A Probabilistic Zero Set Condition for the Bergman Space

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Introduction

A function f, analytic in the open unit disk $\mathbf{D}$, is said to belong to a Bergman space L_a^p, $0<p<\infty$, if

$$\int_{\mathbf{D}} |f(z)|^p \, dA(z) < \infty,$$

where $dA(z)$ is area measure on $\mathbf{D}$. (The space L_a^2 is referred to as the Bergman space, and L_a^∞ is defined to be H^∞.)

Axler [1] gives a short introduction to the Bergman spaces with proofs of the basic facts about these spaces; however, describing the zero sets of the functions in the Bergman spaces remains an unsolved problem.

This paper presents a condition on a sequence $r_1, r_2, \ldots \in [0,1]$ that is weaker than the Blaschke condition, namely,

$$\limsup_{\epsilon \to \infty} \frac{\sum_{j=1}^{\infty}(1-r_j)^{1+\epsilon}}{\log(1/\epsilon)} < \frac{1}{4},$$

that guarantees that a set of points in the disk with moduli r_j and random arguments is almost surely the zero set of a function in L_a^2. An explicit construction of a function with the desired zero set that almost surely belongs to the Bergman space is provided (using Horowitz's generalization of the Blaschke factors).

It is well known (see, e.g., [5, pp. 90–95]) that a countable set $S=\{z_j\}$ of points (assumed to be ordered by magnitude) in $\mathbf{D}$ is a zero set for an H^p function, $0<p\leq\infty$, if and only if the points satisfy the Blaschke condition:

$$\sum_{z_j \in S} (1-|z_j|) < \infty.$$

No such simple condition for the zero sets for L_a^p functions is known. Horowitz obtained many interesting results about zero sets in the Bergman spaces in [4]. There are three results in particular that highlight the differences and similarities between the Hardy spaces and the Bergman spaces:

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1. If $0<p<q\leq\infty$ then there is an L_a^p zero set which is not an L_a^q zero set.
2. If $0<p<\infty$ then the union of two L_a^p zero sets is an $L_a^{(p/2)}$ zero set but not necessarily an L_a^q zero set for any $q>(p/2)$.
3. Subsets of L_a^p zero sets are L_a^p zero sets.

Only the third item listed above is similar to the H^p case; the other two items are in sharp contrast to the simple "one condition holds for all p" Hardy space result.

Shapiro and Shields [7] showed that the zeros of a Bergman function along any radius do satisfy the Blaschke condition. Since for all $1\leq p\leq\infty$ we have $L_a^p\supset H^p$, one might expect that a Blaschke-like condition for the zeros of Bergman space functions exists. The Blaschke condition does generalize to a necessary (but not sufficient) condition for L_a^p zero sets. For $f\in L_a^p$, f not identically zero, let Z be the set of points where f is zero. Then,

$$\forall\epsilon>0,\quad \sum_{z_j\in Z}(1-|z_j|)^{1+\epsilon}<\infty. \tag{1}$$

This can be easily proved (see [2] for an outline). Note that this condition does not involve the argument of the zero points.

The result of Shapiro and Shields shows that no condition on a sequence $\{z_j\}_{j=1}^{\infty}$ in $\mathbf{D}$, weaker than the Blaschke condition and expressible solely in terms of the absolute values $|z_j|$, can guarantee that the sequence is an L_a^p zero set. This motivated the probabilistic approach that is being presented: If the growth of the bounds, in the necessary condition (1) above, is not too fast as ϵ tends to zero then the condition becomes sufficient, almost surely.

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The L^2 Norm of a Blaschke–Horowitz Factor

Blaschke factors play an important role in the function theory in the unit disk.

DEFINITION. For z_0 a nonzero point of $\mathbf{D}$, let

$$B_{z_0}(z)=\frac{|z_0|}{z_0}\cdot\frac{z_0-z}{1-\overline{z}_0 z}$$

be the usual *Blaschke factor* for z_0. The Blaschke factor for the point zero is defined to be $B_0(z)=z$.

