Space-Preserving Composition Operators

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1. Introduction

If φ is an analytic function mapping the unit disk Δ into itself, and if f belongs to the Hardy class H^p, then the composition $(f\circ\varphi)$ belongs to H^p also. This was first pointed out by Littlewood [7]. Our object here is to consider when a reverse implication may hold. That is, let $H(\Delta)=H$ be the topological vector space of functions holomorphic on Δ and let V be a subspace of H. We ask the following question: *What are the holomorphic functions φ mapping Δ into Δ, such that whenever $f\in H$ and $(f\circ\varphi)\in V$ it follows that $f\in V$?* A function φ satisfying this condition will be said to *possess property* $(*)$ *relative to the subspace V.*

It is immediately clear that if φ_1 and φ_2 possess property $(*)$ then so does $(\varphi_1\circ\varphi_2)$. We will show in Example 5 of the next section that φ_1 and $(\varphi_1\circ\varphi_2)$ may possess property $(*)$ even if φ_2 does not. As a first example, a linear fractional transformation mapping Δ onto Δ clearly possesses property $(*)$ relative to the H^p spaces, BMOA, and the disk algebra. Further, if φ_1 is a linear fractional transformation mapping Δ onto Δ, then $(\varphi_1\circ\varphi_2)$ possesses property $(*)$ relative to the H^p spaces, BMOA, or the disk algebra if and only if φ_2 does.

Ryff [9] proved the following theorem related to our question: *Let f be nonconstant and analytic on Δ. Let φ be analytic on Δ with $\varphi(0)=0$ and $|\varphi|<1$. Then $\|f\|_p=\|f\circ\varphi\|_p$ if and only if φ is inner.* Later, Nordgren [8] showed that *if φ is an inner function, then φ possesses property $(*)$ relative to H^p. And, the composition operator C_φ is norm-preserving on H^p ($\|f\|_p=\|f\circ\varphi\|_p$) if and only if $\varphi(0)=0$.*

In this paper we introduce a family of functions φ mapping Δ into Δ for which $(*)$ holds for the H^p spaces, BMOA, and the disk algebra. Our maps can be factored as a finite Blaschke product times a nonconstant outer function, and hence have modulus strictly less than 1 on arcs of $\partial\Delta=\mathbf{T}$ of positive measure. In addition to satisfying $(*)$, the composition operators associated with these maps provide examples illustrating results of spectral properties of C_φ as studied by Cowen [2; 3].

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2. Some Examples

We begin with some examples that illustrate how the geometry of the image of a function φ in the unit ball of H^∞ is related to the question of whether it possesses property $(*)$ relative to the H^p spaces. In fact, in order for a function to possess property $(*)$ relative to the H^p spaces, its image will have to contain most points near the unit circumference. Yet, a function may map onto the entire disk and not possess property $(*)$. A further purpose for including these examples is the motivation they provide for our later construction.

EXAMPLE 1. Let $\varphi(z)=z/2$. Then φ does not possess property $(*)$ since for any $f\in H$, $(f\circ\varphi)$ is continuous on the closed unit disk whereas f may belong to no H^p space. More generally, suppose that φ maps Δ into Δ but omits an entire neighborhood of some boundary point ζ of Δ. If $f(z)=1/(z-\zeta)$, then $(f\circ\varphi)$ is bounded whereas f does not belong to H^1.

EXAMPLE 2. Let r $(0<r<1)$ be fixed and let φ denote a universal covering map of Δ onto the annulus $A=\{r<|z|<1\}$. Let f be analytic on Δ and suppose that $(f\circ\varphi)$ is in H^p for some $p>0$. There exists a harmonic function $v(z)$ on Δ such that $|f\circ\varphi(z)|^p<v(z)$ for all z in Δ. By the theory of covering maps there exists a function V, harmonic on A, such that $V(\varphi)=v$. Thus $|f(w)|^p<V(w)$ and hence f belongs in $H^p(A)$. Gauthier and Hengartner [6] have shown that if a function is locally in H^p at each boundary point of the unit circle, then it belongs to H^p of the entire disk Δ. Hence, any universal covering map of Δ onto the annulus $A=\{r<|z|<1\}$ possesses property $(*)$ relative to H^p, even though its image omits the entire disk $\{|z|\le r\}$.

