Limits of Strongly Irreducible Operators, and the Riesz Decomposition Theorem

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1. Introduction

Let T be a (bounded linear) operator acting on a complex, separable, infinite-dimensional Hilbert space $\mathcal{H}$ and assume that the spectrum of T, $\sigma(T)$, is not connected. The Riesz decomposition theorem says that under these circumstances $\mathcal{H}$ can be written as the algebraic sum $\mathcal{H}_1 \dotplus \mathcal{H}_2$ of two nontrivial invariant subspaces of T; equivalently, T commutes with a nontrivial idempotent operator E. Furthermore, $E=E(T)$ can be written as a certain contour integral, and the upper semicontinuity of separate parts of the spectrum implies that every operator T' close enough to T commutes with a nontrivial idempotent $E'=E(T')$. Moreover, if T has the above property then the same is true for every operator WTW^{-1} similar to T, because $\sigma(WTW^{-1})=\sigma(T)$.

On the other hand, in [6] Gilfeather considered the class of all strongly irreducible operators defined by

$$\mathcal{SI}(\mathcal{H})=\{T\in\mathcal{L}(\mathcal{H}): T \text{ does not commute with any nontrivial idempotent}\}.$$

(Here $\mathcal{L}(\mathcal{H})$ denotes the algebra of all operators acting on $\mathcal{H}$.)

In this note we characterize the norm-closure $\mathcal{SI}(\mathcal{H})^-$ of the class $\mathcal{SI}(\mathcal{H})$. In a certain sense, this characterization can be considered as an "approximate inverse" of the Riesz decomposition theorem. Indeed, we have the following.

THEOREM.
$$\mathcal{SI}(\mathcal{H})^-=\{T\in\mathcal{L}(\mathcal{H}): \sigma(T) \textit{ is connected}\}.$$

Our introductory paragraph indicates that $\sigma(T)$ is necessarily connected for each T in $\mathcal{SI}(\mathcal{H})^-$; moreover, the class $\mathcal{SI}(\mathcal{H})$ (as well as its closure) is invariant under similarity. Thus, we must show only that every T in $\mathcal{L}(\mathcal{H})$ with a connected spectrum can be uniformly approximated by strongly irreducible operators.

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2. Some Examples of Strongly Irreducible Operators

The proof of the reverse inclusion in the Theorem will follow the same lines as the proof of Lemma 5.5 of [4]. Recall that a *Cauchy domain* is a non-empty bounded open subset Ω of $\mathbf{C}$ whose boundary consists of finitely many piecewise disjoint rectifiable Jordan curves; Ω is an *analytic* Cauchy domain if, in addition, the boundary consists of analytic Jordan curves. A connected Cauchy domain is called a *Cauchy region.*

LEMMA 1. *Suppose that*

$$R=\begin{bmatrix} A_1 & & & & & & & \\ & A_2 & & & & & 0 & \\ 0 & & \ddots & & & & & \\ & & & A_n & & & & \\ Q_{11} & Q_{12} & \cdots & Q_{1n} & C_1 & & & \\ Q_{21} & Q_{22} & \cdots & Q_{2n} & & C_2 & & \\ \vdots & \vdots & & \vdots & & 0 & \cdot & \\ Q_{m1} & Q_{m2} & \cdots & Q_{mn} & & & & C_m \end{bmatrix}$$

with respect to the orthogonal direct sum decomposition

$$\mathcal{H}=\left(\sum \bigoplus_{j=1}^{n} \mathfrak{N}_j\right)\oplus\left(\sum \bigoplus_{i=1}^{m} \mathfrak{M}_i\right),$$

where

(i) $\{\sigma(A_j)\}_{j=1}^{n}$ *and* $\{\sigma(C_i)\}_{i=1}^{m}$ *are two familes of pairwise disjoint compact sets such that*

$$\sigma(R)=\left[\bigcup_{j=1}^{n}\sigma(A_j)\right]\cup\left[\bigcup_{i=1}^{m}\sigma(C_i)\right]$$

and $\sigma(R)$ *is connected;*

(ii) interior $\sigma(R)$, interior $\sigma(C_i)$ $(i=1,2,\dots,m)$, *and* interior $\sigma(A_j)$ $(j=1,2,\dots,m)$ *are Cauchy regions;*