In the Bergman space the useful analogue of the Blaschke factor is

$$B_{z_0}(z)\cdot(2-B_{z_0}(z)),$$

which was introduced by Horowitz in [4]. We will refer to such terms as Blaschke–Horowitz factors.

In the proof of the main theorem the following proposition, determining the L^2 norm of a Blaschke-Horowitz factor as the argument of the zero varies over the unit circle, will be useful.

PROPOSITION 1. *For* $0<r<1$,

$$\frac{1}{2\pi}\int_0^{2\pi}|B_{re^{i\theta}}(z)\cdot(2-B_{re^{i\theta}}(z))|^2\,d\theta=\frac{p_1(r)s^6+p_2(r)s^4+p_3(r)s^2+p_4(r)}{(rs-1)^3(rs+1)^3},$$

where

$$p_1(r)=-4r^7+10r^6-4r^5-4r^4+4r^3-r^2,$$
$$p_2(r)=-4r^6+20r^5-26r^4-4r^3+16r^2-4r-1,$$
$$p_3(r)=-r^6-4r^5+16r^4-16r^3+4r^2+8r-4,$$
$$p_4(r)=-r^4+4r^3-4r^2,$$

and $s=|z|$.

Proof. Let $\zeta=re^{i\theta}$ be the zero point of the Blaschke-Horowitz factor. The restriction that $\zeta\neq 0$ is necessary since the definition of $B_{z_0}(z)$ is not continuous as $z_0\to 0$. (Because of this the zeros of a function at the origin are often dealt with separately when using Blaschke factors; fortunately this is usually simple since it merely means dividing out by a high enough power of z to reduce to the case with no zeros at the origin.) Let I be the integral we are interested in evaluating.

We have

$$|B_\zeta(z)\cdot(2-B_\zeta(z))|^2=\frac{(2|\zeta|-1)^2|\zeta-z|^2|\zeta'-z|^2}{|1-\bar{\zeta}z|^4}, \tag{2}$$

where

$$\zeta'=\frac{\zeta}{|\zeta|}\,\frac{1-(|\zeta|/2)}{|\zeta|-(1/2)}.$$

(The variable ζ' was introduced so that the following argument would be more symmetric. The presence of the initial factor, $(2|\zeta|-1)^2$, in equation (2) above ensures that there is no singularity when $|\zeta|=1/2$.)

Since the integration is over the argument of ζ on the entire circle, only the absolute value $s=|z|$, not the argument of z, affects the integral, so we have

$$I=(2r-1)^2\int_0^{2\pi}\frac{|re^{i\theta}-s|^2|r'e^{i\theta}-s|^2}{|1-rse^{i\theta}|^4}\,\frac{d\theta}{2\pi},$$

where $r'=(1-(r/2))/(r-(1/2))=(2-r)/(2r-1)$. Now, since the integral is complicated, we split the integrand into two factors which will make the integration easier.

Let

$$f_1=\left|\frac{re^{i\theta}-s}{1-rse^{i\theta}}\right|^2 \quad\text{and}\quad f_2=\left|\frac{r'e^{i\theta}-s}{1-rse^{i\theta}}\right|^2;$$

then

$$\frac{1}{(2r-1)^2}I=\int_0^{2\pi} f_1\cdot f_2\frac{d\theta}{2\pi}.$$

The simplification will arise since $f_1f_2=1-(1-f_1)-(1-f_2)+(1-f_1)(1-f_2)$, and integrating each part of the right-hand side is straightforward. We are using $\Re z$ to denote the real part of the complex number z in the following. We have

$$1-f_1=1-\left|\frac{re^{i\theta}-s}{1-rse^{i\theta}}\right|^2=\frac{1+r^2s^2-r^2-s^2}{|1-rse^{i\theta}|^2}=\frac{(1-r^2)(1-s^2)}{|1-rse^{i\theta}|^2}$$

and

$$1-f_2=1-\left|\frac{r'e^{i\theta}-s}{1-rse^{i\theta}}\right|^2=\frac{1+r^2s^2-r'^2-s^2-2s(r-r')\Re e^{i\theta}}{|1-rse^{i\theta}|^2}$$
$$=\frac{(1-r^2)(1-s^2)+r^2-r'^2-2s(r-r')\Re e^{i\theta}}{|1-rse^{i\theta}|^2}.$$

We will need the following lemma to simplify the integral.

