On the BP Homology and Cohomology of $P^{2n} \wedge P^{2m}$

GEORGE NAKOS

1. Statement of Results

Let BP be the Brown–Peterson spectrum associated with the prime 2 and let $\mathrm{BP}_*(\ )$ and $\mathrm{BP}^*(\ )$ be the corresponding reduced homology and cohomology theories. Let P^{2n} be the $2n$-dimensional real projective space. There is a Künneth short exact sequence due to Landweber [3] for both $\mathrm{BP}_*(P^{2n} \wedge P^{2m})$ and $\mathrm{BP}^*(P^{2n} \wedge P^{2m})$ which is split exact in this case. For instance, for the BP-cohomology one has

$$\mathrm{BP}^*(P^{2n} \wedge P^{2m}) = \Sigma^{-1} \mathrm{Tor}_{\mathrm{BP}^*}(\mathrm{BP}^*(P^{2n}), \mathrm{BP}^*(P^{2m})) \oplus \mathrm{BP}^*(P^{2n}) \otimes_{\mathrm{BP}^*} \mathrm{BP}^*(P^{2m}). \tag{1}$$

The tensor product module is well understood. It is the ideal generated by xy in the polynomial algebra $\mathrm{BP}^*[x,y]$ modulo the ideal $(([2]x)y, x([2]y))$, where $[2]x$ denotes the two-series in x. Furthermore, the tensor product has been computed as an abelian group in each degree larger than $2\max\{m,n\}$ [1; 2]. This computation has led to a strong non-immersion theorem for real projective spaces into Euclidean spaces [2].

Our goal in this note is to compute the Tor groups as BP-modules. We shall prove the following propositions.

PROPOSITION 1. $\mathrm{BP}^{\mathrm{odd}}(P^{2n} \wedge P^{2m}) = \Sigma^{-1} \mathrm{Tor}_{\mathrm{BP}^*}(\mathrm{BP}^*(P^{2n}), \mathrm{BP}^*(P^{2m}))$ *is isomorphic as a* BP^**-module to a copy of* $\Sigma^{2\max\{m,n\}-1}\mathrm{BP}^*(P^{2\min\{m,n\}})$.

PROPOSITION 2. $\mathrm{BP}_{\mathrm{odd}}(P^{2n} \wedge P^{2m}) = \Sigma^{1} \mathrm{Tor}^{\mathrm{BP}_*}(\mathrm{BP}_*(P^{2n}), \mathrm{BP}_*(P^{2m}))$ *is isomorphic as a* BP_**-module to a copy of* $\Sigma^{2}\mathrm{BP}_*(P^{2\min\{m,n\}})$.

We shall prove Proposition 1 in detail. The dual computation for homology follows the same line of proof and only a brief sketch will be given. As a by-product of the computation we get all of the v_1-torsion of the tensor product. Explicitly, we have the following corollary.

COROLLARY 9. *The* v_1*-torsion submodule of* $\mathrm{BP}^*(P^{2n}) \otimes_{\mathrm{BP}^*} \mathrm{BP}^*(P^{2m})$ *is the ideal generated by* $xy(x-y)$.

Received March 6, 1989. Revision received June 23, 1989.
The author has been partially supported by a Naval Academy Research Center grant.
Michigan Math. J. 37 (1990).

2. Method and Proof

We shall assume that $m \geq n$ throughout this section.

First we recall some standard notation along with some well-known facts [4]. The coefficient ring BP* is isomorphic to $\mathbf{Z}_{(2)}[v_1, v_2, \ldots]$. The degree $(=-\text{codegree})$ of each generator v_n is $-2(2^n-1)$. $\mathrm{BP}^*(CP^\infty)$ is the power series over BP* on a 2-dimensional generator x and $\mathrm{BP}^*(CP^n)$ is a truncated polynomial algebra over BP* with $x^{n+1}=0$. The inclusion $P^{2n} \subseteq P^{2n+1}$ composed with the circle bundle projection $P^{2n+1} \to CP^n$ defines a nontrivial map $P^{2n} \to CP^n$. By using the Atiyah–Hirzebruch spectral sequence, it is easy to see that x maps nontrivially in $\mathrm{BP}^2(P^{2n})$ and that the induced map is an epimorphism in BP*( ). The relation $x^{n+1}=0$ holds in $\mathrm{BP}^*(P^{2n})$ (here, x is the image of x). On the other hand, the composite map

$$P^{2n} \to P^\infty \to CP^\infty \xrightarrow{2} CP^\infty$$

is trivial and x maps to zero and to $[2]x$, the two-series. Thus $[2]x=0$ in $\mathrm{BP}^*(P^{2n})$. It is easy to show that

$$\mathrm{BP}(P^{2n}) = (x) \subseteq \mathrm{BP}^*[x]/(x^{n+1}, [2]x). \tag{2}$$

Full details are in Lemma 3.5 of [1].