(iii) A_j *and* C_i *are strongly irreducible operators such that*

$$A_jX=XC_i\Rightarrow X=0;$$

(iv) *either* $\sigma(A_j)\cap\sigma(C_i)=\emptyset$ *and* $Q_{ji}=0$, *or* $\sigma(A_j)\cap\sigma(C_i)\neq\emptyset$ *and* Q_{ji} *does not belong to the range of the mapping*

$$Y\rightarrow\tau_{C_i,A_j}(Y):=C_iY-YA_j.$$

Then R is strongly irreducible.

The proof follows exactly as in the proof of [4, Lemma 5.5, pp. 67–70]. The condition "$A_jX=XC_i\Rightarrow X=0$" guarantees that every operator E commuting with R necessarily has a lower triangular matrix with respect to the given decomposition; moreover,

$$E = \begin{bmatrix} E_1 & & & & & & & \\ & E_2 & & & & 0 & & \\ & & \ddots & & & & & \\ 0 & & & E_n & & & & \\ & & & & E_1' & & & \\ & & * & & & E_2' & & \\ & & & & & & \ddots & \\ & & & & 0 & & & E_m' \end{bmatrix},$$

where E_j commutes with A_j ($j=1,2,\dots,n$) and E_i' commutes with C_i ($i=1,2,\dots,m$).

Furthermore, since A_j and C_i are strongly irreducible, an idempotent E commuting with R necessarily satisfies $E_j=0$ or 1 and $E_i'=0$ or 1. Finally, condition (iv) guarantees that $E=0$ or 1, whence we conclude that R is strongly irreducible. □

LEMMA 2. *Let $A, C \in \mathcal{L}(\mathcal{H})$. Assume that*

$$\mathcal{H} = \bigvee \{\ker(\lambda - C)^k : \lambda \in \Gamma,\, k \geq 1\}$$

for a certain subset Γ of the point spectrum $\sigma_p(C)$ of C, and $\sigma_p(A)\cap\Gamma=\emptyset$; then

$$AX = XC \Rightarrow X = 0.$$

Proof. Let p be a monic polynomial with zeros in Γ and let $x \in \mathcal{H}$ be any vector such that $p(C)x=0$; then $AX=XC$ implies

$$p(A)Xx = Xp(C)x = X0 = 0.$$

Since $p(A)$ is injective, we infer that $Xx=0$. It readily follows that

$$\ker X \supset \bigvee \{\ker(\lambda - C)^k : \lambda \in \Gamma,\, k \geq 1\} = \mathcal{H}.$$

Hence, $X=0$. □

Recall that $T \in \mathcal{L}(\mathcal{H})$ is semi-Fredholm if $\operatorname{ran} T$ is closed and either $\ker T$ or $\ker T^*$ is finite-dimensional; in this case, we define the index of T by $\operatorname{ind} T = \dim\ker T - \dim\ker T^*$. The reader is referred to [9] for references.

LEMMA 3. *Let Ω be a Cauchy region. Given n ($1 \leq n < \infty$), there exists $A = A(\Omega, -n) \in \mathcal{SI}(\mathcal{H})$ such that $\sigma(A) = \Omega^-$, $\sigma_e(A) = \partial\Omega$, and* $\ker(\lambda - A) = \{0\}$ *and* $\operatorname{ind}(\lambda - A) = -n$ *for all $\lambda \in \Omega$, where $\sigma_e(A)$ denotes the essential spectrum of the operator A.*