EXAMPLE 3. As mentioned earlier, the work of Nordgren [8] shows that all inner functions possess property $(*)$ relative to the H^p spaces. It is also the case that inner functions possess property $(*)$ relative to BMOA. This may be shown using the equivalence of BMO to $(H^1)^*$. Suppose that $g\in H^2$, $f\in H^1$, φ is inner, and $\varphi(0)=0$. We further assume that $(f\circ\varphi)\in$ BMOA. Then

$$\left|\int_{|\zeta|=1}\bar{f}g\,dm\right|=\left|\int_{|\zeta|=1}\bar{f}g\,dm\varphi^{-1}\right|=\left|\int_{|z|=1}(\overline{f\circ\varphi})(g\circ\varphi)\,dm\right|$$
$$\le C(f\circ\varphi)\|g\circ\varphi\|_1=C(f\circ\varphi)\|g\|_1,$$

where $C(f\circ\varphi)$ is a constant depending only on $(f\circ\varphi)$. Since this inequality holds for all $g\in H^2$, we conclude that $f\in$ BMOA. Now if $\varphi(0)\neq 0$ then the above argument, with φ replaced by $\psi=S\circ\varphi$ (where S is a linear fractional transformation taking Δ onto Δ and satisfying $S(\varphi(0))=0$), shows that ψ possesses property $(*)$ relative to BMOA. Since S is invertible, we conclude that φ also possesses property $(*)$ relative to BMOA.

EXAMPLE 4. Let a be fixed, $0<a<\pi$. For z in Δ, define

$$\varphi_a(z)=\frac{K((1+z)/(1-z))^{a/\pi}-1}{K((1+z)/(1-z))^{a/\pi}+1},\quad \text{where } K=e^{i(\pi-a)/2}.$$

Now, φ_a maps Δ in a one-to-one manner onto a lens domain in the disk bounded by the upper half of the unit circle and a circular arc in the disk making an angle a with the unit circle at ± 1. Let $f(w)=1/(w-1)$. The function f is in H^p for all $p<1$, but f is not in H^1. The function

$$f\circ\varphi_a(z)=\frac{1}{\varphi_a(z)-1}\doteq\frac{K((1+z)/(1-z))^{a/\pi}+1}{-2}$$

belongs to H^p for $0<p<\pi/a$, and so each of these functions $(f\circ\varphi_a)$ belongs to H^1 even though f does not. Hence, φ_a does not possess property (*) relative to the H^p spaces.

EXAMPLE 5. Next, let $\psi_a=(\varphi_a)^2$, where φ_a is defined as in Example 4. The function ψ_a maps the disk into the disk and for various choices of a has the following mapping behavior.

(i) If $0<a<\pi/2$ then ψ_a maps Δ onto a crescent with multiple point at $z=1$. The valence is 1 for points in the crescent and 0 for points in the disk but not in the crescent. The angle formed by the unit circle and the internal boundary curve of the crescent is a.

(ii) If $a=\pi/2$ then ψ_a maps Δ onto the disk with the segment $[0,1)$ removed, and each point in the range is covered once.

(iii) If $\pi/2<a<\pi$ then the mapping ψ_a maps the disk onto the disk. There is a crescent in the disk with $\{|z|=1\}$ in the boundary of the crescent and in this crescent the valence of ψ_a is 1. The interior of this crescent is covered twice and the angle formed by the lower boundary curve of this crescent with the upper unit semi-circle is a.

Again, with $f(w)=1/(w-1)$ we see that

$$f(\psi_a)(z)=\frac{1}{[\varphi_a(z)]^2-1}=-\frac{K}{4}\left(\frac{1+z}{1-z}\right)^{a/\pi}-\frac{1}{2}-\frac{1}{4K}\left(\frac{1+z}{1-z}\right)^{a/\pi}.$$

Hence, $f(\psi_a)$ is in H^p for $0<p<\pi/a$ but f is in H^p only for $0<p<1$. Thus ψ_a does not possess property (*) relative to the H^p spaces.

Notice, however, that if $n>2$ then $(\varphi_a)^n$ does possess property (*) relative to the H^p spaces, even though φ_a and $(\varphi_a)^2$ do not. This may be seen as follows. We observe that for $n>2$, each t ($|t|=1$) has at least one preimage point s with the property that $(\varphi)^n$ takes a neighborhood of s to a neighborhood of t in a one-to-one manner. Thus, if $(f\circ\varphi)$ belongs to H^p then f belongs to H^p in some neighborhood of each boundary point, and the work of Gauthier and Hengartner [6] again shows that f belongs to H^p of the entire disk Δ. This implies that $(\varphi_a)^n$ possesses property (*) if $n>2$ relative to the spaces H^p. For example, if $n=3$, $(\varphi_a)^3$ is the composition of the inner function z^3 with φ_a. This composition possesses property (*) relative to the H^p spaces even though φ_a does not.

