Generators of Certain Groups of Semi-free S^1 Actions on Spheres and Splitting of Codimension-3 Knot Exact Sequences

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0. Introduction

Let $\Sigma_k^n(S^1)$ denote the set of oriented equivariant diffeomorphism classes of smooth semi-free S^1 actions on oriented homotopy $(n+2k-1)$-spheres satisfying these properties:

(P) The fixed point set is diffeomorphic to the standard $(n-1)$-sphere S^{n-1}. The normal bundle of the fixed point set is trivial as a complex vector bundle where the complex structure is an induced one from the action (see the Conventions below).

In fact, $\Sigma_k^n(S^1)$ is an abelian group under the equivariant connected sum operation (except for some low-dimensional cases). The group structure is fairly well understood. First, Hsiang [8] noted that $\Sigma_1^n(S^1)=0$. On the other hand it has been observed by many people that $\Sigma_k^n(S^1)$ is nontrivial in many cases; for instance, Browder [3] applied surgery theory to exhibit elements of infinite order for certain values of n and k. Later, Browder and Petrie [4] determined the rank of the free part of $\Sigma_k^n(S^1)$ as follows:

$$(0.1)\qquad \operatorname{rank}_Z \Sigma_k^n(S^1)=\operatorname{rank}_Z H^{4*}(CP^{k-1}\times(D^n,S^{n-1});Z)-\epsilon,$$

where $\epsilon=1$ if $n+2k-2\equiv 0\,(4)$ and $\epsilon=0$ otherwise. In particular it follows that $\operatorname{rank}_Z \Sigma_2^n(S^1)=1$ if and only if $n\equiv 0\,(4)$. In fact, $\Sigma_2^4(S^1)$ is known to be infinitely cyclic (i.e., torsion free).

Under these circumstances Davis [5, Prop. 7.15] has discovered that the generator of $\Sigma_2^4(S^1)$ is given by a semi-free smooth S^1 action defined naturally on an exotic 7-sphere discovered by Milnor [16]. An alternative proof is given in [13]. The result of Davis motivates this question:

What is a generator of the free part of $\Sigma_k^n(S^1)$? In other words, *is there an explicit description for such a generator?*

As is well known, famous Brieskorn spheres support natural semi-free smooth S^1 actions and some of them satisfy Property (P). We can verify that one of them is a generator or twice a generator of the free part of $\Sigma_k^n(S^1)$

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when $n>2k$ (Corollary 4.13). As a matter of fact, we will define an integer-valued homomorphism $\hat{\beta}$ on $\Sigma_k^n(S^1)$ under the assumption $n>2k$ and see that $\hat{\beta}$ takes values 1 or 2 for this example (Theorem 4.7).

The method of defining $\hat{\beta}$ is based on an observation of Montgomery and Yang [19]; namely, given $\Sigma \in \Sigma_k^n(S^1)$, we do surgery equivariantly along the fixed point set producing a manifold with a free smooth S^1 action, and then we take the orbit space. We then assign to Σ an integer determined from the $(n/4)$th Pontrjagin class of the resulting smooth manifold. This definition involves an ambiguity stemming from the choice of an equivariant framing used in the surgery, but the independence can be established with the aid of the G-signature theorem when $n>2k$.

As first observed by Montgomery and Yang [19] and then by Levine [12] in more detail, $\Sigma_2^n(S^1)$ is closely related to the group $\Sigma^{n+2,n-1}$, consisting of isotopy classes of oriented codimension-3 knots in S^{n+2}, by regarding the fixed point set as a knot in the orbit space. Through this correspondence $\hat{\beta}$ induces a homomorphism $\hat{\beta}_3$ from $\Sigma^{n+2,n-1}$ to Z. According to Levine [11], there is a short exact sequence

$$0 \to Z \xrightarrow{\partial_3'} \Sigma^{n+2,n-1} \to \ker \sigma_3(n-1,3) \to 0 \tag{0.2}$$

if $n \equiv 0\,(4)$ (see §5). As far as the author knows, little is known concerning the group extension of this exact sequence (cf. [14], [12, §IV]). We will see that if $n \equiv 0\,(8)$, then the composition $\hat{\beta}_3 \circ \partial_3'$ is the identity map, and hence the above exact sequence is split (Theorem 5.3).

