Associate Harmonic Immersions in 3-Space

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1. Introduction

The immersion of a surface S with definite prescribed metric g is *harmonic* into Euclidean 3-space $E^{3,0}$ (or into Minkowski 3-space $E^{3,1}$) if and only if the three coordinate functions of the immersion satisfy Laplace's equation with respect to coordinates isothermal for g. Similarly, the immersion of a surface S with indefinite prescribed metric g is harmonic into $E^{3,0}$ (or into $E^{3,1}$) if and only if the three coordinate functions of the immersion satisfy the wave equation with respect to coordinates isothermal for g.

Since the immersion of a surface S with definite or indefinite prescribed metric g is harmonic into $E^{3,0}$ if and only if it is harmonic into $E^{3,1}$, we refer throughout this paper to a harmonic immersion $\mathcal{Z}: (S,g) \to E^{3,j}$ where the index j can assume either value 0 or 1. The results below are meant, in part, to illustrate that the Minkowski geometry of a harmonic immersion $\mathcal{Z}: (S,g) \to E^{3,j}$ is at least as interesting as its Euclidean geometry.

An immersion $\mathcal{Z}: S \to E^{3,0}$ is *minimal* if and only if $\mathcal{Z}: (S, I^0) \to E^{3,j}$ is harmonic, where I^0 is the metric induced on S by $E^{3,0}$. Similarly, an immersion $\mathcal{Z}: S \to E^{3,1}$ is minimal if and only if $\mathcal{Z}: (S, I^1) \to E^{3,j}$ is harmonic, where I^1 is the metric induced on S by $E^{3,1}$. Since I^0 and I^1 are seldom proportional, the harmonic immersions into $E^{3,j}$ which are minimal into $E^{3,0}$ differ from those minimal into $E^{3,1}$.

In this paper, we are most concerned with harmonic immersions $\mathcal{Z}: (S,g) \to E^{3,j}$ with indefinite prescribed metric g. We look at harmonic immersions $\mathcal{Z}: (S,g) \to E^{3,j}$ with definite prescribed metric g solely to compare and contrast results. When g is definite, the properties of harmonic $\mathcal{Z}: (S,g) \to E^{3,j}$ tend to generalize the behavior of minimal $\mathcal{Z}: S \to E^{3,0}$ (see [3]). When g is indefinite, the properties of harmonic $\mathcal{Z}: (S,g) \to E^{3,j}$ tend to generalize the behavior of timelike minimal $\mathcal{Z}: (S,g) \to E^{3,1}$ (see [5] and [6]).

In Section 3 we define associate families of harmonic immersions into $E^{3,j}$. For definite g, the construction imitates the classical definition of associate families of minimal immersions in $E^{3,0}$ (see [2]). For indefinite g, the construction specializes to the definition of associate families of timelike minimal immersions into $E^{3,1}$, which are studied more closely in [6].

Received March 28, 1988. Revision received October 6, 1989.
Michigan Math. J. 37 (1990).

We will show that the associate pairing of harmonic immersions into $E^{3,j}$ preserves Euclidean unit normals, Gauss curvature, energy-1 metrics, and equiareal metrics. Wherever $\det I^1 \neq 0$, the associate pairing also preserves Minkowski unit normals, Gauss curvature, energy-1 metrics, and equiareal metrics. This remains true no matter which oriented direction is chosen to be positive timelike when changing $E^{3,0}$ to $E^{3,1}$, for both definite and indefinite prescribed metric g.

The theorem proved is that all immersions associate to an entire harmonic immersion into $E^{3,j}$ are entire over a fixed plane. The result is straightforward when g is definite, following easily from Bers' proof in [1] of Bernstein's theorem. When g is indefinite, more substantial argument is needed, based on the use of global coordinates provided by the Hilbert–Holmgren theorem for harmonic maps in [5].

The results of this paper are used in [6] to generate families of entire timelike minimal surfaces in $E^{3,1}$. We also develop in [6] a construction that assigns, to any harmonic $\mathcal{Z}: (S, g) \to E^{3,j}$ with indefinite g which is locally a graph over some fixed plane, a timelike minimal immersion $\tilde{\mathcal{Z}}: S \to E^{3,1}$ which is locally a graph over the spacelike coordinate plane. (If $\mathcal{Z}$ is entire, then so is $\tilde{\mathcal{Z}}$.) The assignment procedure allows one to define a local Weierstrass representation for any harmonic immersion $\mathcal{Z}: (S, g) \to E^{3,j}$ with indefinite g. The representation is determined by the Euclidean Gauss unit normal map for $\mathcal{Z}$ and the two real-valued Weierstrass functions for $\tilde{\mathcal{Z}}$. For details and examples, see [6].

2. Background

View $E^{3,j}$ as R^3 with the scalar product

$$\langle V, W \rangle^j = v_1 w_1 + v_2 w_2 + (-1)^j v_3 w_3,$$

where $j = 0$ gives the Euclidean and $j = 1$ the Minkowski scalar product. A vector V in $E^{3,1}$ is *spacelike* if $\langle V, V \rangle^1 > 0$, *timelike* if $\langle V, V \rangle^1 < 0$, and *null* if $\langle V, V \rangle^1 = 0$.

