A Picard Theorem for Projective Varieties

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In an earlier paper [3], the author proved a Picard theorem for maps of the entire plane $\mathbf{C}$ into a certain algebraic variety. The method of that paper seemed not to go to the heart of the matter, and it was obscure how far the theorem could be generalized. In the present paper we prove a more general Picard theorem by a method that seems to be the appropriate one for the problem, and we give examples to show that our result is nearly the best possible. The main element of our new method is a parametrization of a monomial variety that is familiar to students of toric varieties ([5, Introduction], [9, Prop. 1.2]). The use we make of this parametrization is of course completely different from theirs.

Let V be an irreducible algebraic variety in $\mathbf{CP}^n$, possibly singular. Let $\Pi_0, \dots, \Pi_n$ be independent hyperplanes of $\mathbf{CP}^n$. Consider a holomorphic map $f: \mathbf{C} \to V$ such that the image of f does not meet $\Pi_0, \dots, \Pi_n$. Suppose that none of the sections $V \cap \Pi_k$, $k = 0, \dots, n$, is contained in the union of the others. Then a theorem of Green [2, p. 66] (for related results see Lang [6; 7]) asserts that f is *algebraically degenerate* in the sense that its image lies in a proper hypersurface section of V.

When can this conclusion be strengthened to say that the image of f lies in a section by a hyperplane? Green gives an example [2, p. 62], similar to our Example 1, to show that this is not always so. In the present paper Theorem 2 gives a sufficient condition in terms of the intersections of V with $\Pi_0, \dots, \Pi_n$, and Theorem 3 relates this to our earlier paper [3]. The technical result needed to obtain Theorem 2 is Theorem 1. We apply the Borel lemma to show that the Zariski closure Z of $f(\mathbf{C})$ is a monomial variety and then parametrize it in the manner to which we referred above. Our result is almost that Z is a *toric variety* [9, Thm. 1.4], but the definition of a toric variety requires such a variety to be normal, which is irrelevant for our purposes. It would be possible to apply Theorem 1 to obtain refinements of Theorem 2 that referred to higher orders of contact, but in the absence of applications we have not discussed these.

When I wrote the first version of this paper I was not aware that the construction was standard in the theory of toric varieties. I am indebted to an anonymous referee for the suggestion that my argument must correspond to

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something in the literature. For my invitation to Washington University, where the first version was written, I thank G. R. Jensen. The 1989 Roever Lectures at Washington University, in which W. Fulton spoke on toric varieties, stimulated my interest in the subject.

The proof of Theorem 1 will rely on value-distribution theory, but will not require any new estimates. The analytical part is contained in the Borel lemma in its original version. We remark that Picard theorems for other domains can be obtained by applying other versions of the Borel lemma, as Green has done [2]. We now state the Borel lemma in the form in which we shall use it.

BOREL LEMMA ([7, Chap. VII, Thm. 1.1], [8, Art. 57]). *Let* $g_0, \dots, g_m$: $\mathbf{C} \to \mathbf{C}^*$ *be nowhere-zero holomorphic functions satisfying the identity*

$$g_0 + \cdots + g_m = 0. \tag{1}$$

Then there is a partition of the indices $0, \dots, m$ *such that each subset* E *of the partition has these properties:*

(1) *There exists a function* $h_E: \mathbf{C} \to \mathbf{C}^*$ *such that, for each* $j \in E$, $g_j = a_j h_E$ *identically for some constant* a_j.
(2) *There holds the identity*

$$\sum_{j \in E} g_j = \sum_{j \in E} a_j h_E = 0. \tag{2}$$

Briefly, the Borel lemma asserts that the only identities of the form (1) among nowhere-zero holomorphic functions are the trivial ones.

THEOREM 1. *Let* $\Pi_0, \dots, \Pi_n$ *be* $n+1$ *independent hyperplanes in* $\mathbf{CP}^n$ *and let* $f: \mathbf{C} \to \mathbf{CP}^n$ *be a holomorphic curve that does not meet* $\Pi_0, \dots, \Pi_n$. *Let* Z *be the Zariski closure of* $f(\mathbf{C})$. *Then there is an algebraic torus* T *and an isomorphism*

$$i: T \to Z - \bigcup_{k=0}^{n} \Pi_k$$

such that the translations of T *extend to an action of* T *on* Z.

An *algebraic torus* is by definition the Cartesian product of finitely many copies of $\mathbf{C}^*$, considered as an algebraic group ([4, p. 104], [5, p. 1], [9, p. 4]).