In a manner similar to that used in the definition of $\hat{\beta}$, one can define other invariants. These will be discussed in [15] in connection with equivariant inertia groups.

This paper is organized as follows. In Section 1 we define an invariant $\beta\colon \Sigma_k^n(S^1) \to Z$ as explained above for $\hat{\beta}$. In Section 2 we see that β turns out to be a homomorphism. In general, β is far from surjective. In Section 3 we consider the largest integer dividing each element in the image of β and define $\hat{\beta}$ to be β divided by that largest integer. In Section 4 we carry out an explicit computation of $\hat{\beta}$ for a Brieskorn sphere with a semi-free smooth S^1 action. Section 5 treats the splitting problem of the knot exact sequence (0.2).

Throughout this paper every action will be smooth and the following conventions will be used unless otherwise stated.

CONVENTIONS. (1) S^1 will denote the circle group considered as the multiplicative group of elements with unit length in the complex numbers.

(2) Given an oriented manifold W, the boundary ∂W will be oriented as follows. Let $(w_1, \ldots, w_m)$ be an orthonormal frame such that the m-form $w_1 \wedge \cdots \wedge w_m$ represents the orientation of W and w_m is outward normal to W. Then we orient ∂W by the $(m-1)$-form $w_1 \wedge \cdots \wedge w_{m-1}$.

(3) For a manifold M with a semi-free S^1 action, the normal bundle ν of the fixed point set F is equipped with the complex structure defined as follows: for $g \in S^1 \subset C$ and $u \in \nu$ the complex multiplication gu is defined by $g_* u$, where g_* denotes the differential of the diffeomorphism g. In particular, ν has a natural orientation induced from the complex structure. If M is oriented, then we give an orientation to F compatible with those of M and ν.

1. An Invariant β

As outlined in the Introduction, we shall define an invariant $\beta: \Sigma_k^n(S^1) \to Z$ under the assumption $n > 2k$.

Let Σ be an element of $\Sigma_k^n(S^1)$. Let $\tilde{D}^{2k}$ denote the unit disk of C^k with the S^1 action induced from the complex multiplication. Property (P) implies that there is an equivariant imbedding $\psi: S^{n-1} \times \tilde{D}^{2k} \to \Sigma$, where the trivial S^1 action is considered on S^{n-1}. We do surgery on ψ equivariantly to obtain an S^1 manifold $\Sigma(\psi)$ with the orientation inherited from Σ. Note that the S^1 action on $\Sigma(\psi)$ is free; so the natural projection map from $\Sigma(\psi)$ to the orbit space $\Sigma(\psi)/S^1$ becomes an S^1 bundle. Since S^1 acts naturally on C as complex multiplication, one gets a complex line bundle associated with the S^1 bundle. Let $x(\psi)$ (resp. $D(\Sigma, \psi)$) denote the first Chern class (resp. the unit disk bundle) of the complex line bundle. Clearly $\partial D(\Sigma, \psi) = \Sigma(\psi)$. Since $\Sigma(\psi)$ is already oriented, we give a compatible orientation to $D(\Sigma, \psi)$. Moreover $\Sigma(\psi)/S^1$ is the base space of the disk bundle, and the fiber and the total space are oriented; so we again give a compatible orientation to $\Sigma(\psi)/S^1$. With these understood, we have the following.