The surface S is assumed to be C^∞, oriented, and connected. Given a C^∞ immersion $\mathcal{Z}: S \to R^3$, we also write $\mathcal{Z}: S \to E^{3,0}$ and $\mathcal{Z}: S \to E^{3,1}$ since the same underlying map is involved. To study immersions $\mathcal{Z}: S \to E^{3,j}$ for $j = 0, 1$, we use the fundamental forms

$$I^j = \langle d\mathcal{Z}, d\mathcal{Z} \rangle^j, \qquad II^j = -\langle d\mathcal{Z}, d\nu^j \rangle^j,$$

where the unit normal ν^j is given in terms of local coordinates x, y on S by

$$\sqrt{|\det I^j|}\, \nu^j = \begin{vmatrix} \vec{i} & \vec{j} & (-1)^j \vec{k} \\ & \mathcal{Z}_x & \\ & \mathcal{Z}_y & \end{vmatrix}.$$

Gauss curvature K^j and mean curvature H^j are given by

$$K^j = \det II^j / \det I^j, \qquad H^j = \operatorname{tr}_{I^j}(II^j).$$

Definition of ν^1, II^1, K^1, and H^1 above requires that $\det I^1 \neq 0$. Thus we study the Minkowski geometry of an immersion $\mathcal{Z}: S \to E^{3,j}$ only where $\det I^1 \neq 0$. In particular, the condition $\det I^1 \neq 0$ is to be understood in any statement involving ν^1, II^1, K^1, or H^1. If $\det I^1 > 0$, $\mathcal{Z}: S \to E^{3,1}$ is called *spacelike*. If $\det I^1 < 0$, $\mathcal{Z}: S \to E^{3,1}$ is called *timelike*.

Assume always that the metric g prescribed on S is nondegenerate, so that $\det g \neq 0$. A mapping $\mathcal{Z}: (S, g) \to E^{3,j}$ is harmonic if and only if

$$\mathcal{Z}_{xx} \pm \mathcal{Z}_{yy} \equiv 0 \tag{1}$$

for all local coordinates x, y on S, in terms of which

$$g = \lambda(dx^2 \pm dy^2) \tag{2}$$

for some $\lambda = \lambda(x, y)$, where the choice of signs in (1) and (2) must match. Coordinates x, y on S for which (2) is valid are called *g-isothermal.*

When g is indefinite, it is more useful to observe that $\mathcal{Z}: (S, g) \to E^{3,j}$ is harmonic if and only if $\mathcal{Z}_{xy} \equiv 0$ for all local coordinates x, y on S, in terms of which $g = 2\mu\, dx\, dy$ for some $\mu = \mu(x, y)$. Such coordinates are called *g-null.* Thus, for indefinite g, $\mathcal{Z}: (S, g) \to E^{3,j}$ is harmonic if and only if $\mathcal{Z}$ has the expression

$$\mathcal{Z}(x, y) = \mathcal{X}(x) + \mathcal{Y}(y) \tag{3}$$

near any point in the domain of local g-null coordinates x, y on S. If x, y are also Euclidean arc length parameters for $\mathcal{X}(x)$ and $\mathcal{Y}(y)$ respectively, then x, y are called I^0-Tchebychev g-null coordinates. Such I^0-Tchebychev g-null coordinates are always available locally, given a harmonic immersion $\mathcal{Z}: (S, g) \to E^{3,j}$ with indefinite g. Moreover, x, y are uniquely determined over their domain on S up to additive constants and switches to $y, -x$ or $-x, -y$ or $-y, x$.

An immersion $\mathcal{Z}: S \to E^{3,j}$ is *entire* if $\mathcal{Z}(S)$ is the graph of a C^∞ real-valued function over some whole plane. An entire immersion is always an imbedding, with S simply connected. We recall the following theorem from [5] in which $\mathfrak{M}^n$ is an arbitrary pseudo-Riemannian manifold of dimension $n \geq 2$.

HILBERT-HOLMGREN THEOREM FOR HARMONIC MAPS. *If $\mathcal{Z}: (S, g) \to \mathfrak{M}^n$ is harmonic with g indefinite and the metric h induced on S by $\mathfrak{M}^n$ complete and Riemannian, then the universal cover $\tilde{S}$ of S with the lift $\tilde{g}$ of g is conformally equivalent to the Minkowski 2-plane $E^{2,1}$. Moreover, the intrinsic curvature of h cannot be bounded away from zero.*

REMARK 1. The proof of the Hilbert-Holmgren theorem in [5] establishes a conformal diffeomorphism between the x, y-plane with the metric $dx\, dy$ and $(\tilde{S}, \tilde{g})$, under which x, y become global $\tilde{h}$-Tchebychev $\tilde{g}$-null coordinates on $\tilde{S}$. For an entire harmonic $\mathcal{Z}: (S, g) \to E^{3,j}$ with indefinite g, I^0 is complete and $S = \tilde{S}$, so there always exist global I^0-Tchebychev g-null coordinates x, y on S with x and y assuming all real values. It follows that every entire harmonic $\mathcal{Z}\colon (S,g)\to E^{3,j}$ with indefinite g has a global expression of the form (3), with x and y Euclidean arc length parameters for $\mathcal{X}(x)$ and $\mathcal{Y}(y)$ respectively. Classically, $\mathcal{Z}(S)$ is referred to as a *translation surface,* since all of $\mathcal{Z}(S)$ is swept out by translating the curve $\mathcal{X}(x)$ by the fixed vector $\mathcal{Y}(k)$ for all real values of the constant k. However, we prefer to think of (3) expressed in terms of I^0-Tchebychev g-null coordinates x,y as the *Weierstrass representation* of the immersion. We show in [6] that $\mathcal{X}(x)$ and $\mathcal{Y}(y)$ in (3) are locally determined by $\nu^0(x,y)$ and the Weierstrass functions $A(x)$ and $B(y)$ for the "assigned" timelike minimal immersion. (See §4 of [6] for details.) Note meanwhile that any entire timelike minimal immersion $\mathcal{Z}\colon S\to E^{3,1}$ has a representation of the form (3) over the whole x,y-plane with x,y I^0-Tchebychev I^1-null coordinates on S.