LEMMA 2.

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-rse^{i\theta}|^2}\,d\theta=\frac{1}{1-r^2s^2},$$
$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-rse^{i\theta}|^4}\,d\theta=\frac{1+r^2s^2}{(1-r^2s^2)^3},$$
$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\Re e^{i\theta}}{|1-rse^{i\theta}|^2}\,d\theta=\frac{rs}{1-r^2s^2},$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\Re e^{i\theta}}{|1-rse^{i\theta}|^4}\,d\theta=\frac{2rs}{(1-r^2s^2)^3}.$$

These equalities are quite simple to establish, using the Taylor series for $1/(1-x)$ and the orthonormality of the exponentials in $L^2[0,2\pi]$.

Returning to the proof of the proposition, we can now do the computations.

$$\frac{1}{(2r-1)^2}I=\int_0^{2\pi} f_1\cdot f_2\frac{d\theta}{2\pi}$$
$$=\int_0^{2\pi}1\frac{d\theta}{2\pi}-\int_0^{2\pi}(1-f_1)\frac{d\theta}{2\pi}-\int_0^{2\pi}(1-f_2)\frac{d\theta}{2\pi}$$
$$+\int_0^{2\pi}(1-f_1)(1-f_2)\frac{d\theta}{2\pi}$$
$$=1-\frac{2(1-r^2)(1-s^2)}{1-r^2s^2}-\frac{(r-r')(r+r')}{1-r^2s^2}+\frac{2s(r-r')rs}{1-r^2s^2}$$
$$+\frac{(1-r^2)^2(1-s^2)^2(1+r^2s^2)}{(1-r^2s^2)^3}+$$

$$+(1-r^2)(1-s^2)(r^2-r'^2)\frac{1+r^2s^2}{(1-r^2s^2)^3}$$
$$-(1-r^2)(1-s^2)(r-r')2s\frac{2rs}{(1-r^2s^2)^3}.$$

Recalling that $r'=(2-r)/(2r-1)$ and combining terms, we obtain the proposition. □

It is possible to verify Lemma 2 and to calculate the L^2 norm of the Blaschke-Horowitz factor directly using the residue theorem. Although the calculations in the latter case do become tedious, the denominator does factor which makes things easier.

A Blaschke Type Condition for Probable Zero Sets

After a few necessary definitions the main theorem can now be stated.

DEFINITION. Let $\Omega=\prod_{j=1}^{\infty}[0,2\pi)$. Let μ_j be normalized Lebesgue measure on the jth factor of Ω. Then let μ be the probability measure on Ω such that $\mu=\prod_{j=1}^{\infty}\mu_j$.

(See [3, p. 157], for example, for a proof of the existence of such a probability measure.)

For $\{r_j\}_{j=1}^{\infty}$ an ordered sequence (allowing repetition) in $(0,1)$, consider the map of Ω into $H(\mathbf{D})$ defined by $\omega\mapsto B_\omega(z)$, where $\omega=(\varphi_1,\varphi_2,\varphi_3,\dots)$ and

$$B_\omega(z)=\prod_{j=1}^{\infty}B_{r_je^{i\varphi_j}}(z)\cdot(2-B_{r_je^{i\varphi_j}}(z)). \tag{3}$$

$B_\omega(z)$ will be called a Blaschke-Horowitz product. Horowitz [4, p. 705] showed that if $\sum_{j=1}^{\infty}(1-r_j)^2<\infty$ then $B_\omega(z)$ is indeed in $H(\mathbf{D})$. As was stated in equation (1), this sum is bounded for a Bergman zero set.