Our approach to computing the Tor modules is a direct one. We use the definition of the Tor and tensor products. Equation (2) implies that the map $f\colon B^0 \to B^1$ is a BP*-free resolution for $\mathrm{BP}^*(P^{2n})$, where $f(x^i) = ([2]x)x^{i-1}$ with $B^0 = B^1$ BP*-free on x^i, $i = 1, 2, \ldots, n$. It is very convenient to maintain the multiplicative notation throughout (so xx^k is x^{k+1}, etc.). If we tensor with $\mathrm{BP}^*(P^{2m})$ over BP*, we have

$$d \equiv f \otimes 1\colon A^0 \equiv B^0 \otimes_{\mathrm{BP}^*} \mathrm{BP}^*(P^{2m}) \to A^1 \equiv B^1 \otimes_{\mathrm{BP}^*} \mathrm{BP}^*(P^{2m}). \tag{3}$$

The kernel and cokernel of d are

$$\mathrm{Tor}_{\mathrm{BP}^*}(\mathrm{BP}^*(P^{2n}), \mathrm{BP}^*(P^{2m})) \quad \text{and} \quad \mathrm{BP}^*(P^{2n}) \otimes_{\mathrm{BP}^*} \mathrm{BP}^*(P^{2m}),$$

respectively. We may define the complex $(A^*, d)\colon 0 \to A^0 \to A^1 \to 0$. Then what we need is the homology $H(A^*, d)$.

B^1 and B^0 are formally isomorphic and all the groups are finite in each degree. Therefore the orders of Tor and $\otimes$ are equal in each degree. One way of proving Proposition 1 is to produce enough elements of Tor and then compare the order of the submodule generated by these elements with the order of $\otimes$ in each degree. If they are equal, then one has all of Tor. Even more easily, one can filter the complex (A^*, d), compute the homology of the associated graded object, and "lift" all the cycles of the graded d to cycles of d itself.

Let $\mathrm{BP}^*(P^{2m}) = \mathrm{BP}^*[y]/(y^{m+1}, [2]y)$, and let F^kA^e $(e = 0, 1)$ be the BP*-submodule of A^e generated by x^iy^j, $i + j \geq k$. This defines a finite decreasing filtration of A^*:

$$F^{n+m+1}A^* = \{0\} \subseteq \cdots \subseteq F^2A^* = A^*.$$

In (3), d is filtration-preserving and we derive the usual spectral sequence (E_r^{**}, d_r) of the filtered complex. This spectral sequence is of cohomological type with differentials of bidegree $(r, 1-r)$. Its E_∞-term is the graded module of $H(A^*, d)$ with respect to the induced filtration. In standard notation, we have

(4) $$E_0^{k,l} = F^k A^{k+l}/F^{k+1}A^{k+l}, \qquad E_1^{k,l} = H^{k+l}(F^k A^*/F^{k+1}A^*),$$

and

(5) $$E_\infty^{k,l} = F^k H^{k+l}(A^*, d)/F^{k+1}H^{k+l}(A^*, d).$$

Note that $E_r^{k,l} = 0$ if $k+l \neq 0$ or 1. So we may assume that $k+l$ is either 0 or 1.

All the differentials are induced by d. More precisely, they all follow from relation (7) below. If $[2]y = \sum_{0 \le s} a_s y^{s+1}$, then we know that

(6) $$a_s \in \mathrm{BP}_{-2s}, \qquad a_0 = 2, \qquad a_1 \equiv v_1 \bmod (2).$$

For $x^i y^j \in A^0$ we have

(7) $$\begin{aligned} d(x^i y^j) &= ([2]x)x^{i-1}y^j = \Big(\sum_{0 \le s} a_s x^{s+1}\Big)x^{i-1}y^j \\ &= \Big(\sum_{1 \le s} a_s x^s\Big)x^i y^j + 2x^i y^j = x^i y^j \sum_{1 \le s} a_s (x^s - y^s). \end{aligned}$$

Therefore, if $i+j=k$ then

(8) $$d(x^i y^j) = v_1 x^i y^j (x-y) + P(x,y),$$

where $P(x,y)$ is a polynomial in x and y with terms of filtration greater than $k+1$.