Proof. Let $H^2(\partial\Omega)$ denote the closure in $L^2(\partial\Omega, dm)$ (dm = linear Lebesgue measure on the boundary of Ω) of the rational functions with poles outside Ω^-, and let $M_+(\partial\Omega)$ = "multiplication by λ" on $H^2(\partial\Omega)$. It is well known (see, e.g., [8, Chap. 3]) that

$$\sigma(M_+(\partial\Omega))=\Omega^-, \qquad \sigma_e(M_+(\partial\Omega))=\partial\Omega,$$

and

$$\ker(\lambda-M_+(\partial\Omega))=\{0\} \quad\text{and}\quad \operatorname{ind}(\lambda-M_+(\partial\Omega))=-1 \quad \text{for all } \lambda\in\Omega.$$

Moreover, by Yoshino's theorem, the commutant $\mathfrak{A}'(M_+(\partial\Omega))$ of $M_+(\partial\Omega)$ consists of all operators of multiplication by functions in $H^\infty(\partial\Omega)$ [10]. Since Ω is a connected Cauchy region, it is not difficult to deduce that $M_+(\partial\Omega)$ is strongly irreducible. Thus, if $n=1$ then we can take $A=M_+(\partial\Omega)$.

Suppose $1<n<\infty$. In this case we define

$$A=\begin{bmatrix} B & 1 & & & & 0 \\ & B & 1 & & & \\ & & B & \ddots & & \\ & & & \ddots & & \\ & 0 & & & B & 1 \\ & & & & & B \end{bmatrix}$$

(with respect to the orthogonal direct sum $H^2(\partial\Omega)^{(n)}$ of n copies of $H^2(\partial\Omega)$), where $B=M_+(\partial\Omega)$.

Suppose $L=(L_{ij})_{i,j=1}^n\in\mathfrak{A}'(A)$; then

$$\begin{aligned}0&=AL-LA\\ &=\begin{bmatrix} [B,L_{11}]+L_{21} & & & & \\ [B,L_{21}]+L_{31} & & & * & \\ \vdots & & & & \\ [B,L_{n-1,1}]+L_{n1} & & & & \\ [B,L_{n1}] & [B,L_{n2}]-L_{n1} & [B,L_{n3}]-L_{n2} & \cdots & [B_1L_{nn}]-L_{n,n-1}\end{bmatrix},\end{aligned}$$

where $[B,C]=BC-CB$ and the (i,j)-entry for $1\le i<n$ and $1<j\le n$ is equal to $[B,L_{ij}]+L_{i+1,j}-L_{i,j-1}$.

The $(n,1)$-entry indicates that $L_{n1}\in\mathfrak{A}'(B)$, and the $(n,2)$-entry shows that

$$L_{n1}=[B,L_{n2}]=\delta_B(L_{n2})\in\operatorname{ran}\delta_B,$$

where δ_B is the inner derivation induced by B. Thus

$$L_{n1}\in\mathfrak{A}'(B)\cap\operatorname{ran}\delta_B.$$

We shall see later (Lemma 4 below) that $\mathfrak{A}'(B)\cap\operatorname{ran}\delta_B=\{0\}$, and therefore $L_{n1}=0$. Now the $(n-1,1)$- and $(n,2)$-entries show that $L_{n-1,1}$ and L_{n2} commute with B. By induction, we deduce that

$$L_{n1}=L_{n-1,1}=\cdots=L_{21}=L_{n2}=L_{n3}=\cdots=L_{n,n-1}=0$$

and $L_{11},L_{nn}\in\mathfrak{A}'(B)$. By a formal repitition of the same arguments, we infer that

$$L_{ij}=0 \quad \text{for } 1\le j<i\le n$$

and that

$$L_{ii}\in\mathfrak{A}'(B) \quad \text{for all } i=1,2,\dots,n.$$

Suppose that $E \in \mathcal{A}'(B)$ is idempotent. By replacing (if necessary) E by $1-E$, we can directly assume that $E_{11} \neq 0$. Since $E=E^2$ implies $E_{ii}=E_{ii}^2$ for all i, and since B is strongly irreducible, we deduce that $E_{ii}=1$ or 0 for all $i=1,2,\dots,n$; in particular, $E_{11}=1$.