Proof. Take homogeneous coordinates $x_0, \dots, x_n$ on $\mathbf{CP}^n$ so that the coordinate hyperplanes are $\Pi_0, \dots, \Pi_n$. We shall show that the Zariski closure Z of $f(\mathbf{C})$ can be defined by *monomial equations* in $x_0, \dots, x_n$, where a monomial equation is one having exactly two terms.

If $Z = \mathbf{CP}^n$ the result is immediate. Otherwise, take any finite set of equations defining Z. Consider any one of these equations, say of degree d,

$$\sum_{p(0)+\cdots+p(n)=d} a_{p(0),\dots,p(n)} x_0^{p(0)} \cdots x_n^{p(n)} = 0. \tag{3}$$

Let $(f_0, \ldots, f_n)$ be an expression for f in the coordinates $(x_0, \ldots, x_n)$. Since $f(\mathbf{C})$ does not meet the coordinate planes, each f_k is a nowhere-zero holomorphic function. Therefore any product of d of the f_k is a nowhere-zero holomorphic function. Since $f(\mathbf{C})$ lies in Z, the f_k satisfy (3) at every point, and we have the identity

$$\sum_{p(0)+\cdots+p(n)=d} a_{p(0),\ldots,p(n)} f_0^{p(0)} \cdots f_n^{p(n)} = 0. \tag{4}$$

We apply the Borel lemma to (4), letting each g_j be a term in (4) with a nonzero coefficient. The Borel lemma yields a set of identities in the f_k, each of which is of the form

$$f_0^{p(0)} \ldots f_n^{p(n)} = c f_0^{q(0)} \ldots f_n^{q(n)}, \tag{5}$$

where c is a constant and the sets of indices $(p(0), \ldots, p(n))$ and $(q(0), \ldots, q(n))$ are distinct. The identity (5) gives the equation of an algebraic locus

$$x_0^{p(0)} \ldots x_n^{p(n)} = c x_0^{q(0)} \ldots x_n^{q(n)} \tag{6}$$

that contains $f(\mathbf{C})$. From each such identity we obtain a locus; because of the identity (2) in the Borel lemma, the intersection of these loci is contained in the hypersurface defined by the original equation (3).

Let

$$Y = Z - \bigcup_{k=0}^{n} \Pi_k.$$

We now proceed to show that Y can be identified with an algebraic torus. The argument that follows appears to be standard in the theory of toric varieties [5, Introduction], but we have not found a convenient reference for it.

We start from a finite set of monomial equations defining Z and successively eliminate the x_k. The typical monomial equation is equation (6), in which we may assume without loss of generality that $p(n) > q(n)$. Solving for x_n, we have

$$x_n = (c x_0^{q(0)-p(0)} \ldots x_{n-1}^{q(n-1)-p(n-1)})^{1/(p(n)-q(n))}. \tag{7}$$

If there are any more equations, we substitute for x_n in them using equation (7), obtaining a new set of monomial equations in $x_0, \ldots, x_{n-1}$ which may contain fractional powers. The effect of substituting for x_n may be to make some of the equations trivial, in the sense that they now have the same monomial on each side. If any equations remain after discarding these trivial equations, we repeat the process to eliminate another variable (without loss of generality) x_{n-1}; then we eliminate x_{n-2}, and so on.

At the end of this process we have a set of equations

$$\begin{aligned} x_k &= b_k x_0^{r(k,0))} \ldots x_{k-1}^{r(k,k-1)} \\ x_{k+1} &= b_{k+1} x_0^{r(k,0)} \ldots x_k^{r(k+1,k)} \\ &\vdots \\ x_n &= b_n x_0^{r(n,0)} \ldots x_{n-1}^{r(n,n-1)} \end{aligned}$$

for some k, $1 \leq k \leq n$. We may substitute back to eliminate $x_k, \ldots, x_{n-1}$ from the right-hand side and obtain equations of the form

$$
\begin{aligned}
x_k &= c_k x_0^{s(k,0)} \ldots x_{k-1}^{s(k,k-1)} \\
x_{k+1} &= c_{k+1} x_0^{s(k+1,0)} \ldots x_{k-1}^{s(k+1,k-1)} \\
&\vdots \\
x_n &= c_n x_0^{s(n,0)} \ldots x_{k-1}^{s(n,k-1)}.
\end{aligned} \tag{8}
$$

