, f is a Bloch function, since no disk on R_f may have radius greater than 7.

It remains to show that Δ/Γ, considered as a Riemann surface, is regular. But Δ/Γ is equivalent to the surface $T-\bigcup D_{n,k}$, where each boundary segment is identified with the boundary segment either directly above it or directly below it. Clearly, every boundary point of this surface belongs to a continuum of boundary points except for the point at ∞. Thus, the only point which needs to be checked for regularity is the point at ∞. Applying the mapping $w(z)=1/z$ to $T-\bigcup D_{n,k}$, the image is a region T^* for which every boundary point belongs to a continuum except for the point at $z=0$. We note that the boundary of T^* contains curves E_n joining the point $z=(2^n-2^{-3n}i)/(2^{2n}+2^{-6n})$ to the point $(2^n-2^{-3(n+1)}i)/(2^{2n}+2^{-6(n+1)})$ for each positive integer n. We wish to apply the Wiener criterion (see [3, p. 298] and [9]). According to the Wiener criterion, the point $z=0$ is a regular point for T^* (meaning that the Green's function will converge to zero as that boundary point is approached) if and only if the series $\sum_{n=1}^{\infty} W(F_n)\log 2^n$ diverges, where $W(F_n)$ is the Wiener capacity of the set $F_n = \{z \in \partial T^* : 2^{-n-1} \le |z| < 2^{-n}\}$. Here, $W(F_n)$ is related to the logarithmic capacity $L(F_n)$ by the relationship $W(F_n) = 1/\log(1/L(F_n))$. We note that the set E_n is a subset of F_n even when T^* is considered to be a Riemann surface with the identifications of top and bottom edges. Further, E_n can be approximated as the line segment from $2^{-n} - 2^{-5n}i$ to $2^{-n} - 2^{-5n-3}i$, a segment with length $2^{-5n} - 2^{-5n-3}$. (This approximation is sufficiently accurate for our purposes, and it simplifies the calculations greatly.) Since the logarithmic capacity of a line segment is one-fourth of its length, we have that $L(E_n)$ is approximately 2^{-5n-2} and $W(E_n)$ is approximately $((5n+2)\log 2)^{-1}$; that is, $W(E_n)$ is of the order of $1/n$. Clearly, $\sum_{n=1}^{\infty}(\log 2^n)/n$ diverges. It follows that T^*, with appropriate boundary identifications, is a regular Riemann surface, and hence so is $R = \Delta/\Gamma$. This completes the proof. □

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