If $\mathcal{Z}\colon (S,g)\to E^{3,j}$ is harmonic, then whether g is definite or indefinite, $\mathcal{Z}\colon (S,\sigma g)\to E^{3,j}$ is harmonic for any function σ which never vanishes. Convenient choices of σ lead to useful conformal normalizations of g, among them the energy-1 metric Γ^j and the equiareal metric Π^j, which are described as follows. As always, $j=0,1$.

For any immersion $\mathcal{Z}\colon (S,g)\to E^{3,j}$, define $\mathcal{K}^j$ and $\mathcal{H}^j$ by

$$\mathcal{K}^j=\mathcal{K}(g,I^j)=\det I^j/\det g,\qquad 2\mathcal{H}^j=2\mathcal{H}(g,I^j)=\operatorname{tr}_g I^j.$$

The *energy-1 metric* Γ^j is given wherever $\mathcal{H}^j\neq 0$ by

$$\Gamma^j=\mathcal{H}^j g.$$

In any statement involving Γ^j, we presume that $\mathcal{H}^j\neq 0$. The *equiareal metric* Π^j is given wherever $\mathcal{K}^j\neq 0$ by

$$\Pi^j=|\mathcal{K}^j|^{1/2}g.$$

In any statement involving Π^j, we presume that $\mathcal{K}^j\neq 0$.

The induced metric I^j of an immersion $\mathcal{Z}\colon S\to E^{3,j}$ is given in terms of local coordinates x,y on S by

$$I^j=E^j\,dx^2+2F^j\,dx\,dy+G^j\,dy^2,$$

where

$$E^j=\langle \mathcal{Z}_x,\mathcal{Z}_x\rangle^j,\qquad F^j=\langle \mathcal{Z}_x,\mathcal{Z}_y\rangle^j,\qquad G^j=\langle \mathcal{Z}_y,\mathcal{Z}_y\rangle^j.$$

If g is definite, use of g-isothermal coordinates x,y gives

$$\begin{gathered} g=\lambda(dx^2+dy^2),\\ \lambda^2\mathcal{K}^j=E^jG^j-(F^j)^2,\qquad 2\lambda\mathcal{H}^j=E^j+G^j, \end{gathered}\tag{4}$$

so that

$$\begin{gathered} 2\Gamma^j=(E^j+G^j)(dx^2+dy^2),\\ \Pi^j=\operatorname{sign}\lambda\,|E^jG^j-(F^j)^2|^{1/2}(dx^2+dy^2). \end{gathered}\tag{5}$$

If g is indefinite, use of g-null coordinates x,y gives

(6)
$$g = 2\mu \, dx \, dy,$$
$$-\mu^2 \mathcal{K}^j = E^j G^j - (F^j)^2, \qquad \mu \mathcal{H}^j = F^j,$$

so that

(7)
$$\Gamma^j = 2F^j \, dx \, dy,$$
$$\Pi^j = 2 \operatorname{sign} \mu |E^j G^j - (F^j)^2|^{1/2} \, dx \, dy.$$

It is easy to check that $\mathcal{K}^j \equiv \mathcal{H}^j \equiv 1$ if $g = I^j$, so that $\Gamma^j \equiv \Pi^j \equiv I^j$. Thus it is no surprise that properties of Γ^j and Π^j for a harmonic immersion $\mathcal{Z}$: $(S, g) \to E^{3,j}$ often mimic the properties that I^j has for a minimal immersion $\mathcal{Z}: S \to E^{3,j}$. For general information about energy-1 and equiareal metrics, see [3] or [4].

3. Associate Harmonic Immersions

Consider first the case in which the prescribed metric g on S is definite. Given a harmonic $\mathcal{Z}: (S, g) \to E^{3,j}$, use g-isothermal coordinates x, y locally on S so that

$$\mathcal{Z}_{z\bar{z}} \equiv \mathcal{Z}_{xx} + \mathcal{Z}_{yy} \equiv 0,$$

where $2\partial/\partial z = \partial/\partial x - i\partial/\partial y$ and $2\partial/\partial \bar{z} = \partial/\partial x + i\partial/\partial y$. Then

$$\phi \stackrel{\text{def}}{=} 2\mathcal{Z}_z$$

is holomorphic in $z = x + iy$, and $\mathcal{Z}$ has the local expression

$$\mathcal{Z} = \operatorname{Re} \int_{z_0}^{z} \phi \, dz + C$$

for a constant vector C (see [3]).