THEOREM 3. *If*

$$\limsup_{\epsilon\to 0}\frac{\sum_{j=1}^{\infty}(1-r_j)^{1+\epsilon}}{\log(1/\epsilon)}<\frac{1}{4}$$

and $B_\omega(z)$ is as in equation (3), *then $B_\omega(z)\in L_a^2$ a.e.* $[\mu]$.

Proof. Consider

$$\int_\Omega\int_{\mathbf{D}}|B_\omega(z)|^2\,dA\,d\mu.$$

Call this integral I. We claim that this integral is finite; hence the inner integral is finite, and $B_\omega(z)\in L_a^2$, except perhaps for a set of zero measure with respect to μ, that is, a.e. $[\mu]$ as required. In the following, for simplicity we will write B_j for $B_{r_je^{i\varphi_j}}(z)$.

Since μ is a product measure we have

$$\begin{aligned} I &= \int_{\mathbf{D}}\int_{\Omega} |B_\omega(z)|^2\, d\mu\, dA \\ &= \int_{\mathbf{D}}\int_{\Omega} \left| \prod_{j=1}^{\infty} B_j\cdot(2-B_j) \right|^2 d\mu\, dA \\ &= \int_{\mathbf{D}}\int_{\Omega} \prod_{j=1}^{\infty} |B_j\cdot(2-B_j)|^2\, d\mu\, dA \\ &= \int_{\mathbf{D}} \prod_{j=1}^{\infty} \int_0^{2\pi} |B_j\cdot(2-B_j)|^2\, d\mu_j\, dA \\ &= \int_{\mathbf{D}} \prod_{j=1}^{\infty} f(r_j, s)\, dA, \end{aligned}$$

where $f(r_j, s)$ is the rational function described in Proposition 1 (we are using s for $|z|$, as before). Thus,

$$\begin{aligned} I &= \int_{\mathbf{D}} \prod_{j=1}^{\infty} f(r_j, s)\, dA \\ &= \int_{\mathbf{D}} \exp\left(\log \prod_{j=1}^{\infty} f(r_j, s) \right) dA \\ &= \int_{\mathbf{D}} \exp\left(\sum_{j=1}^{\infty} \log f(r_j, s) \right) dA. \end{aligned}$$

Now, by Proposition 4 (below) we have

$$\log f(r, s) \le 4(1-r)^{2-s}$$

for r and s sufficiently close to 1, so, by the assumption on the growth in the theorem,

$$\begin{aligned} \exp\left(4 \sum_{j=j_0}^{\infty} (1-r_j)^{2-s} \right) &\le \exp\left((1-\delta) \log \frac{1}{1-s} \right) \\ &= \frac{1}{(1-s)^{1-\delta}} \end{aligned}$$

for some j_0 large enough, some $\delta \in (0, 1)$, and s close to 1. This gives us our result, since

$$\int_0^1 \frac{1}{(1-s)^{1-\delta}}\, ds < \infty. \qquad \square$$

The next proposition describes the limiting behavior of the rational function from Proposition 1, which we will call $f(r, s)$.

PROPOSITION 4. *For $f(r, s)$ as in Proposition 1,*

$$\log f(r, s) \le 4(1-r)^{2-s}$$

for r and s sufficiently close to and less than 1.