DEFINITION 1.1. Suppose $n \equiv 0\,(4)$. Then we define

$$\beta(\Sigma, \psi) = (-x(\psi))^{k-1} \cup p(\Sigma(\psi)/S^1)[\Sigma(\psi)/S^1] \in Z,$$

where $p(\ )$ denotes the total Pontrjagin class and $[\ ]$ denotes the fundamental class.

REMARK 1.2. Obviously only $(n/4)$th Pontrjagin class contributes to the definition. One can define more invariants making use of other Pontrjagin classes.

It is not difficult to see that when $n \leq 2k$, $\beta(\Sigma, \psi)$ actually depends on the choice of ψ for a linear S^1 action on S^{n+2k-1}. The next theorem establishes the invariance of $\beta(\Sigma, \psi)$ when $n > 2k$.

THEOREM 1.3. *If $n > 2k$, then $\beta(\Sigma, \psi)$ is independent of a choice of ψ; hence $\beta(\Sigma, \psi)$ is an invariant of Σ.*

The rest of this section is devoted to the proof of this theorem. The G-signature theorem plays a role in the proof.

If X is an oriented manifold, then the notation $-X$ will be used for the same manifold with opposite orientation. We choose an orientation on the interval $[-1,1]$ such that $\partial[-1,1]=\{1\}\cup-\{-1\}$, where the natural point orientations are considered on $\{1\}$ and $\{-1\}$. The direct product $\Sigma\times[-1,1]$ is then an oriented S^1 manifold such that $\partial(\Sigma\times[-1,1])=\Sigma\times\{1\}\cup-\Sigma\times\{-1\}$. We identify $\Sigma\times\{1\}$ and $\Sigma\times\{-1\}$ with Σ and regard ψ as an imbedding to $\Sigma\times\{1\}$. Now choose another equivariant imbedding $\psi': S^{n-1}\times\tilde{D}^{2k}\to\Sigma=\Sigma\times\{-1\}$ and glue two copies of $D^n\times\tilde{D}^{2k}$ to $\Sigma\times[-1,1]$ via ψ and ψ'. The resulting oriented S^1 manifold $\Sigma(\psi,\psi')$ has $\Sigma(\psi)$ and $-\Sigma(\psi')$ as boundaries. Hence one can glue $-D(\Sigma,\psi)$ and $D(\Sigma,\psi')$ to $\Sigma(\psi,\psi')$ along their boundary. The resulting space,

$$W=D(\Sigma,\psi')\cup\Sigma(\psi,\psi')\cup(-D(\Sigma,\psi)),$$

is a closed oriented manifold with a semi-free S^1 action.

We shall apply the G-signature theorem to this W. For that purpose we observe the following facts:

(1.4) The S^1-fixed point set of W consists of three connected components: two of them are $\Sigma(\psi)/S^1$ and $-\Sigma(\psi')/S^1$, and the other one, denoted by F, is diffeomorphic to S^n.

(1.5) The total Chern class of the complex normal bundle of F is trivial because $k<n/2$; that is, the rank of the complex normal bundle is less than half of the dimension of F $(=S^n)$. Moreover, the Hirzebruch L-class of F is trivial since F is S^n.

(1.6) Since S^1 is a connected group, the induced action of S^1 on cohomology groups of W is trivial, and hence the S^1 signature of W is equal to Sign W, the signature of W.

Let t denote the complex 1-dimensional standard S^1 module. Using (1.4), (1.5), (1.6), and the G-signature theorem (see [7, p. 50] or [1]), we get an identity of rational functions of t:

$$\begin{aligned}\text{Sign}\, W=&L(\Sigma(\psi')/S^1)(te^{2x(\psi')}+1)/(te^{2x(\psi')}-1)[\Sigma(\psi')/S^1]\\ &-L(\Sigma(\psi)/S^1)(te^{2x(\psi)}+1)/(te^{2x(\psi)}-1)[\Sigma(\psi)/S^1],\end{aligned}\tag{1.7}$$

where $L(\ )$ denotes the Hirzebruch L-class.