The above matricial computation shows that

$$0=AE-EA=\begin{bmatrix} 0 & [B,E_{12}]+E_{22}-E_{11} & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix},$$

so that $E_{22}-E_{11}=E_{22}-1 \in \mathcal{A}'(B) \cap \operatorname{ran} \delta_B$. Once again, we deduce that $E_{22}=1$ (and therefore $E_{12}=0$ because E is idempotent).

By another inductive argument, we conclude that $E_{ii}=1$ for all i, and hence $E=1$. It follows that $A \in \mathcal{SI}(\mathcal{H})$. The other properties follow by straightforward computations. □

In order to complete the proof of Lemma 3, we must show that

$$\mathcal{A}'(B) \cap \operatorname{ran} \delta_B = \{0\},$$

where $B=M_+(\partial\Omega)$. Indeed, we have a stronger result.

LEMMA 4. *$\mathcal{A}'(B)$ is orthogonal to* $\operatorname{ran} \delta_B$, *in the sense of Banach spaces; that is,*

$$\|R-\delta_B(L)\| \geq \|R\|$$

for all $R \in \mathcal{A}'(B)$ and all $L \in \mathcal{L}(\mathcal{H})$.

Proof. Clearly, $\ker \delta_B = \mathcal{A}'(B)$, and (by using Yoshino's theorem [10]) this algebra contains no nonzero compact operators. Furthermore, if $\pi: \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ denotes the canonical projection of $\mathcal{L}(\mathcal{H})$ onto the quotient Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, and if $R=$ "multiplication by ϕ" ($\phi \in H^\infty(\partial\Omega)$) commutes with B, then

$$\|\pi(R)\|=\|R\|=\|\phi\|_\infty.$$

(Here $\mathcal{K}(\mathcal{H})$ denotes the ideal of all compact operators.)

Recall that $B=M_+(\partial\Omega)$ is a rationally cyclic subnormal operator [8, Chap. 3]. The Berger–Shaw theorem implies that B is essentially normal, that is, $m=\pi(B)$ is a normal element of the Calkin algebra [3].

Let $\rho: \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_\rho)$ be a faithful unital $*$-representation. Since $M=\rho(m)$ is normal, a result of Anderson indicates that

$$\|A\| \leq \|A-\delta_M(X)\|$$

for all $A \in \mathcal{A}'(M)$ and all $X \in \mathcal{L}(\mathcal{H}_\rho)$ [1, Thm. 1.7].

It readily follows that for each $R \in \mathcal{A}'(B)$ and each $L \in \mathcal{L}(\mathcal{H})$,

$$\begin{aligned} \|R\| = \|\rho\circ\pi(R)\| &\leq \|\rho\circ\pi(R)-\delta_M(\rho\circ\pi(L))\| \\ &= \|\rho\circ\pi[R-\delta_B(L)]\| \leq \|R-\delta_B(L)\|. \end{aligned}$$

Hence, $\mathcal{A}'(B)$ is orthogonal to $\operatorname{ran} \delta_B$. □

LEMMA 5. *Given an analytic Cauchy region* Ω *and* $\eta > 0$ *small enough so that the complements of* Ω^- *and*

$$\Lambda(\Omega, \eta) := \bigcup\{\Omega^- + r\eta : 0 \le r \le 1\}$$

have exactly the same number of components, there exists $A = A(\Omega, \eta, -\infty) \in \mathcal{SI}(\mathcal{H})$ *such that* $\sigma(A) = \Lambda(\Omega, \eta)$, *the left essential spectrum* $\sigma_{le}(A)$ *of* A *coincides with* $\bigcup\{\partial\Omega + r\eta : 0 \le r \le 1\}$, *and*

$$\ker(\lambda - A) = \{0\} \quad \textit{and} \quad \operatorname{ind}(\lambda - A) = -\infty$$

for all $\lambda \in \Omega \setminus \sigma_{le}(A)$.