Proof. We claim that $(f(r,s)-1) \le 4(1-r)^{2-s}$, which gives the proposition since $\log(x) \le (x-1)$ for all positive x. In order to prove the claim we will define a new function $g(r,s)$ as follows:

$$g(r,s) = \frac{f(r,s)-1}{4(1-r)^{2-s}}.$$

In order to prove the proposition we will show that $g(r,s) \le 1$ for all (r,s) close to $(1,1)$ in the unit square ($[0,1]\times[0,1]$). It is easy to check that $g(r,1)=1$ for $r \in [0,1)$ (and for $r=1$ in the limit), so we will show that g is a nondecreasing function of s near $(1,1)$. We have

$$\frac{\partial g}{\partial s}(r,s) = -\frac{(1-r)^s(p(r,s)\log(1-r)+q(r,s))}{4(rs-1)^4(rs+1)^4},$$

where

$$\begin{aligned} p(r,s) = {} & (rs-1)(rs+1) \\ & \times (4r^5s^6 - r^4s^6 - 2r^3s^6 + r^2s^6 + 4r^4s^4 - 12r^3s^4 - 5r^2s^4 \\ & \quad + 6rs^4 + s^4 + r^4s^2 + 6r^3s^2 - 5r^2s^2 + 4s^2 + r^2 - 2r - 1), \end{aligned}$$

and

$$q(r,s) = -4(r-1)^2(r+1)^2 s(2r^2s^4 + r^2s^2 + 6rs^2 + s^2 + 2).$$

We claim that $p(r,s)\log(1-r)+q(r,s)$ is negative, which gives the result. Clearly $q(r,s)$ is always negative, as is $\log(1-r)$, so if $p(r,s) \ge 0$ then we are done. We will analyze the behavior of $p(r,s)$ in the unit square quite explicitly in order to show that the claim is still true even when $p(r,s)$ is negative.

The polynomial $p(r,s)$ is zero in the unit square only when $r=s=1$ or when (r,s) is a root of the third factor of $p(r,s)$. Upon examining this third factor more closely, one notices that it is a cubic equation in s^2 with coefficients which are functions of r. In fact, the leading coefficient of the cubic in s^2 is $r^2(4r^3-r^2-2r+1)$, which is always positive for $r \in (0,1)$. So for each fixed r, there are at most three roots of the equation for $s \in [0,1]$. Actually, there is exactly one root in this interval for this factor. To see this note that $p(r,0) = -(r^2-2r-1)$ is positive, $p(r,1) = 4(r-1)^3(r+1)^4$ is negative (or zero when $r=1$), and $p(r,\infty)$ is $+\infty$ (the leading coefficient of the factor is positive). So, there is at least one root in $[1,\infty]$ and since it is not possible for there to be two roots in the interval $[0,1]$, given the values at the endpoints, there must be exactly one root. For points in the unit square above this path (i.e., for fixed r, values of s greater than the value of the root), $p(r,s)$ is always negative; below, $p(r,s)$ is always positive. It is only in this *negative region* for $p(r,s)$ that the claim that $\partial g/\partial s$ is positive is left to be checked.

Figure 1 shows this important negative region (the curve Γ, indicated in the figure, is defined below).