For any real θ, define the *associate immersion* $\mathcal{Z}_\theta: (D, g) \to E^{3,j}$ by setting

(8)
$$\mathcal{Z}_\theta = \operatorname{Re} e^{i\theta} \int_{z_0}^{z} \phi \, dz + C$$

over the domain D of x, y on S. Of course, $\mathcal{Z} = \mathcal{Z}_\theta$ for $\theta = 0 \bmod 2\pi$. Here

$$\phi_\theta = e^{i\theta} \phi = 2(\mathcal{Z}_\theta)_z = (\mathcal{Z}_\theta)_x - i(\mathcal{Z}_\theta)_y$$

gives

(9)
$$(\mathcal{Z}_\theta)_x = \cos\theta \, \mathcal{Z}_x + \sin\theta \, \mathcal{Z}_y,$$
$$(\mathcal{Z}_\theta)_y = -\sin\theta \, \mathcal{Z}_x + \cos\theta \, \mathcal{Z}_y,$$

so $(\mathcal{Z}_\theta)_{xx} + (\mathcal{Z}_\theta)_{yy} \equiv 0$ and $\mathcal{Z}_\theta: (D, g) \to E^{3,j}$ is harmonic. By (4), (5), and (9), we have

$$\det I_\theta^j = \det I^j,$$
$$\mathcal{K}_\theta^j = \mathcal{K}(g, I_\theta^j) = \mathcal{K}^j, \qquad \mathcal{H}_\theta^j = \mathcal{H}(g, I_\theta^j) = \mathcal{H}^j,$$
$$\Gamma_\theta^j = \Gamma^j, \qquad \Pi_\theta^j = \Pi^j.$$

Since $(\mathcal{Z}_\theta)_x$ and $(\mathcal{Z}_\theta)_y$ span the same oriented plane as $\mathcal{Z}_x$ and $\mathcal{Z}_y$, the unit normals $\nu_\theta^j = \nu^j$ do not vary with θ. Thus the second fundamental form of $\mathcal{Z}_\theta$ is given by

$$\Pi_\theta^j = \langle (\mathcal{Z}_\theta)_{xx}, \nu^j\rangle^j\, dx^2 + 2\langle (\mathcal{Z}_\theta)_{xy}, \nu^j\rangle^j\, dx\, dy + \langle (\mathcal{Z}_\theta)_{yy}, \nu^j\rangle^j\, dy^2,$$

and $(\mathcal{Z}_\theta)_{xx} = -(\mathcal{Z}_\theta)_{yy}$ yields

$$\det \Pi_\theta^j = \det \Pi^j \leq 0.$$

Finally, the Gauss curvatures satisfy $K_\theta^j = K^j$, with sign $K^j = -\text{sign}(\det I^j)$ wherever $K^j \neq 0$.

REMARK 2. If $\mathcal{Z}: S \to E^{3,0}$ is minimal then x, y are I^0-isothermal, and $I^0 = \Gamma^0 = \Pi^0$ in (5) shows that the associate immersions $\mathcal{Z}_\theta: D \to E^{3,0}$ given by (8) have $I_\theta^0 = \Gamma_\theta^0 = \Pi_\theta^0 = I^0$, making them minimal and isometric in $E^{3,0}$. Similarly, if $\mathcal{Z}: S \to E^{3,1}$ is spacelike and minimal, then the associate immersions $\mathcal{Z}_\theta: S \to E^{3,1}$ are spacelike, minimal, and isometric in $E^{3,1}$. (See [6] for details.)

Consider next the case in which the prescribed metric g on S is indefinite. Given a harmonic immersion $\mathcal{Z}: (S, g) \to E^{3,j}$, use g-null coordinates locally on S so that (3) holds. For any constant $c > 0$, define the associate immersion $\mathcal{Z}_c: (S, g) \to E^{3,j}$ by setting

$$\mathcal{Z}_c(x, y) = c\mathcal{X}(x) + \mathcal{Y}(y)/c \tag{10}$$

over the domain D of x, y so that $\mathcal{Z}_c = \mathcal{Z}$ for $c = 1$. Here $\mathcal{Z}_c: (D, g) \to E^{3,j}$ is harmonic simce $(\mathcal{Z}_c)_{xy} \equiv 0$.

REMARK 3. The choice of different g-null coordinates on D may reparametrize the associate family (sending $\mathcal{Z}_c$ to $\mathcal{Z}_{1/c}$) but the same set of immersions is determined. Nonetheless, we assume a fixed choice of null coordinates x, y on S when discussing $\mathcal{Z}_c$.