Proof. The operator L of [7, Lemma 3] satisfies all our requirements. Indeed, the operator constructed in this reference has the right spectral properties and its double commutant, $\mathcal{A}''(L) = [\mathcal{A}'(L)]'$, is a maximal Abelian strictly cyclic subalgebra of $\mathcal{L}(\mathcal{H})$. More precisely: (1) $\mathcal{A}''(L) = \mathcal{A}'(L)$ coincides with the algebra of all multiplications by the elements of a suitable Hilbert space of smooth functions in two variables t, λ ($0 \le t \le 1$, and λ runs over a certain compact subset of $\mathbf{C}$); (2) $L =$ "multiplication by λ" on this space; and (3) $\mathcal{H} = \mathcal{A}''(L)e_0$, where $e_0(t, \lambda) \equiv 1$ is a strictly cyclic vector for the algebra. Moreover, if Ω is connected, then $\mathcal{H}$ is a space of continuous functions defined on a *connected* subset of $[0,1] \times \Lambda(\Omega, \eta)$, whence we readily infer that L commutes with no nontrivial idempotent. Hence, L is strongly irreducible. □

COROLLARY 6. *Let* Ω *be an analytic Cauchy region. Given* n ($1 \le n < \infty$), *there exists* $A = A(\Omega, n) \in \mathcal{SI}(\mathcal{H})$ *such that* $\sigma(A) = \Omega^-$, $\sigma_e(A) = \partial\Omega$, *and*

$$\ker(\lambda - A)^* = \{0\} \quad \textit{and} \quad \operatorname{ind}(\lambda - A) = n$$

for all $\lambda \in \Omega$.

Moreover, if $\eta > 0$ *is small enough to guarantee that the complements of* Ω^- *and* $\Lambda(\Omega, \eta)$ *have exactly the same number of components, then there exists* $A = A(\Omega, \eta, \infty) \in \mathcal{SI}(\mathcal{H})$ *such that* $\sigma(A) = \Lambda(\Omega, \eta)$,

$$\sigma_{re}(A) = \bigcup\{\partial\Omega + r\eta : 0 \le r \le 1\},$$

and

$$\ker(\lambda - A)^* = \{0\} \quad \textit{and} \quad \operatorname{ind}(\lambda - A) = \infty$$

for all $\lambda \in \Omega \setminus \sigma_{re}(A)$.

Proof. For $1 \le n < \infty$, define $A(\Omega, n) = A(\Omega^*, -n)^*$, where $\Omega^* = \{\bar{\lambda} : \lambda \in \Omega\}$ and $A(\Omega^*, -n)$ is defined as in Lemma 3. For $n = \infty$, $A(\Omega, \eta, \infty)$ is similarly defined by using Lemma 5. □

LEMMA 7. *Let* Ω *be an analytic Cauchy region. Then there exists* $A = A(\Omega, 0) \in \mathcal{SI}(\mathcal{H})$ *such that* $\sigma(A) = \Omega^-$, $\sigma_e(A) = \partial\Omega$, *and*

$$\dim\ker(\lambda - A) = \dim\ker(\lambda - A)^* = 1$$

for all $\lambda \in \Omega$.

Proof. Define

$$A=\begin{pmatrix} M_+(\partial\Omega^*)^* & X \\ 0 & M_+(\partial\Omega) \end{pmatrix}$$

(with respect to $H^2(\partial\Omega)\oplus H^2(\partial\Omega)$). Since

$$\sigma_e(M_+(\partial\Omega^*)^*)=\sigma_e(M_+(\partial\Omega))=\partial\sigma_e(M_+(\partial\Omega^*)^*)$$
$$=\partial\sigma_e(M_+(\partial\Omega))=\partial\Omega,$$

it follows from [5, Thm. 5] that X can be chosen so that

$$X\notin \operatorname{ran}\tau_{M_+(\partial\Omega^*)^*,M_+(\partial\Omega)}$$

($\tau_{B,C}$ is defined exactly as in Lemma 1).