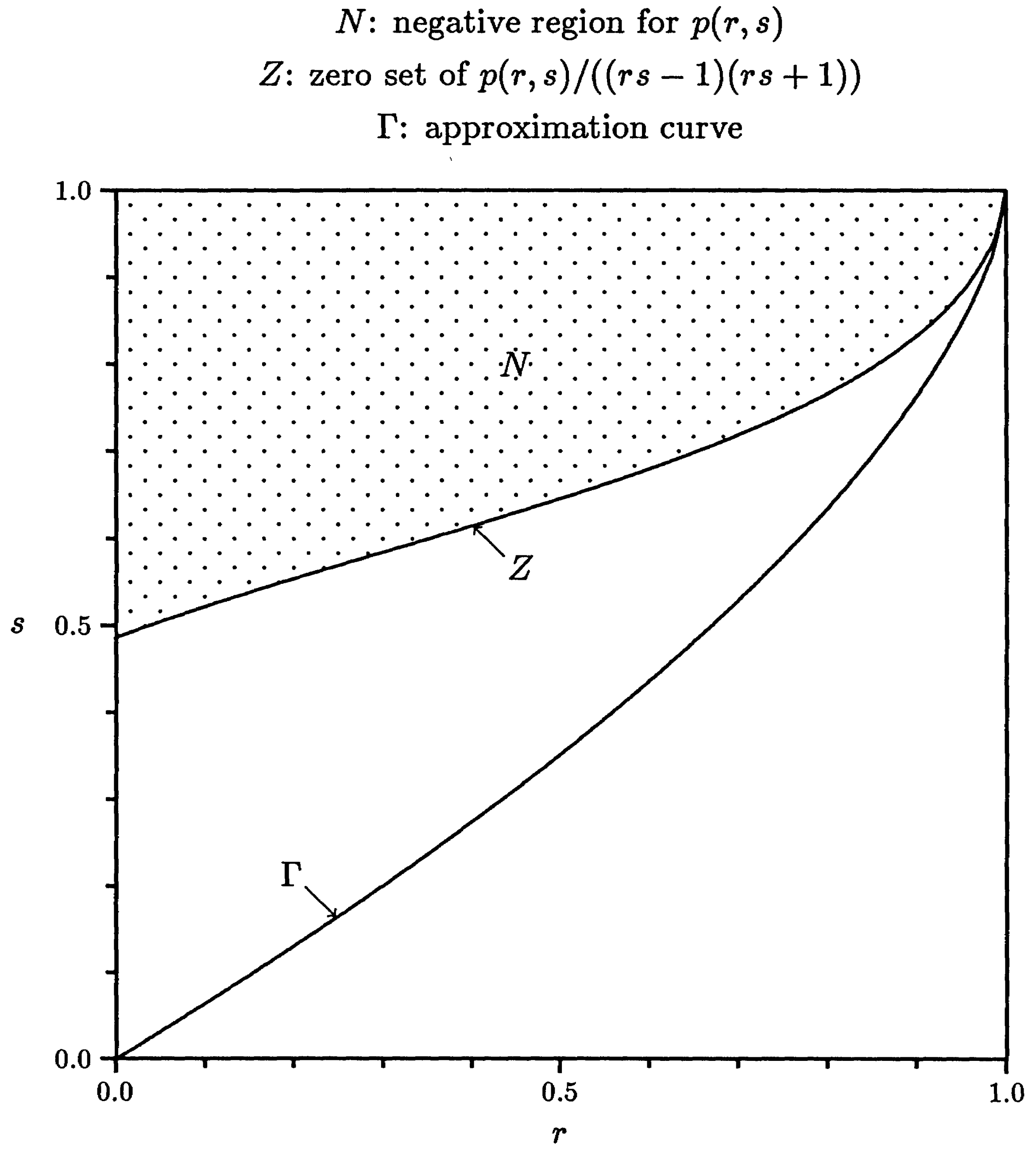

Figure 1

Now consider the curve Γ defined by

$$(1-s)^8=(1-r)^5$$

in the unit square. Along Γ, p is positive near $(1,1)$. To see this write p as a function of $((1-r),(1-s))$ (call it $\hat{p}(x,y)$ where $x=(1-r)$ and $y=(1-s)$):

$$\begin{aligned}\hat{p}(x,y)= &-(x-1)^4(4x^3-11x^2+8x-2)y^8\\ &+8(x-1)^4(4x^3-11x^2+8x-2)y^7\\ &-8(x-1)^3(14x^4-53x^3+66x^2-32x+6)y^6\\ &+16(x-1)^3(14x^4-54x^3+65x^2-26x+4)y^5-\end{aligned}$$