LEMMA 1.8. *Let ρ stand for either ψ or ψ'. Then*

$$L(\Sigma(\rho)/S^1)(te^{2x(\rho)}+1)/(te^{2x(\rho)}-1)[\Sigma(\rho)/S^1]$$

has a pole of order k at $t=1$, and the coefficient of $1/(t-1)^k$ is a multiple of $\beta(\Sigma,\rho)$ that is independent of the choice of ρ. In fact, the multiple is

$$2^{2q+k}(2^{2q-1}-1)B_q/(2q)!,$$

where $q=n/4$ and B_q is the qth Bernoulli number (see Appendix B of [18]).

Theorem 1.3 will follow immediately from Lemma 1.8 and (1.7). In fact, (1.7) implies that the right-hand side of (1.7) cannot admit a pole, in particular, a pole of order k at $t=1$. Hence, by Lemma 1.8, $\beta(\Sigma, \psi)$ must coincide with $\beta(\Sigma, \psi')$.

Now we shall prove Lemma 1.8. First we note

(1.9) $\Sigma(\rho)/S^1$ has the form $D^n \times CP^{k-1} \cup D^n \times CP^{k-1}$ (see [3]);

hence it has the same cohomology ring as $S^n \times CP^{k-1}$ when $n > 2k$ (this is where the hypothesis $n > 2k$ enters). In particular, $x(\rho)^m = 0$ for $m \geq k$. Since we have

$$\begin{aligned}(te^{2x(\rho)}-1)^{-1} &= \frac{1}{t-1}\left\{1-\frac{t(1-e^{2x(\rho)})}{t-1}\right\}^{-1}\\ &= \frac{1}{t-1}\sum_{m=0}^{k-1}\left(\frac{t}{t-1}\right)^m(1-e^{2x(\rho)})^m \quad \text{by (1.9)},\end{aligned}$$

the expansion of $(te^{2x(\rho)}+1)/(te^{2x(\rho)}-1)$ with respect to $(t-1)$ has the pole of order k, and the coefficient of $1/(t-1)^k$ is $2^k(-x(\rho))^{k-1}$. Therefore, the coefficient of $1/(t-1)^k$ in $L(\Sigma(\rho)/S^1)(te^{2x(\rho)}+1)/(te^{2x(\rho)}-1)[\Sigma(\rho)/S^1]$ is

$$L_q(\Sigma(\rho)/S^1)2^k(-x(\rho))^{k-1}[\Sigma(\rho)/S^1],$$

where $L_q(\ )$ denotes the factor of $L(\ )$ with cohomology degree $4q$. Since $L_q(\ )$ is a polynomial of Pontrjagin classes with total cohomology degree $4q=n$, only the qth Pontrjagin class $p_q(\Sigma(\rho)/S^1)$ survives in $L_q(\Sigma(\rho)/S^1)$ by (1.9). As is well known, the coefficient of $p_q(\ )$ in $L_q(\ )$ is

$$2^{2q}(2^{2q-1}-1)B_q/(2q)!$$

(see Problem 19-C of [18]). The lemma follows from these observations.

2. Additivity of β

By virtue of Theorem 1.3 we may abbreviate $\beta(\Sigma, \psi)$ as $\beta(\Sigma)$. Remember that $\Sigma_k^n(S^1)$ is an abelian group (provided $n \geq 5$) under the equivariant connected sum operation #. In this section we prove the following theorem.

THEOREM 2.1. *Suppose $n > 2k$ as before. Then $\beta: \Sigma_k^n(S^1) \to Z$ is a homomorphism; that is,*

$$\beta(-\Sigma) = -\beta(\Sigma) \quad \textit{and} \quad \beta(\Sigma_1 \# \Sigma_2) = \beta(\Sigma_1) + \beta(\Sigma_2)$$

for $\Sigma, \Sigma_1, \Sigma_2 \in \Sigma_k^n(S^1)$.