Since $M_+(\partial\Omega)$ and $M_+(\partial\Omega^*)^*$ are strongly irreducible, it follows from Lemmas 1 and 2 that so is A. (Indeed, $\sigma_p(M_+(\partial\Omega))=\phi$, while $H^2(\partial\Omega)=\bigvee\{\ker(\omega-M_+(\partial\Omega^*)^*)^k: k\geq 1\}$ for each $\omega\in\Omega$.) □

We close this section with a standard result on approximation of operators (see [8, Chap. 3]).

LEMMA 8. *Given $T\in\mathcal{L}(\mathcal{H})$ and $\epsilon>0$, there exists $T_\epsilon\in\mathcal{L}(\mathcal{H})$ such that $\|T-T_\epsilon\|<\epsilon$, $\sigma_{lre}(T_\epsilon):=\sigma_{le}(T)\cap\sigma_{re}(T)$ is the closure of an analytic Cauchy domain Φ such that*

$$\sigma_{lre}(T)\subset\Phi\subset\{\lambda\in\mathbf{C}:\operatorname{dist}[\lambda,\sigma_{lre}(T)]\leq\epsilon\},$$

and

$$\operatorname{ind}(\lambda-T_\epsilon)=\operatorname{ind}(\lambda-T),$$
$$\dim\ker(\lambda-T_\epsilon)^k=\dim\ker(\lambda-T)^k,\quad \textit{and}$$
$$\dim\ker(\lambda-T_\epsilon)^{*k}=\dim\ker(\lambda-T)^{*k}$$

for all $\lambda\notin\sigma_{lre}(T_\epsilon)$ and all $k\geq 1$.

In particular, the number of components of $\sigma(T_\epsilon)$ is finite and cannot exceed the number of components of $\sigma(T)$.

3. Proof of the Main Result

Now we are in a position to prove the Theorem. Suppose $T\in\mathcal{L}(\mathcal{H})$ and $\sigma(T)$ is a connected set. Given $\epsilon>0$, we construct T_ϵ and Φ as in Lemma 8. Clearly, $\sigma(T_\epsilon)$ is connected. Let Ω be an analytic Cauchy domain such that $\Omega^-\subset\Phi$ and $\mathbf{C}\backslash\Omega^-$ and $\mathbf{C}\backslash\Phi^-$ have exactly the same number of components.

Let $\Omega_1,\Omega_2,\ldots,\Omega_n$ be an enumeration of the components of $\sigma(T_\epsilon)\backslash\Omega^-$. Let η $(0<\eta<\epsilon)$ be small enough so that $(\Omega_j)_\eta\cap(\Omega_k)_\eta=\emptyset$. By using Lemmas 3, 5, and 7 and Corollary 6, we define A_j as follows: if

$$n=\operatorname{ind}(\lambda-T)=\operatorname{ind}(\lambda-T_\epsilon)\neq 0,\pm\infty\quad\text{for all }\lambda\in\Omega_j,$$

then $A_j=A(\Omega_j,n)$; if $\operatorname{ind}(\lambda-T_\epsilon)=0(\lambda\in\Omega_j)$ then $A_j=A(\Omega_j,0)$; and finally, if $\operatorname{ind}(\lambda-T_\epsilon)=-\infty$ (resp., ∞) for all $\lambda\in\Omega_j$, then $A_j=A(\Omega_j,\eta,-\infty)$ (resp., $A_j=A(\Omega_j,\eta,\infty)$).

Observe that if $j \neq k$ then

$$\sigma(A_j) \cap \sigma(A_k) \subseteq \Lambda(\Omega_j, \eta) \cap \Lambda(\Omega_k, \eta) \subset (\Omega_j)_\eta \cap (\Omega_k)_\eta = \emptyset.$$