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$$S = \mathbf{CP}^n - \bigcup_{k=0}^{n} \Pi_k \cong (\mathbf{C}^*)^n$$

as an algebraic torus with $(1, \ldots, 1, c_k, \ldots, c_n)$ as the unit element. We use the equations (8) to define an algebraic homomorphism $\phi: (\mathbf{C}^*)^k \to S$ such that $\operatorname{im} \phi = Y$. For $j = 0, \ldots, k-1$, let $\delta(j)$ be the lowest common denominator of $s(k, j), \ldots, s(n, j)$. The components of ϕ in the coordinate system $x_0, \ldots, x_n$ will be denoted $\phi_0, \ldots, \phi_n$. We define ϕ by setting

$$\phi_i((t_0, \ldots, t_{k-1})) = \begin{cases} t_i^{\delta(i)}, & i = 0, \ldots, k-1, \\ c_i t_0^{\delta(0)s(i,0)} \ldots t_{k-1}^{\delta(k-1)s(i,k-1)}, & i = k, \ldots, n. \end{cases} \tag{9}$$

Then ϕ is an algebraic homomorphism, and so ϕ induces an isomorphism of algebraic varieties between Y and $T = (\mathbf{C}^*)^k / \ker \phi$, which is an algebraic torus [4, p. 101]. It is clear from the formulas (9) that the translations of T extend to an action of T on Z. □

THEOREM 2. *Let $V \subset \mathbf{CP}^n$ be an irreducible algebraic variety. Let $f: \mathbf{C} \to V$ be a holomorphic curve that does not meet $n+1$ independent hyperplanes $\Pi_0, \ldots, \Pi_n$ of $\mathbf{CP}^n$. Suppose that the image of f does not lie in any proper linear subspace of $\mathbf{CP}^n$. Then, after reordering the hyperplanes $\Pi_0, \ldots, \Pi_n$, the line L_i has contact of order at least 2 with V at the point p_i, for $i = 0, 1$, where*

$$L_0 = \bigcap_{k=2}^{n} \Pi_k, \qquad L_1 = \bigcap_{k=0}^{n-2} \Pi_k,$$

$$p_0 = \bigcap_{k=1}^{n} \Pi_k, \qquad p_1 = \bigcap_{k=0}^{n-1} \Pi_k.$$

Before proving Theorem 2, we give three examples showing that the conclusion cannot be greatly improved. Note that in particular the conclusion implies that the intersection with V of the union of the Π_k, counting multiplicities, cannot be a divisor with normal crossings.

Example 1. The twisted cubic given with respect to the inhomogeneous parameter t by $(1, t, t^2, t^3)$ lies on the smooth quadric surface $Q \subset \mathbf{CP}^3$ defined by the equation

$$x_0 x_2 - x_1^2 + x_0 x_3 - x_1 x_2 - x_1 x_3 + x_2^2 = 0.$$

By setting $t=e^z$ we obtain a full holomorphic curve $f:\mathbf{C}\to Q$ such that $f(\mathbf{C})$ does not meet the coordinate planes. In this example none of the sections of Q by the coordinate planes is contained in the union of the others. The tangent planes to the surface Q at $(1,0,0,0)$ and $(0,0,0,1)$ do not coincide with any of the coordinate planes, which shows that the conclusion of Theorem 2 cannot be strengthened to include a condition on tangent planes.

Example 2. The construction of Example 1 can be performed with curves other than the twisted cubic. The cuspidal quartic $(1,t,t^2,t^4)$ lies on the smooth quadric surface Q given by

$$x_0x_2-x_1^2+x_0x_3-x_2^2=0,$$

and the holomorphic curve determined by setting $t=e^z$ does not meet the coordinate planes. However, the configuration of this example is somewhat special in that the plane $x_0=0$ is tangent to Q at $(0,0,0,1)$.

Example 3. We now describe an example where the Zariski closure of the holomorphic curve is 2-dimensional. The Veronese surface V is the embedding of $\mathbf{CP}^2$ in $\mathbf{CP}^5$ given in terms of the inhomogeneous coordinates (s,t) on $\mathbf{CP}^2$ by $(1,s,t,s^2,st,t^2)$. The surface V lies on the smooth quadric hypersurface $Q\subset\mathbf{CP}^5$ given by

$$x_0x_3-x_1^2+x_0x_5-x_2^2+x_3x_5-x_4^2=0.$$

Setting $s=e^z$ and $t=e^{\sqrt{2}z}$, we obtain a holomorphic curve $f:\mathbf{C}\to Q$ such that $f(\mathbf{C})$ does not meet the coordinate hyperplanes and the Zariski closure of $f(\mathbf{C})$ is V.

Proof of Theorem 2. We might use the map ϕ constructed in the proof of Theorem 1, but it seems better to derive Theorem 2 from Theorem 1 itself.

Let ϵ be the composite map

$$T\xrightarrow{i}Z-\bigcup_{k=0}^{n}\Pi_k\subset\mathbf{CP}^n.$$

Take homogeneous coordinates $(x_0,\dots,x_n)$ on $\mathbf{CP}^n$ so that the coordinate hyperplanes are $\Pi_0,\dots,\Pi_n$. Let $(\epsilon_0,\dots,\epsilon_n)$ be an expression for ϵ in the coordinates $(x_0,\dots,x_n)$. Since ϵ does not meet $\Pi_0,\dots,\Pi_n$, the regular functions $\epsilon_0,\dots,\epsilon_n$ have no zeros, and so they are constant multiples of characters on T.