3. Our Construction

3.1. *A 2-valent example.* For w in the upper half-plane, define

$$\psi(w)=k\left(w-\frac{1}{w}\right)+\operatorname{Log}(w).$$

Then a simple calculation shows that $\operatorname{Im}(\psi(w))>0$, so ψ maps the upper half-plane into the upper half-plane. For w on the positive real axis, $\psi(w)$ is real while $\operatorname{Im}(\psi(w))=\pi$ on the negative real axis. In order that the image of ψ contain the entire upper half-plane, we fix $k>\frac{1}{2}$. Then the real part of ψ increases from $-\infty$ to $+\infty$ as w increases from $-\infty$ to 0. As w increases from 0 to $+\infty$, the real part of ψ again increases from $-\infty$ to $+\infty$. Thus, for w in the upper half-plane, $\psi(w)$ covers the strip $\{0<\operatorname{Im}(\zeta)\leq\pi\}$ exactly once and the half-plane above the line $\{\operatorname{Im}(\zeta)=\pi\}$ exactly twice. To get a function from Δ to Δ, put $\varphi=U\circ\psi\circ S$, where

$$S(z)=i\left(\frac{1+z}{1-z}\right) \quad\text{and}\quad U(\zeta)=S^{-1}(\zeta)=\left(\frac{\zeta-i}{\zeta+i}\right).$$

Then φ takes the lower semi-circle in a one-to-one manner onto the entire boundary of Δ, whereas the upper semi-circle is taken by φ to a circle lying inside Δ which is tangent to the boundary at the point $z=1$ (the image under U of the line $\{\operatorname{Im}(\zeta)=\pi\}$). Points inside this internally tangent circle are taken on twice, while points inside the unit disk but outside this internally tangent circle are covered exactly once. A straightforward computation shows that

$$\varphi'(z)=\frac{4(iz^2+4kz-i)}{[(-2k-1)+(-2k+1)z^2+i(1-z^2)\operatorname{Log}(i[(1+z)/(1-z)])]^2}.$$

The denominator is both bounded and bounded away from zero, and thus φ' is bounded. In order for φ' to equal zero, z must equal $(2k\pm\sqrt{4k^2-1})i$. Therefore, if k is real and $k>\frac{1}{2}$, then φ' is never zero on $|z|=1$. Consequently, on the boundary of the unit disk, the derivative of φ is both bounded and bounded away from zero.

We now show that the function φ just constructed possesses property (*) relative to the H^p spaces. So, suppose that f is analytic on Δ and that $(f\circ\varphi)$ belongs to H^p for some $p>0$. We wish to prove that f belongs to H^p. Define

$$N(|f|^p,e^{i\theta})=\sup\{|f(z)|^p\colon z\in S_\rho(e^{i\theta},\gamma)\},$$

where

$$S_\rho(e^{i\theta},\gamma)=\{z\colon(1-|z|)\leq\rho \quad\text{and}\quad |\operatorname{Arg}(1-e^{-i\theta}z)|\leq\gamma\}.$$

Then, for fixed ρ and γ, $f\in H^p$ if and only if $N(|f|^p,\cdot)\in L^1$. So we will compare $\int N(|f|^p,\cdot)$ with $\int N(|f\circ\varphi|^p,\cdot)$.

Since $\varphi(\{e^{i\alpha}: -\pi \le \alpha \le 0\}) = \{e^{i\theta}: 0 \le \theta \le 2\pi\}$, we have

$$\int_{-\pi}^{0} N(|f\circ\varphi|^p, e^{i\alpha})\, d\alpha = \int_0^{2\pi} N(|f\circ\varphi|^p, \varphi^{-1}e^{i\theta}) \frac{1}{|\varphi'(\varphi^{-1}(e^{i\theta}))|}\, d\theta.$$

Now $|\varphi'|$ is bounded. To complete the argument, we compare

$$N(|f\circ\varphi|^p, \varphi^{-1}e^{i\theta}) \quad \text{with} \quad N(|f|^p, e^{i\theta}).$$

To do this, we will show that

$$\varphi^{-1}[S_r(e^{i\theta}, \beta)] \subset S_\rho(\varphi^{-1}(e^{i\theta}), \gamma)$$

for appropriate choices of r, β, ρ, and γ. This is routine when $e^{i\theta}$ is bounded away from 1 since φ is analytic and one-to-one in a full neighborhood of $\varphi^{-1}(e^{i\theta})$.