The first fundamental form I^j for $\mathcal{Z}_c: (S, g) \to E^{3,j}$ is given in terms of the g-null coordinates x, y over D by

$$I_c^j = c^2 E^j\, dx^2 + 2F^j\, dx\, dy + (G^j/c^2)\, dy^2,$$

so that

$$\det I_c^j = \det I^j.$$

Application of (6) and (7) to $\mathcal{Z}_c$ gives

$$\mathcal{K}_c^j = \mathcal{K}(g, I_c^j) = \mathcal{K}^j, \qquad \mathcal{H}_c^j = \mathcal{H}(g, I_c^j) = \mathcal{H}^j,$$
$$\Gamma_c^j = \Gamma^j, \qquad \Pi_c^j = \Pi^j.$$

Since $c\mathcal{X}'(x)$ and $\mathcal{Y}'(y)/c$ span the same oriented plane as $\mathcal{X}'(x)$ and $\mathcal{Y}'(y)$, the unit normals $\nu_c^j = \nu^j$ do not vary with c. Thus the second fundamental form of $\mathcal{Z}_c$ is

$$II_c^j = \langle c\mathcal{X}'', \nu^j\rangle^j\, dx^2 + \langle \mathcal{Y}''/c, \nu^j\rangle^j\, dy^2,$$

so that

$$\det II_c^j = \det II^j,$$

and the Gauss curvatures satisfy $K_c^j = K^j$. Here det II^j can have any sign, so the sign of K^j is not determined by the sign of det I^j, even if $g = I^1$, so that $\mathcal{Z}: S \to E^{3,1}$ is minimal.

REMARK 4. If $\mathcal{Z}: S \to E^{3,1}$ is a timelike minimal immersion then x, y are I^1-null, and $I^1 = \Gamma^1 = \Pi^1$ in (7) shows that the associate immersions $\mathcal{Z}_c: D \to E^{3,1}$ given by (10) have $I_c^1 = \Gamma_c^1 = \Pi_c^1 = I^1$, making them timelike, minimal, and isometric in $E^{3,1}$. By Remark 1, the timelike minimal immersions $\mathcal{Z}_c$: $S \to E^{3,1}$ can be globally defined on S in case the timelike minimal immersion $\mathcal{Z}: S \to E^{3,1}$ is entire over some plane.

4. Entire Harmonic Immersions

We now show that immersions associate to an entire immersion are themselves entire. Assume henceforth that u, v, w are fixed Cartesian coordinates in $E^{3,j}$ so that the u, v-coordinate plane is the spacelike coordinate plane $\mathcal{P}$ in $E^{3,1}$. Let $T: E^{3,j} \to \mathcal{P}$ denote orthogonal projection onto $\mathcal{P}$. The lemma below is needed to handle the case in which the metric g prescribed on S is definite.

LEMMA. *If a harmonic $\mathcal{Z}: (S, g) \to E^{3,j}$ with definite g is entire over $\mathcal{P}$, then there are global g-isothermal coordinates x, y on S assuming all real values in terms of which*

$$\mathcal{Z}(x,y) = (ax + by + t, \alpha x + \beta y + \tau, w(x,y)), \tag{11}$$

where $w(x,y)$ is a harmonic function and the constants $a, b, t, \alpha, \beta, \tau$ satisfy $a\beta - \alpha b \neq 0$.

Proof. Here (S, g) is conformally equivalent to some simply connected domain $\mathfrak{D}$ in the x, y-plane. Using x, y over $\mathfrak{D}$ as global g-isothermal coordinates on S, $Z = T \circ \mathcal{Z}$ is a harmonic diffeomorphism $Z: (\mathfrak{D}, g) \to \mathcal{P}$. If we write $Z(x,y) = (u, v)$, then Lemma 3.2 in [1] implies that the harmonic functions u and v are each the real parts of one-to-one holomorphic functions on $\mathfrak{D}$. Moreover, by Lemma 3.3 in [1], $\mathfrak{D}$ must be the whole x, y-plane. Since the only one-to-one holomorphic maps from $E^{2,0}$ into $E^{2,0}$ are onto and linear, u and v are linear. □

THEOREM. *If a harmonic immersion $\mathcal{Z}: (S, g) \to E^{3,j}$ is entire over a plane, then its associate immersions are globally defined on S and are entire over the same plane.*

Proof. Since our arguments make no reference to Minkowski geometry, there is no loss of generality in assuming that $\mathcal{Z}$ is entire over the u, v-coordinate plane $\mathcal{P}$ in u, v, w-space $E^{3,0}$, with T the orthogonal projection of $E^{3,0}$ onto $\mathcal{P}$. Then $Z = T \circ \mathcal{Z}$ is a diffeomorphism of S onto $\mathcal{P}$.

If g is definite, use the g-isothermal coordinates x, y provided by the lemma. For each fixed value of θ, (8), (9), and (11) show that $Z_\theta = T \circ \mathcal{Z}_\theta$ is given by

$$Z_\theta = (ax+by,\ \alpha x+\beta y)\cos\theta + (bx-ay,\ \beta x-\alpha y)\sin\theta + C_\theta$$

over the whole x, y-plane, with C_θ a constant vector in $\mathcal{P}$. Since Z_θ is linear with nonvanishing Jacobian, it is onto $\mathcal{P}$, making $\mathcal{Z}_\theta$ entire.

If g is indefinite, introduce global I^0-Tchebychev g-null coordinates x, y on S so that (3) holds, as described in Remark 1. Then the diffeomorphism $Z = T \circ \mathcal{Z}$ of the x, y-plane onto $\mathcal{P}$ has the form

$$Z(x,y) = X(x) + Y(y). \tag{12}$$

With no loss of generality, assume that $\mathcal{Z}(0,0) = \mathcal{X}(0) = \mathcal{Y}(0) = (0,0,0)$ so that $Z(0,0) = X(0) = Y(0) = (0,0)$. Reorient S if necessary so the normals $\nu^0(x,y)$ for $\mathcal{Z}$ point upward. Because Z is a diffeomorphism of the x, y-plane onto $\mathcal{P}$, we know the following.