The ratios $\rho_{jk}=\epsilon_k/\epsilon_j$ $(0\le j<k\le n)$ are constant multiples of characters on T, and by the assumption that $f(\mathbf{C})$ is not contained in a proper linear subspace of $\mathbf{CP}^n$ none of the ρ_{jk} is constant. Let $U\subset T$ be a 1-parameter subgroup such that none of the ρ_{jk} is constant on U. Let u be a coordinate on $U\cong\mathbf{C}^*$. We have

$$\epsilon_i(u)=B_iu^{\beta(i)},\quad i=0,\dots,n, \tag{10}$$

for some nonzero constants B_i and integers $\beta(i)$.

Since none of the ρ_{jk} is constant on U, the $\beta(i)$ are all distinct. Reorder the indices $(0, \ldots, n)$ so that the $\beta(i)$ are a strictly increasing sequence. As in the theorem, let

$$L_0 = \bigcap_{k=2}^{n} \Pi_k = \{(x_0, \ldots, x_n) : x_2 = \cdots = x_n = 0\},$$

$$L_1 = \bigcap_{k=0}^{n-2} \Pi_k = \{(x_0, \ldots, x_n) : x_0 = \cdots = x_{n-2} = 0\};$$

$$p_0 = \bigcap_{k=1}^{n} \Pi_k = (1, 0, \ldots, 0),$$

$$p_1 = \bigcap_{k=0}^{n-1} \Pi_k = (0, \ldots, 0, 1).$$

We see from (10) that the Zariski closure ζ of $\epsilon(U)$ passes through p_0 and its tangent at p_0 is L_0. Since ζ lies on V, the ideal of V is contained in the ideal of ζ; this implies that the tangent to ζ at p_0 is tangent to V at p_0. Therefore L_0 has contact of order at least 2 with V at p_0. By a similar argument, L_1 has contact of order at least 2 with V at p_1. This completes the proof of Theorem 2. □

In our work on the Gauss map of a minimal surface, the conclusion was that a certain holomorphic curve lay in a linear subspace of codimension 2. We now prove a more general theorem with that conclusion, from which Theorem 2 of [3] may be derived. We shall need an application of the Borel lemma due to Green.

THEOREM [1, Thm. 2]. *Let $h: \mathbf{C} \to \mathbf{CP}^m$ be a holomorphic curve that omits $m+2$ distinct hyperplanes. Then the image of h lies in a proper linear subspace of $\mathbf{CP}^m$.*

THEOREM 3. *Let $V \subset \mathbf{CP}^n$ be an irreducible algebraic variety and let $\Pi_0, \ldots, \Pi_n$ be independent hyperplanes of $\mathbf{CP}^n$. Suppose that V does not pass through the intersection of any n of $\Pi_0, \ldots, \Pi_n$. Let $f: \mathbf{C} \to V$ be a holomorphic curve that does not meet $\Pi_0, \ldots, \Pi_n$. Then the image of f lies in a linear space of $\mathbf{CP}^n$ of codimension 2.*

Proof. By Theorem 2, $f(\mathbf{C})$ lies in a hyperplane Λ of $\mathbf{CP}^n$. We must show that $f(\mathbf{C})$ is a linearly dependent subset of Λ.

Each of the hyperplanes Π_k $(k = 0, \ldots, n)$ meets Λ in a hyperplane P_k of Λ. The image of f does not meet any of the P_k. If the P_k are all distinct, then the theorem of Green given above proves that the image of f is a linearly dependent subset of Λ.

Hence if $f(\mathbf{C})$ does not lie in a hyperplane of Λ then two of the P_k must coincide, say $P_0 = P_1$. Then $P_1, \ldots, P_n$ are independent, because $\Pi_0, \ldots, \Pi_n$ were independent. We may now consider f as a holomorphic curve mapping into some component W of $V \cap \Lambda$ and not meeting the linear spaces

$P_1, \ldots, P_n$. By Theorem 2, if $f(\mathbf{C})$ does not lie in a hyperplane of Λ, the variety W must pass through two distinct points p_0 and p_1, each of which is the intersection of $n-1$ of the linear spaces $P_1, \ldots, P_n$. Either p_0 or p_1 must lie on P_1 and hence be the intersection of n of the linear spaces $\Pi_0, \ldots, \Pi_n$. Because this contradicts the hypotheses, Theorem 3 follows. □

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