$$
\begin{aligned}
&-(280x^7-1951x^6+5452x^5-7796x^4+6016x^3-2416x^2+448x-32)y^4\\
&+4(x-2)x(56x^5-287x^4+542x^3-456x^2+160x-16)y^3\\
&-2(x-2)^2x^2(8x-13)(7x^2-12x+4)y^2\\
&+4(x-2)^3x^2(8x^2-13x+2)y\\
&-4(x-2)^4x^3\\
=&-(xy-y-x)(xy-y-x+2)\\
&\times(4x^5y^6-19x^4y^6+34x^3y^6-29x^2y^6+12xy^6-2y^6-24x^5y^5\\
&\qquad+114x^4y^5-204x^3y^5+174x^2y^5-72xy^5+12y^5+60x^5y^4\\
&\qquad-289x^4y^4+514x^3y^4-418x^2y^4+156xy^4-24y^4-80x^5y^3\\
&\qquad+396x^4y^3-696x^3y^3+512x^2y^3-144xy^3+16y^3+60x^5y^2\\
&\qquad-310x^4y^2+544x^3y^2-352x^2y^2+48xy^2-24x^5y+132x^4y\\
&\qquad-240x^3y+144x^2y+4x^5-24x^4+48x^3-32x^2).
\end{aligned}
$$

Now we are interested in showing that $\hat{p}(y^8,y^5)$ is positive for y near 0. As it happens $\hat{p}(y^8,y^5)$ is a complicated, but not intractable, single-variable high-degree polynomial:

$$
\begin{aligned}
&-y^{20}(y^8-y^3-1)(y^{13}-y^8-y^5+2)\\
&\quad\times(4y^{55}-24y^{50}-19y^{47}+60y^{45}+114y^{42}-80y^{40}+34y^{39}-289y^{37}\\
&\qquad+60y^{35}-204y^{34}+396y^{32}-29y^{31}-24y^{30}+514y^{29}-310y^{27}\\
&\qquad+174y^{26}+4y^{25}-696y^{24}+12y^{23}+132y^{22}-418y^{21}+544y^{19}\\
&\qquad-72y^{18}-24y^{17}+512y^{16}-2y^{15}-240y^{14}+156y^{13}-352y^{11}\\
&\qquad+12y^{10}+48y^9-144y^8+144y^6-24y^5+48y^3-32y+16).
\end{aligned}
$$

Examining this formula we have that, for y near 0, it is positive.

Consider any point (r,s) in the negative region for p described above and close to $(1,1)$. Since this point is *above* Γ, $(1-r)^5>(1-s)^8$, or in terms of x and y, $y<x^{5/8}$. Now $p(r,s)=\hat{p}(x,y)$ so using the expression for $\hat{p}$ to determine the behavior for small x (i.e. for r near 1), we obtain

$$
\begin{aligned}
-p(r,s)&=-\hat{p}(x,y)\\
&=k_1x^3+k_2x^2y+k_3xy^3+k_4y^4+O(y^5)\\
&\le k_1x^3+k_2x^{21/8}+k_3x^{23/8}+k_4x^{5/2}+O(x^4)\\
&\le k_5x^2x^{1/2}+k_6x^4
\end{aligned}
$$

for some strictly positive constants k_1, k_2, k_3, k_4, k_5, and k_6. Also, as is obvious from its definition, $-q(r,s)\ge k_7x^2$ for $k_7>0$, so we have that

$$
\begin{aligned}
-p(r,s)\log(1-r)-q(r,s)&\ge(k_5x^2x^{1/2}+k_6x^4)\log x+k_7x^2\\
&=x^2[(k_5+k_6x^{3/2})(x^{1/2})\log x+k_7]\\
&\ge 0
\end{aligned}
$$

for small x, since the function $x^{1/2}\log x$ tends to zero as x tends to zero. □

An Example of the New Condition

A simple example of an increasing sequence of moduli, which is not a Blaschke sequence but which satisfies the condition given in the corollary to the main theorem, can be given.

For $x \in [1, \infty)$ let

$$f(x) = \frac{1}{k} \int_1^x \frac{e^y}{y}\, dy,$$

where k is any constant greater than 4. Then $f: [1, \infty) \to [0, \infty)$ is onto and has a positive derivative. Hence f has a differentiable increasing inverse function, which we will call g. Now $g: [0, \infty) \to [1, \infty)$, and the r_j will be defined by:

$$r_j = 1 - e^{-g(j)}.$$

Clearly the r_j increase monotonically to 1.