Proof. The first property is obvious from the definition, because the fundamental class of $\Sigma(\psi)/S^1$ is reversed if we alter the orientation of Σ.

We shall verify the second property. The method is almost the same as that followed in Theorem 1.3. Take the product $\Sigma_i \times [-1, 1]$ for $i = 1, 2$, and form the equivariant boundary connected sum of them along $\Sigma_1 \times \{1\}$ and

$\Sigma_2 \times \{1\}$. Clearly the resulting S^1 manifold $W(\Sigma_1, \Sigma_2)$ gives an equivariant oriented cobordism between $\Sigma_1 \# \Sigma_2$ and $\Sigma_1 \cup \Sigma_2$. Let $\psi_i : S^{n-1} \times \tilde{D}^{2k} \to \Sigma_i$ and $\psi : S^{n-1} \times \tilde{D}^{2k} \to \Sigma_1 \# \Sigma_2$ be equivariant imbeddings. As discussed in the proof of Theorem 1.3, we glue three copies of $D^n \times \tilde{D}^{2k}$ to $W(\Sigma_1, \Sigma_2)$ using those imbeddings. The boundary of the resulting S^1 manifold is the disjoint union of $\Sigma_1 \# \Sigma_2(\psi)$, $-\Sigma_1(\psi_1)$, and $-\Sigma_2(\psi_2)$. Hence one can glue $-D(\Sigma_1 \# \Sigma_2, \psi)$, $D(\Sigma_1, \psi_1)$, and $D(\Sigma_2, \psi_2)$ along their boundary to obtain a closed oriented manifold W' with a semi-free S^1 action. Applying the G-signature theorem to this W' as before, we get a similar identity to (1.7):

$$\begin{aligned} \text{Sign}\, W' = {} & \sum_{i=1}^{2} L(\Sigma_i(\psi_i)/S^1)(te^{2x(\psi_i)}+1)/(te^{2x(\psi_i)}-1)[\Sigma(\psi_i)/S^1] \\ & -L(\Sigma_1 \# \Sigma_2(\psi)/S^1)(te^{2x(\psi)}+1)/(te^{2(\psi)}-1)[\Sigma_1 \# \Sigma_2(\psi)/S^1]. \end{aligned} \tag{2.2}$$

Repeat the argument done in the proof of Theorem 1.3 using Lemma 1.8. Then one can deduce the additivity of β from (2.2). □

3. Divisibility of β

In this section we shall investigate the largest integer that divides each element in the image of the homomorphism β. We deduce the divisibility condition from two sources: one is the Atiyah–Singer index theorem for a twisted Dirac operator and the other is obstruction theory. The reader will find that our method is essentially the same as in [9].

Suppose $n \equiv 0\,(4)$ and set $n = 4q$. The following lemma is a consequence of the Atiyah–Singer index theorem.

LEMMA 3.1. *$\beta(\Sigma)B_q/(2q)!\,2$ is an integer.*

Proof. Let η be the complex line bundle over $\Sigma(\psi)/S^1$ with the first Chern class $x(\psi)$. Since $H^3(\Sigma(\psi)/S^1; Z)$ vanishes by (1.9), $\Sigma(\psi)/S^1$ admits a Spin^c structure. Choosing a Spin^c structure gives rise to a Dirac operator on $\Sigma(\psi)/S^1$. We consider the index of the Dirac operator twisted by $(\eta-1)^{k-1}$. The Atiyah–Singer index theorem (see, e.g., [6, §26]) then yields the integrality of the number

$$\text{ch}(\eta-1)^{k-1} e^{c/2} \hat{A}(\Sigma(\psi)/S^1)[\Sigma(\psi)/S^1], \tag{3.2}$$