⟨1⟩ The curve $X(x)$ (resp. $Y(y)$) is simple, regular, and divides $\mathcal{P}$ into two distinct nonempty, open "half-planes" whose union is the complement of $X(x)$ (resp. $Y(y)$) in $\mathcal{P}$.

⟨2⟩ The "rays" of $X(x)$ or $Y(y)$ over $[0,\infty)$ and $(-\infty,0]$ are divergent, leaving every compact subset of $\mathcal{P}$.

Because $X(0) = Y(0) = (0,0)$, $X(x)$ and $Y(y)$ cannot lie to opposite sides of any line ℓ in $\mathcal{P}$. Suppose now that $X(x)$ and $Y(y)$ both lie to one (open) side of a line ℓ in $\mathcal{P}$. Then $(0,0)$ cannot lie on ℓ. Let W be the vector joining $(0,0)$ to the point on ℓ closest to $(0,0)$. If λ_x and λ_y are the components of $X(x)$ and $Y(y)$ in the W direction, then $\lambda_x < |W|$ and $\lambda_y < |W|$ imply that $2W$ is not of the form $X(x)+Y(y)$ for any value of x, y. Since Z is onto $\mathcal{P}$, this contradiction proves the following.

⟨3⟩ Either $X(x)$ or $Y(y)$ has a point on any line in $\mathcal{P}$.

Let C be the circle $u^2+v^2=1$ in $\mathcal{P}$ with C_x and C_y the nonempty arcs on C containing the oriented directions of $X'(x)$ and $Y'(y)$ respectively. Since Z is a diffeomorphism, $X'(x)$ and $Y'(y)$ are never parallel. Thus the four arcs $\pm C_x, \pm C_y$ are pairwise disjoint, putting C_x and C_y together within a half-closed semi-circular arc on C. Rotate the u, v-plane as needed so the midpoint of C_y lies at $u=0$, $v=1$. Thus $v>0$ on C_y, and since the normals $\nu^0(x,y)$ for $\mathcal{Z}$ point upward (with $\mathcal{X}'(x)$, $\mathcal{Y}'(y)$, and $\nu^0(x,y)$ a right-handed triple), $u>0$ must hold on C_x.

Because $u>0$ on C_x, $X(x)$ describes the graph of some function $v=f(u)$ in $\mathcal{P}$. The function $f(u)$ is defined for all real values of u. Otherwise, $X(x)$ lies to one side of a line ℓ parallel to the v axis in $\mathcal{P}$. Suppose that $X(x)$ lies to the left of ℓ, since the argument is identical if $X(x)$ lies to the right of ℓ. Because $X(0)=(0,0)$, ℓ lies to the right of $(0,0)$. If C_y reduces to a point, ⟨1⟩ and $Y(0)=(0,0)$ imply that $Y(y)$ describes the v axis, so that $X(x)$ and $Y(y)$ are to the same side of ℓ, contradicting ⟨3⟩. If C_y has positive length, $X(x)$ cannot lie outside of the closed trianguar region Δ bounded by ℓ and the lines joining $(0,0)$ to the endpoints of C_x for any particular value $\hat{x}>0$. Otherwise, for some $\bar{x}$ between 0 and $\hat{x}$, $X'(\bar{x})$ would have an oriented direction outside of C_x, contradicting the definition of C_x. But then the ray of $X(x)$ over $[0,\infty)$ lies in Δ, contradicting $\langle 2\rangle$. Note that $f'(u)$ for any u is always the slope of some $X'(x)$. If C_y has positive length then there must be a constant $M>0$, so that $|f'(u)|\leq M$ for all u.

The argument showing that $Y(y)$ describes the graph of some function $u=g(v)$ in $\mathcal{P}$ for all values of v is identical, except that C_x might consist of a single point other than $u=1$, $v=0$. If so, since C_y is centered at $u=0$, $v=1$, the length of C_y is less than π, showing that $|g'(v)|$ must be bounded. As above, $\langle 1\rangle$ implies that $g(v)$ is defined for all values of v. We summarize as follows.

$\langle 4\rangle$ $X(x)$ is the graph of a function $x=f(u)$ defined for all u, and unless C_y reduces to the point $u=0$, $v=1$, $|f'(u)|<M$ for some constant $M>0$.

$\langle 5\rangle$ $Y(y)$ is the graph of a function $u=g(v)$ defined for all v, and unless C_x reduces to the point $u=1$, $v=0$, $|g'(v)|<1/M$ for some constant $M>0$.

Note that unless $C_x=\{(1,0)\}$ or $C_y=\{(0,1)\}$, the same constant $M>0$ can be used in $\langle 4\rangle$ and $\langle 5\rangle$.

Consider now the mapping $Z_c=T\circ\mathcal{Z}_c$ from the x,y-plane into $\mathcal{P}$, where $\mathcal{Z}_c\colon(S,g)\to E^{3,j}$ is the associate immersion to $\mathcal{Z}$ globally defined on S for some fixed $c>0$ by (10). Then (12) provides the expression

$$Z_c(x,y)=cX(x)+Y(y)/c=(cX)(x)+(Y/c)(y) \tag{13}$$

for Z_c over the whole x,y-plane. Note that

$\{1\}$ $(cX)'(x)$ and $(Y/c)'(y)$ have the same oriented directions as $X'(x)$ and $Y'(y)$ respectively, so that Z_c is a local diffeomorphism.