Now

$$\sum_{j=1}^{\infty} (1 - |r_j|)^{1+\epsilon} = \sum_{j=1}^{\infty} e^{-g(j)(1+\epsilon)}$$

$$\leq \int_0^{\infty} e^{-g(x)(1+\epsilon)}\, dx.$$

Making the substitution $y = g(x)$ gives

$$\sum_{j=1}^{\infty} (1 - |r_j|)^{1+\epsilon} \leq \int_1^{\infty} e^{-y(1+\epsilon)} \frac{e^y}{ky}\, dy$$

$$= \frac{1}{k} \int_1^{\infty} \frac{e^{-y\epsilon}}{y}\, dy.$$

Now make the substitution $w = \epsilon y$:

$$\sum_{j=1}^{\infty} (1 - |r_j|)^{1+\epsilon} \leq \frac{1}{k} \int_{\epsilon}^{\infty} \frac{\epsilon e^{-w}}{w} \frac{1}{\epsilon}\, dw$$

$$= \frac{1}{k} \left[\int_{\epsilon}^{1} \frac{e^{-w}}{w}\, dw + \int_1^{\infty} \frac{e^{-w}}{w}\, dw \right]$$

$$< \frac{1}{k} \left[\int_{\epsilon}^{1} \frac{1}{w}\, dw + \int_1^{\infty} e^{-w}\, dw \right]$$

$$= \frac{1}{k} \left[\log\left(\frac{1}{\epsilon}\right) + \frac{1}{e} \right].$$

So,

$$\frac{\sum_{j=1}^{\infty} (1 - r_j)^{1+\epsilon}}{\log(1/\epsilon) + (1/e)} < \frac{1}{k}$$

and hence

$$\limsup_{\epsilon \to 0} \frac{\sum_{j=1}^{\infty} (1 - r_j)^{1+\epsilon}}{\log(1/\epsilon)} \leq \frac{1}{k} < \frac{1}{4}.$$

This shows that the $\{r_j e^{i\theta_j}\}_{j=1}^{\infty}$ will be an L_a^2 zero set for almost all choices of the arguments θ_j.

All that remains is to check that the sequence of r_j's defined above is not a Blaschke sequence. This is quite similar to the first part. We have

$$\sum_{j=1}^{\infty}(1-r_j)=\sum_{j=1}^{\infty}e^{-g(j)}$$
$$\geq \int_1^{\infty} e^{-g(x)}\,dx.$$

As above, using the substitution $y=g(x)$ gives

$$\sum_{j=1}^{\infty}(1-r_j)\geq \int_{g(1)}^{\infty} e^{-y}\frac{e^y}{ky}\,dy$$
$$=\frac{1}{k}\int_{g(1)}^{\infty}\frac{1}{y}\,dy$$
$$=\infty.$$

Thus the set $\{r_j e^{i\theta_j}\}_{j=1}^{\infty}$ is not a Blaschke sequence for any choice of the arguments θ_j.

Conclusion

One might ask whether the hypotheses in Theorem 3 are best possible. Proving that a constant greater than 1/4 can be used in the theorem would enlarge the zero sets considered. There are many obvious extensions of this result that come to mind. Are there easy generalizations to other Bergman spaces L_a^p? What are the corresponding results for the *weighted* Bergman spaces: those analytic functions f, defined for $0<p<\infty$ and $\alpha>-1$, such that

$$\int_{\mathbf{D}}|f(z)|^p(1-|z|)^{\alpha}\,dA(z)<\infty?$$

Of course, the fundamental question is: What are necessary and sufficient conditions for L_a^p zero sets? That will require a new method since the arguments of the zeros enter in an essential manner. There have been very few results in this area. Korenblum [6] has managed to describe the zero sets of some spaces related to the Bergman spaces, and perhaps his methods will apply.

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