Let $\mathfrak{N}$ be a small neighborhood of 1 intersected with $\{|w|<1\}$. Let $\mathcal{R}^*$ and $\mathcal{B}^*$ be the two components of $\varphi^{-1}(\mathfrak{N})$ (near -1 and $+1$, respectively), and put $\mathcal{R} = \varphi(\mathcal{R}^*)$ and $\mathcal{B} = \varphi(\mathcal{B}^*)$. Then $\partial\mathcal{R} \cap \{|w|=1\}$ is the arc from 1 to $e^{i\alpha*}$ $(\alpha* > 0)$ and $\partial\mathcal{B} \cap \{|w|=1\}$ is the arc from 1 to $e^{-i\alpha*}$.

Let $g = (\varphi|_{\mathcal{R}^*})^{-1}$, the inverse of the restriction of φ to $\mathcal{R}^*$. Then Theorem IX.6 of Tsuji [12, p. 358] implies that $\arg(\varphi')$ is continuous on $\mathcal{R}^*$ away from the "corners" of $\partial\mathcal{R}^*$. Tsuji uses this theorem to prove Lemma 1 [12, p. 359] which, in our situation, implies that $g(S_\rho(e^{i\theta}, \gamma)) \supset S_\delta(g(e^{i\theta}), \gamma - \epsilon)$, where δ and ϵ can be chosen independently of θ, $0 \le \theta \le \alpha*/2$. A small modification of his proof also shows that $g(S_r(e^{i\theta}, \beta)) \subset S_\rho(g(e^{i\theta}), \beta + \epsilon)$, where ρ and ϵ can be chosen independently of θ, $0 \le \theta \le \alpha*/2$.

To estimate $N(|f|^p, e^{i\theta})$ for $-\alpha*/2 \le \theta \le 0$, use $\mathcal{B}$ and $\mathcal{B}^*$ in place of $\mathcal{R}$ and $\mathcal{R}^*$.

The obvious estimate now shows that $\int N(|f|^p, e^{i\theta})\, d\theta < +\infty$, and $f \in H^p$ as we wanted.

3.2. *Examples with higher valence.* To produce examples like φ which have property $(*)$ but have higher valence, replace $\psi(w)$ in the above construction by

$$\psi_n(w) = \sum_{k=1}^{n} \left\{ c_k \left(\frac{w-a_k}{w-a_{k+1}} - \frac{w-a_{k+1}}{w-a_k} \right) + \alpha_k \operatorname{Log}\left(\frac{w-a_k}{w-a_{k+1}} \right) \right\},$$

where $a_{n+1} < a_n < \cdots < a_2 < a_1$ are real and $2c_k > \alpha_k > 0$ for $k = 1, 2, \ldots, n$. Observe that each term of this sum is a positive constant that is multiplied by $\psi((w-a_k)/(w-a_{k+1}))$. Thus, each term has the same mapping behavior as ψ except that, in its domain, the negative and positive real axes are replaced by the interval $[a_{k+1}, a_k]$ and its complement, respectively. The imaginary part of the kth term equals $\pi\alpha_k$ on (a_{k+1}, a_k) and 0 on the complement of $[a_{k+1}, a_k]$. Thus, if we require that $0 < \alpha_n < \alpha_{n-1} < \cdots < \alpha_1$, then the valence of F_n is 1 on the strip $\{0 < \operatorname{Im}(\zeta) \le \pi\alpha_n\}$, 2 on the strip $\{\pi\alpha_n < \operatorname{Im}(\zeta) \le \pi\alpha_{n-1}\}, \ldots$, and n on the half-plane above the line $\{\operatorname{Im}(\zeta) = \pi\alpha_1\}$. Now put $\varphi_n = U \circ F_n \circ S$, where S and U are as before. There are now n nested circles, internally tangent to Δ at the point $z = 1$, such that the valence of φ_n goes from 1 to n as one goes step by step from the boundary of the unit circle into each successive circle. The derivative of φ_n is still bounded and bounded away from zero on the boundary of the unit disk, and φ_n again possesses property (*) relative to the H^p spaces.