To show that $\mathcal{Z}_c$ is entire, we will prove that Z_c is one-to-one and onto $\mathcal{P}$.

Suppose then that $Z_c(x_1,y_1)=Z_c(x_2,y_2)$ for some $(x_1,y_1)\neq(x_2,y_2)$, so that (13) gives

$$(cX)(x_1)-(cX)(x_2)=(Y/c)(y_2)-(Y/c)(y_1). \tag{14}$$

By $\langle 1\rangle$, $(cX)(x)$ and $(Y/c)(y)$ are simple curves, so that neither side in (14) can vanish unless $x_1=x_2$ and $y_1=y_2$, a contradiction. Thus both $x_1\neq x_2$ and $y_1\neq y_2$, so there must exist an $\bar{x}$ between x_1 and x_2 and a $\bar{y}$ between y_1 and y_2 with $(cX)'(\bar{x})$ parallel to $(Y/c)'(\bar{y})$, which contradicts $\{1\}$. We conclude that

$\{2\}$ Z_c is one-to-one and a diffeomorphism of the x,y-plane onto its image in $\mathcal{P}$. □

REMARK 5. It may seem that $Z_c\circ Z^{-1}\colon\mathcal{P}\to\mathcal{P}$ must be a *quasi-conformal* diffeomorphism onto its image (which must therefore be $\mathcal{P}$), since the map stretches by constant nonzero amounts in the direction $X'(x)$ and $Y'(y)$ respectively. However, the angle between $X'(x)$ and $Y'(y)$ can be arbitrarily close to zero. Consider, for example, the case in which $X(x)$ is a $(1/\sqrt{2})$-speed parametrization of $v=u^2$ and $Y(y)$ a $(1/\sqrt{2})$-speed parametrization of the v axis, with $Y'(y)$ upward in $\mathcal{P}$. Then $X'(x)$ approaches $Y'(y)$ as $x \to \infty$. Of course, the map $Z_c \circ Z^{-1}$ given by (12) and (13) is a diffeomorphism of $\mathcal{P}$ onto $\mathcal{P}$, but it is not quasi-conformal. Moreover, using (1) and (10) in [6], one easily constructs a timelike minimal $\mathcal{Z}: x, y\text{-plane} \to E^{3,1}$ that is entire over the spacelike coordinate plane $\mathcal{P}$ and gives rise to the $X(x)$ and $Y(y)$ just described.

Since $(cX)(x)$ (resp. $(Y/c)(y)$) is the image of $X(x)$ (resp. $Y(y)$) under the diffeomorphism of $\mathcal{P}$ onto itself which stretches $\mathcal{P}$ by the amount c (resp. $1/c$), ⟨1⟩ and ⟨2⟩ yield the following.

{3} The simple, regular curve $cX(x)$ (resp. $(Y/c)(y)$) divides $\mathcal{P}$ into two open "half-planes" whose union is the complement of $(cX)(x)$ (resp. $(Y/c)(y)$) in $\mathcal{P}$.

{4} The "rays" of $(cX)(x)$ and $(Y/c)(y)$ over $[0,\infty)$ and $(-\infty, 0]$ are divergent, leaving any compact subset of $\mathcal{P}$.

{5} $(cX)(x)$ is the graph of a function $v=F(u)$ defined for all u, and unless C_y reduces to the point $u=0$, $v=1$, $|F'(u)|<M$ for some constant $M>0$.

{6} $(Y/c)(y)$ is the graph of a function $u=G(v)$ defined for all v, and unless C_x reduces to the point $u=1$, $v=0$, $|G'(v)|<1/M$ for some constant $M>0$.

Again, unless $C_x=\{(1,0)\}$ or $C_y=\{(0,1)\}$, the same constant $M>0$ can be used in {5} and {6}.

Because $(cX)(0)=(Y/c)(0)=(0,0)$, $(cX)(x)$ and $(Y/c)(y)$ cannot lie to opposite open sides of any line ℓ in $\mathcal{P}$. If $(cX)(x)$ and $(Y/c)(y)$ both lie to one open side of ℓ in $\mathcal{P}$, then $(0,0)$ lies to that side of ℓ too. Arguing as in the proof of ⟨3⟩, one sees that $2(c+1/c)W$ is not of the form $X(x)+Y(y)$ for any value of x, y, a contradiction. Thus we have the following.

{7} Either $(cX)(x)$ or $(Y/c)(y)$ has a point on any line in $\mathcal{P}$.

It is easy to check that Z_c is onto $\mathcal{P}$ if C_x and C_y both reduce to a point. Then $(cX)(x)$ and $(Y/c)(y)$ describe the whole lines through $(0,0)$ of different slopes, so movement of $(cX)(x)$ parallel to itself with $(cX)(0)$ going to $(Y/c)(y)$ clearly sweeps out all of $\mathcal{P}$. If just one arc C_x or C_y reduces to a point, we can assume it is C_y. (Otherwise, work from the outset with the coordinates $-y, x$ in place of x, y.) Then C_y is the point $u=0$, $v=1$, so that $(Y/c)(y)$ describes the whole v axis. If Z_c is not onto $\mathcal{P}$ then there is a vertical line ℓ in $\mathcal{P}$ which does not meet $cX(x)$ or $(Y/c)(y)$, contradicting {7}. Again, Z_c is one-to-one.