LEMMA 1. *If $k+l=0$ or 1, then $E_0^{k,l} \approx E_1^{k,l}$ is isomorphic to a* BP*/(2)-*free module generated by $x^i y^{k-i}$, with i and k in the following ranges:*

(i) $1 \le i \le k-1$ *if* $k \le m, n+1$;
(ii) $1 \le i \le n$ *if* $n+1 < k \le m$; *and*
(iii) $k-m \le i \le n$ *if* $m < k \le m+n$.

Proof. It is trivial to check that the $x^i y^{k-i}$'s generate $E_0^{k,l}$ (for $k+l=0$ or 1) in the above ranges. Moreover, the only relations come from $[2]y=0$ in $\mathrm{BP}^*(P^{2m})$. More precisely, (8) shows that $d_0 \colon E_0^{k,-k} \to E_0^{k,1-k}$ is zero and $2x^i y^{k-i} = 0$ in $E_0^{k,l} \approx E_1^{k,l}$.

Next, we need to compute the E_2^{**}-term of the spectral sequence. We shall fix k and let $(r,t) \in \{(1,k-1),(1,n),(k-m,n)\}$. Let $a_{r,t} = \sum_{i=r}^{t} b_i x^i y^{k-i}$ denote the general element of $E_1^{k,-k}$. The sum is homogeneous and $b_i \in \mathrm{BP}^*$. By (8) we have

(9) $$\begin{aligned} d_1(a_{r,t}) &= \sum_{i=r}^{t} b_i d_1(x^i y^{k-i}) = v_1 \sum_{i=r}^{t} b_i x^i y^{k-i}(x-y) \\ &= v_1 \sum_{i=r}^{t-1} (b_i - b_{i+1}) x^{i+1} y^{k-i} + b_t x^{t+1} y^{k-t} - b_r x^r y^{k-r+1}. \qquad \square \end{aligned}$$

LEMMA 2. *If $k \leq m$ then $d_1 | E_1^{k,-k}$ is a monomorphism.*

Proof. If $k \leq m$ and $n+1$, then a typical element in $E_1^{k,-k}$ is an element of the form $a_{1,k-1}$. Since $x^{i+1}y^{k-i} \neq 0$ for $1 \leq i \leq k-2$, $b_1 = b_2 = \cdots = b_{k-1} \in \mathrm{BP}^*/(2)$ by Equation (9) and Lemma 1. But $xy^k \neq 0$, so $b_1 = 0 \in \mathrm{BP}^*/(2)$. If $n+1 < k \leq m$ then one uses the same argument for an $a_{1,n}$. □

LEMMA 3. *If $m < k \leq m+n$ then $(x-y)\sum_{i=k-m}^{n} x^i y^{k-i} = 0$.*

Proof. The left-hand side is

$$x^{k-m}y^{k-n}(x-y)\sum_{j=0}^{m+n-k} x^j y^{m+n-k-j} = x^{k-m}y^{k-n}(y^{m+n-k+1} - x^{m+n-k+1}) = 0. \qquad \square$$

LEMMA 4. *For each k in $m < n \leq m+n$, the kernel of $d_1 | E_1^{k,-k}$ is generated by $g_k = \sum_{i=k-m}^{n} x^i y^{k-i}$ over* $\mathrm{BP}^*/(2)$.

Proof. If $a_{r,t} \in \mathrm{kernel}(d)$, then Equation (9) holds for $(r,t) = (k-m, n)$ and $x^{i+1}y^{k-i} \neq 0$ for $k-m \leq i \leq n-1$. Hence $b_{k-m} = \cdots = b_m$ and $a_{k-m,n}$ is a $\mathrm{BP}^*/(2)$-multiple of g_k. This in turn is in the kernel of d_1 by Lemma 3. □

Now, Lemmas 2 and 4 immediately yield the following lemma.