The open set $\Omega \backslash \sigma(\sum \bigoplus_{j=1}^n A_j)$ has finitely many components, $\Psi_1, \Psi_2, \ldots, \Psi_m$, and these components are (not necessarily analytic) Cauchy regions. Define $C_i = M_+(\Psi_i^*)^*$, $i = 1, 2, \ldots, m$; if $i \neq h$, then

$$\sigma(C_i) \cap \sigma(C_h) = \Psi_i^- \cap \Psi_h^- = \emptyset.$$

If R is defined as in Lemma 1 (for some $Q = (Q_{ji})$), then

$$\sigma(R) = \left[\bigcup_{j=1}^{n} \sigma(A_j)\right] \cup \left[\bigcup_{i=1}^{m} \sigma(C_i)\right],$$

and this set is connected and coincides with $\Omega^- \cup [\sigma(T_\epsilon) \backslash \sigma_{lre}(T_\epsilon)]$. Thus, $\{\sigma(A_j)\}_{j=1}^m$, $\{\sigma(C_i)\}_{i=1}^m$, and R satisfy (i) and (ii) of Lemma 1.

Since $H^2(\partial \Psi_i) = \bigvee \{\ker(\omega_i - C_i)^k : k \geq 1\}$ and $\omega_i \notin \sigma(A_j)$ for each $\omega_i \in \Psi_i$, we deduce from Lemma 2 that (iii) of Lemma 1 is satisfied.

By construction, $\sigma(A_j) \cap \sigma(C_i) = \partial\sigma(A_j) \cap \partial\sigma(C_i) = \sigma_l(A_j) \cap \sigma_r(C_i)$, where $\sigma_l(\cdot)$ and $\sigma_r(\cdot)$ denote (respectively) the left and right spectrum of the operator. Thus, by using [5] (or [8, Thm. 3.19]), we can construct $Q = (Q_{ji})$ so that (iv) of Lemma 1 is also satisfied.

It readily follows that $R \in \mathcal{SI}(\mathcal{H})$. Moreover, our construction shows that

$$\sigma(R) \subset \sigma(T_\epsilon) \quad \text{and} \quad \sigma_{lre}(R) \subset \sigma_{lre}(T_\epsilon),$$

$\sigma(T_\epsilon)$ and $\sigma(R)$ are connected sets, and each component of $\sigma_{lre}(T_\epsilon)$ meets $\sigma_{lre}(R)$. Further,

$$\operatorname{ind}(\lambda - R) = \operatorname{ind}(\lambda - T_\epsilon), \qquad \dim \ker(\lambda - R)^k \leq \dim \ker(\lambda - T_\epsilon)^k,$$

and

$$\dim \ker(\lambda - R)^{*k} \leq \dim \ker(\lambda - T_\epsilon)^{*k}$$

for all $\lambda \in \sigma(T_\epsilon) \backslash \sigma_{lre}(T_\epsilon)$ and all $k \geq 1$.

The similarity orbit theorem [2, Thm. 9.2] implies that T_ϵ can be uniformly approximated by operators R_ϵ similar to R. Hence, there exists R_ϵ similar to R such that

$$\|T - R_\epsilon\| \leq \|T - T_\epsilon\| + \|T_\epsilon - R_\epsilon\| < 2\epsilon.$$

Since $R_\epsilon \in \mathcal{SI}(\mathcal{H})$ and ϵ can be chosen arbitrarily small, we conclude that $T \in \mathcal{SI}(\mathcal{H})^-$. □

If $T \in \mathcal{L}(\mathbf{C}^d)$ and $\sigma(T) = \{\lambda\}$, then T belongs to the closure of the similarity orbit of $\lambda + q_n$, where q_n denotes the nilpotent Jordan cell of order n, which is strongly irreducible [8, Chap. 2]. By combining this observation with the Theorem and Lemma 8, we can easily derive the following consequence.

COROLLARY 9. *Given $T \in \mathcal{L}(\mathcal{H})$, there exists a sequence $\{T_n\}_{n=1}^\infty$ in $\mathcal{L}(\mathcal{H})$ such that $\|T - T_n\| \to 0 (n \to \infty)$, and T_n is similar to a finite direct sum of strongly irreducible operators; moreover, if $\sigma(T)$ only has a finite number* *m of components, then the T_n's can be chosen so that each of them has exactly m direct summands.*

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