More notation is needed to complete the proof if C_x and C_y both have positive length. Let $R^1=R^1(x)$ and $R^2=R^2(y)$ be the rays of $(cX)(x)$ and $(Y/c)(y)$ respectively over $[0,\infty)$. Let $R^3=R^3(x)$ and $R^4=R^4(y)$ be the rays of $(cX)(x)$ and $(Y/c)(y)$ respectively over $(-\infty, 0]$. For any real constant k, consider the ray

$$R_k^i = \begin{cases} R_k^i(x) = (cX)(x) + (Y/c)(k) & \text{if } i = 1, 3, \\ R_k^i(y) = (cX)(k) + (Y/c)(y) & \text{if } i = 2, 4, \end{cases}$$

so that $R_0^i = R^i$ for $i = 1, 2, 3, 4$. By {4}, we know that the rays R_k^i are all divergent. Moreover, since translation leaves the length of an arc unchanged, the following holds.

{8} The length of R_k^i over any fixed interval $[a, b]$ is independent of k for each $i = 1, 2, 3, 4$.

Since $(cX)(x)$ and $(Y/c)(y)$ meet exactly once and transversally at $(0, 0)$, the intersection of the half-planes determined by $(cX)(x)$ and $(Y/c)(y)$ in {3} yields four disjoint, nonempty, open "quadrants" $\mathcal{Q}^1$, $\mathcal{Q}^2$, $\mathcal{Q}^3$, and $\mathcal{Q}^4$ whose union is the complement of the union of $(cX)(x)$ and $(Y/c)(y)$ in $\mathcal{P}$. We index these quadrants so $\mathcal{Q}^1$ is bounded by R^1 and R^2, $\mathcal{Q}^2$ by R^2 and R^3, $\mathcal{Q}^3$ by R^3 and R^4, and $\mathcal{Q}^4$ by R^4 and R^1. By {7}, no line in $\mathcal{P}$ is completely contained in a single quadrant $\mathcal{Q}^i$.

Assuming that C_x and C_y have positive length, let M be the constant in {5} and {6}. Let $\mathcal{C}^1$, $\mathcal{C}^2$, $\mathcal{C}^3$, and $\mathcal{C}^4$ be the closed sectors of $\mathcal{P}$ bounded by the lines $v = \pm Mu$, indexed so that R^i lies in $\mathcal{C}^i$ for each $i = 1, 2, 3, 4$. Let $\mathcal{C}^i(q)$ denote the parallel translate of $\mathcal{C}^i$ taking $(0, 0)$ to q.

To show that any p in $\mathcal{Q}^1$ lies in $Z_c(S)$, let ℓ be the line of slope $-M$ through p. Since R^1 lies in $\mathcal{C}^1$ and R^2 in $\mathcal{C}^2$, p lies to the right of the line $v = -Mu$ in $\mathcal{P}$, as does all of ℓ. Suppose ℓ hits R^2. (Otherwise, {7} forces ℓ to hit R^1, and the argument is identical.) If ℓ hits R^2 only at points below p, there is a greatest value $\bar{y}$ of y for which $R^2(y)$ lies on ℓ. By {6}, $R^2(y)$ for $y > \bar{y}$ must stay in $\mathcal{C}^2(R^2(\bar{y}))$ without hitting ℓ. Thus $R^2(y)$ for $y > \bar{y}$ stays to the right of p, putting p in $\mathcal{Q}^2$, a contradiction. We conclude that ℓ hits R^2 at a point $R^2(\bar{y})$ above p.

If p lies on $R^1(\bar{y})$, we are done. Otherwise, $R^1(\bar{y})$ lies above p, since $R_{\bar{y}}^1$ is contained in $\mathcal{C}^1(R^2(\bar{y}))$. Choose $\bar{x} > 0$ so large that the distance of $R^1(\bar{x})$ from the circle $\mathcal{C}$ through p centered at $(0, 0)$ is greater than the length of any R_k^2 over $[0, \bar{y}]$. Then the ray $R_{\bar{x}}^2$ which meets $R_{\bar{y}}^1$ at the point $R_{\bar{x}}^2(\bar{y}) = R_{\bar{y}}^1(\bar{x})$ has length over $[0, \bar{y}]$ less than the distance of its initial point $R^1(\bar{x})$ from $\mathcal{C}$, and p must lie to the left of $R_{\bar{x}}^2$ in the region R bounded by R^1, $R_{\bar{y}}^1$, R^2, and $R_{\bar{x}}^2$. Since all points on the simple closed boundary of R lie in the simply connected set $Z_c(S)$, p must lie in $Z_c(S)$ as well.

Identical reasoning shows that each p in $\mathcal{Q}^3$ lies in $Z_c(S)$. For p in $\mathcal{Q}^2$ or $\mathcal{Q}^4$, one uses the analogous argument, taking a line ℓ through p with slope M rather than $-M$. Because every p in $\mathcal{P}$ lies in a $\mathcal{Q}^i$ or on $(cX)(x)$ or on $(Y/c)(y)$, we are done. □

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