LEMMA 5. (a) *If $2 \leq k \leq m$ then $E_2^{k,-k} = 0$. If $m < k \leq m+n$ then $E_2^{k,-k}$ is the* $\mathrm{BP}^*/(2)$*-free module generated by g_k.*

(b) *$E_2^{k,1-k}$ is the* $\mathrm{BP}^*/(2)$*-module generated by $x^i y^{k-i}$, with the relations $v_1(x^{i+1}y^{k-i-1} - x^i y^{k-i}) = 0$ for $k \geq 3$.*

LEMMA 6. *The n elements*

$$g_k = \sum_{i=k-m}^{n} x^i y^{k-i}$$

are in $\mathrm{Tor}_{\mathrm{BP}^*}(\mathrm{BP}^*(P^{2n}), \mathrm{BP}^*(P^{2m}))$.

Proof. By (7) we have

$$d(g_k) = \sum_{i=k-m}^{n} x^i y^{k-i} \sum_{i \leq s} a_s(x^s - y^s) \tag{10}$$
$$= \sum_{i=k-m}^{n} x^i y^{k-i}(x-y) \sum_{1 \leq s} a_s(x^{s-1} + \cdots + y^{s-1}),$$

which is zero by Lemma 3 and the fact that $-y^{s+1} = (x-y)(x^s + \cdots + y^s)$ for $s \geq n$. □

COROLLARY 7. *The spectral sequence collapses and $E_2^{**} \approx E_\infty^{**}$.*

Proof. The differentials raise the codegree by 1, so $E_r^{k,-k}$ can only be the source and not the target of a differential. But all of $E_2^{k,-k}$ consists of permanent cycles, by Lemmas 5 and 6. □

Actually, we can account for all relations in the Tor module.

LEMMA 8. *The* g_k *satisfy* $\sum_{0\le s} a_s g_{k+s}=0$.

Proof.

$$\sum_{0\le s} a_s g_{k+s}=\sum_{0\le s} a_s \sum_{j=k+s-m}^{n} x^j y^{k+s-j}=\sum_{0\le s} a_s x^{s+1}\sum_{l=k-m}^{n} x^{l-1}y^{k-l},$$

since $x^{n+1}=0$. But the last expression is zero by relation (7) and Lemma 3. □

Lemmas 5, 6, and 8 show that we have enough generators and relations to get all of Tor. More precisely,

$$\mathrm{Tor}_{\mathrm{BP}^*}(\mathrm{BP}^*(P^{2n}),\mathrm{BP}^*(P^{2m}))\approx\Sigma^{2\max\{m,n\}}(\mathrm{BP}^{2\min\{m,n\}}),$$

and Proposition 1 follows from (1).

In homology the situation is completely dual. For instance, we have a BP_*-free resolution as follows: $g:\Sigma^{-1}\mathrm{BP}_*(CP^n)\to\Sigma^{-1}\mathrm{BP}_*(CP^n)$, with generators $z_1,z_2,\ldots,z_n$ in degrees $1,3,\ldots,2n-1$ and $g(z_i)=\sum_{0\le j}a_j z_{i-j}$. After tensoring with $\mathrm{BP}_*(P^{2m})$ we need (as before) the kernel and cokernel of $g\otimes 1$. A filtration for both the domain and the range of $g\otimes 1$ is defined dually; that is, at the kth stage we only keep the generators $z_i w_j$ with $i+j\le k$ (the w_j's generate $\mathrm{BP}_*(CP^m)$). Again we have the dual spectral sequence of the filtered complex (of homological type). The ranges of the indices of the generators are conveniently arranged as follows:

(i) $1\le i\le k-1$ if $k\le n+1$;
(ii) $1\le i\le n$ if $n+1<k\le m$; and
(iii) $k-m\le i\le n$ if $m,n+1<k\le n+m$.

The first differential d^1 is a monomorphism if $n+1<k\le m$, or if m and $n+1<k\le m+n$. If $2\le k\le n+1$, then the kernel of d^1 is generated by $\sum_{i=1}^{k-1}z_i w_{k-i}$. One easily computes the $E^2_{*,*}$-term and proves collapse by noting that the above elements are elements in $\mathrm{Tor}^{\mathrm{BP}_*}(\mathrm{BP}_*(P^{2n}),\mathrm{BP}_*(P^{2m}))$. The BP_*-module relations for $\Sigma^1\mathrm{BP}_*(P^{2\min\{m,n\}})$ are easily verified. Proposition 2 now follows from the Landweber split exact sequence in homology.

3. The v_1-Torsion in the Tensor Product

Lemmas 5 and 7 give us a nice description of the tensor product. One can read off all of the v_1-torsion part in it.