3.3. *Spectral properties of* C_φ. We shall now point out some spectral and smoothness properties of the composition operators C_φ (defined by $C_\varphi(f)=f\circ\varphi$) for the functions $\varphi=\varphi_n$ just constructed. The operator C_φ is one-to-one and bounded below on H^p. The functions φ all have radial derivatives on some arcs of the unit circle and so, by Shapiro and Taylor [11], the composition operators C_φ are not compact. We have the following information relating to the work of Cowen [2; 3]. If we choose φ so that φ has no fixed point in the unit disk and say $\varphi(1)=1$, then we know [2, p. 89] that $C_\varphi(f)=\lambda f$ has an infinite number of eigenvectors for each complex λ. So, for example, if $S(z)=i((1+z)/(1+z))$, $U=S^{-1}$, and $F(w)=w-1/w+\text{Log}(w)$, then $\varphi(z)=U(F(S(z)))$ maps the disk into the disk, with $\varphi(1)=1$. We claim φ has no fixed point in the disk. To see this, observe that it is sufficient to prove that F has no fixed point in the upper half-plane. This is readily checked. We define $\mathbf{T}_j$ inductively as follows: $\mathbf{T}_0=\{t:|t|=1, 0<\arg(t)<\pi\}$ and, for $j>0$, $\mathbf{T}_j=\varphi^{-1}(\mathbf{T}_{j-1})$. We see that any eigenvectors for this problem are analytic on $\bigcup_0^\infty \mathbf{T}_j$.

If, however, we define $G(w)=1/(2+\pi/2)F(w)$ (where F is as in the preceding paragraph) and then set $\psi(z)=U(G(S))(z)$, then $\psi(0)=0$. Further, $\psi'(0)=-(i/(2+\pi/2))$ and so $0<|\psi'(0)|<1$. Thus zero is an attractive fixed point. Again by a result in Cowen [2, p. 89], the eigenvalues are given by $\lambda=(\psi'(0))^n$. Once more, the eigenvectors in this case are analytic over $\bigcup_0^\infty \mathbf{T}_j$.

3.4. *Property* (*) *for BMOA and the disk algebra.* We show in this section that the functions we have constructed possess property (*) relative to both the disk algebra and the class BMOA (the linear BMO space of analytic functions on Δ, which is the dual of H^1). Let φ be a mapping as constructed above.

PROPOSITION. *If* $(f\circ\varphi)$ *is in the disk algebra, then* f *is in the disk algebra.*

Proof. The proof is the observation that f is clearly continuous at any point $t\neq 1$ since φ has a unique inverse near such a point. If $t=1$, we check that the cluster set of f at 1 has at most n points. Since the cluster set must be connected, it is a singleton and hence f is continuous at 1. □

To show that a similar result holds for BMOA, we recall a criterion that implies that a holomorphic function on the disk belongs to BMOA. Let $\Omega(h,t)$ be a Carleson set of length h and with center point t on the unit circle. A function F is in BMOA if

$$\iint_{\Omega(h,t)}(1-|z|^2)|F'(z)|^2\,dx\,dy$$

is uniformly bounded for all $h>0$ and all $|t|=1$ [5, p. 240].

THEOREM. *The function $f\circ\varphi(z)=F(z)$ is in BMOA if and only if f is in BMOA.*

Proof. To show that if a function f belongs to BMOA then so does $(f\circ\varphi)$, use the fact that an analytic function belongs to BMOA if and only if it may be expressed as the sum g_1+ig_2, where g_1 and g_2 are analytic and both $\mathrm{Re}(g_1)$ and $\mathrm{Re}(g_2)$ are bounded. Upon taking the composition, one concludes that $(f\circ\varphi)$ belongs to BMOA also.

To prove the converse, suppose that $(f\circ\varphi)$ belongs to BMOA. It suffices to check the integral condition for f over Carleson sets $\Omega(h,1)$. Divide the Carleson set into two pieces, say Ω^+ and Ω^-, where

$$\Omega^+=\{z\in\Omega(h,1):\mathrm{Im}(z)>0\}$$

and Ω^- is defined analogously. Consider the integral

$$\int\int_{\Omega^+}(1-|w|^2)|f'(w)|^2\,du\,dv.$$

We choose a single-valued inverse of φ on this set and let Λ^+ be the preimage of Ω^+ under this inverse. The set Λ^+ is contained in a Carleson rectangle, say $\Omega(h',s)$, and we may choose h' (by the distortion bounds on φ) so that $0<a<|(h'/h)|<(1/a)$, where a depends on φ. Now

$$\begin{aligned}&\int\int_{\Omega^+}(1-|w|^2)|f'(w)|^2\,dx\,dy\\ &\quad=\int\int_{\Lambda^+}\left(\frac{1-|\varphi(z)|^2}{1-|z|^2}\right)(1-|z|^2)|f'(\varphi(z))|^2|\varphi'(z)|^2\,du\,dv\\ &\quad\le\mathrm{Const.}\int\int_{\Omega(h',s)}(1-|z|^2)|F'(z)|^2\,du\,dv.\end{aligned}$$

By our assumption, $F=(f\circ\varphi)$ is in BMOA and so this quantity is bounded. A similar estimate holds on Ω^-, and we conclude that f belongs to BMOA, as was to be proved. □

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