COROLLARY 9. *The* v_1*-torsion in* $\mathrm{BP}_*(P^{2n})\otimes_{\mathrm{BP}_*}\mathrm{BP}_*(P^{2m})$ *is the ideal* $(xy(x-y))$.

Proof. Let $h=xy(x-y)$. Then relation (10), together with

$$a_1\equiv v_1 \bmod(2)\quad\text{and}\quad a_1h=-a_2h(x+y)-a_3h(x^2+xy+y^2)-\cdots,$$

show that v_1h is a sum of elements of higher filtration (with a factor of h). But the filtration is finite, so h is v_1-torsion.

On the other hand, by the relations $\cdots = v_1 x^i y^{k-i} = v_1 x^{i-1} y^{k-i+1} = \cdots$ of Lemma 5, we see that $v_1 x^i y^{k-i}$ is zero in E_∞^{**} if and only if $k \geq n+2$ (by checking in the ranges of Lemma 1). We conclude that $x^i y^{k-i}$ is v_1-torsion if $k \geq n+2$. Alternatively, we can see this from

$$x^i y^{k-i} = x^i y^{k-n-1}(y^{n-i+1} - x^{n-i+1}) \in (h).$$

However, if $k \leq n+1$ then the generators $x^i y^{k-i}$ are not v_1-torsion (since $v_1^r x^i y^{k-i}$ is nonzero even modulo filtration for any r in N). □

Let q be a polynomial in x and y in the tensor product that is v_1-torsion. Let k be the minimal degree of the monomials $x^i y^j$ in q. Then $k \geq 3$. We denote by $\bar{q}$ the image of q in $E_\infty^{k,-k}$, which we may assume nontrivial or we can go up in filtration. Let $\bar{q} = \sum_i a_i x^i y^{k-i}$; the coefficients may be taken to be 0 or 1. Since $x^i y^{k-i}$ is v_1-torsion for $k \geq n+2$, we may assume that $k \leq n+1$. We then have

$$a_i x^i y^{k-i} + a_{i+1} x^{i+1} y^{k-i-1} \equiv (a_i + a_{i+1}) x^{i+1} y^{k-i-1} \operatorname{mod}(h).$$

By applying the above relation repeatedly we see that

$$\bar{q} = (a_1 + \cdots + a_{k-1}) x^{k-1} y \operatorname{mod}(h).$$

If the sum $a_1 + \cdots + a_{k-1}$ is even, everything in filtration k is in (h) and multiplication by v_1 will bring us to higher filtrations where we can repeat the process. If it is odd then we get a contradiction, since even though q is v_1-torsion, $v_1^r x^{k-1} y \neq 0$ modulo filtration in the current range.

COROLLARY 10. *There is a* BP*-*module filtration of*

$$\mathrm{BP}^*(P^{2n}) \otimes_{\mathrm{BP}^*} \mathrm{BP}^*(P^{2m})$$

such that the associated graded module is $\mathrm{BP}^*/(2, v_1)$-*free on* $x^i y^{k-i}$ *in the range* $k \geq \min\{m, n\} + 2$.

Since the Tor product is (up to filtration) $\mathrm{BP}^*/(2)$-free on $g_{\max\{m,n\}+1}, \ldots, g_{m+n}$, counting orders in the tensor product becomes very easy. One can easily verify the orders of some of the groups in [4] for BP or BP2 without the use of ku^* or of the tensor product.

ACKNOWLEDGMENT. Our original approach to the problem was using the Adams spectral sequence. We would like to thank the referee for suggesting the above direct approach; one did not need the extra notation after all. Finally, we would like to thank Steve Wilson for pointing out an error in the original statement of Corollary 10.

References

1. L. Astey, *Geometric dimension of bundles over real projective spaces,* Quart. J. Math. Oxford Ser. (2) 31 (1980), 139–155.

2. D. Davis, *A strong non-immersion theorem for real projective spaces,* Ann. of Math. (2) 120 (1984), 517–528.
3. P. S. Landweber, *Künneth formulas for bordism theories,* Trans. Amer. Math. Soc. 121 (1966), 242–256.
4. W. S. Wilson, *Brown–Peterson homology: an introduction and sampler,* CBMS Regional Conf. Ser. in Math., Conf. Board Math. Sci., Washington, DC, 1982.

Department of Mathematics
U.S. Naval Academy
Annapolis, MD 21402