where ch denotes the Chern character, c is the first Chern class of the complex line bundle associated with the chosen Spin^c structure, and $\hat{A}$ denotes the $\hat{A}$ class. By (1.9), $\text{ch}(\eta-1)^{k-1} = x(\psi)^{k-1}$ and hence only the $p_q(\Sigma(\psi)/S^1))$ term in $\hat{A}(\Sigma(\psi)/S^1)$ contributes to the computation of (3.2). Here the coefficient of $p_q(\ )$ in the $\hat{A}$ class is $-B_q/(2q)!\,2$. Hence (3.2) implies that

$$\{-B_q/(2q)!\,2\} x(\psi)^{k-1} p_q(\Sigma(\psi)/S^1)[\Sigma(\psi)/S^1] \in Z.$$

This, together with Definition 1.1, proves the lemma. □

The following lemma is a consequence of obstruction theory.

LEMMA 3.3. *$\beta(\Sigma)$ is divisible by $(2q-1)!\,a_q$, where a_q equals 1 or 2 depending on whether q is (respectively) even or odd.*

Proof. The Gysin exact sequence for the S^1 bundle $\pi: \Sigma(\psi) \to \Sigma(\psi)/S^1$ yields an isomorphism $\pi^*: H^{4q}(\Sigma(\psi)/S^1; Z) \to H^{4q}(\Sigma(\psi); Z)$. On the other hand, the tangent bundle of $\Sigma(\psi)$ is isomorphic to the Whitney sum of the pull-back of the tangent bundle of $\Sigma(\psi)/S^1$ by π and the bundle tangent along the fibers. The latter is a real line bundle, so its total Pontrjagin class is trivial. Since $H^*(\Sigma(\psi)/S^1; Z)$ has no 2-torsion by (1.9), the above total Pontrjagin classes behave multiplicatively with respect to the Whitney sum (see [18, Thm. 15.3]). Hence

$$\pi^* p(\Sigma(\psi)/S^1) = p(\Sigma(\psi)).$$

In the sequel it suffices to verify that $p_q(\Sigma(\psi))$ is divisible by $(2q-1)!\,a_q$.

Stabilize the tangent bundle of $\Sigma(\psi)$ by adding a trivial bundle of large dimension and consider its associated principal SO_ℓ (ℓ: large) bundle ξ, where SO_ℓ denotes the special orthogonal group on ℓ-dimensional Euclidean space. We undertake to construct a cross-section of ξ over $\Sigma(\psi)$. Since $\Sigma(\psi)$ has the same cohomology as $S^{4q} \times S^{2k-1}$ (cf. (1.9)) and $2k-1 < 4q$ by the assumption, the primary obstruction θ lies in $H^{2k-1}(\Sigma(\psi); \pi_{2k-2}(\mathrm{SO}_\ell))$. We note that $\Sigma(\psi)$ contains $D^{4q} \times S^{2k-1}$ as a submanifold via ψ and that the tangent bundle of $D^{4q} \times S^{2k-1}$ is trivial. Hence the restriction of θ to $D^{4q} \times S^{2k-1}$ vanishes. However, the restriction map induces an isomorphism between cohomology groups of degree $2k-1$; hence θ itself vanishes.

Thus the secondary obstruction θ_q emerges in $H^{4q}(\Sigma(\psi); \pi_{4q-1}(\mathrm{SO}_\ell))$, which is isomorphic to $H^{4q}(\Sigma(\psi); Z)$ because $\pi_{4q-1}(\mathrm{SO}_\ell) \cong Z$ when ℓ is large. Through this identification, θ_q is related to $p_q(\Sigma(\psi))$ as follows:

$$p_q(\Sigma(\psi)) = (2q-1)!\,a_q \theta_q,$$

which is established in (ii) of Lemma 1.1 of [9]. This completes the proof of the Lemma. □

Decompose $B_q\beta(\Sigma)/(2q)!\,2$ into two factors as follows:

$$B_q\beta(\Sigma)/(2q)!\,2 = \beta(\Sigma)/(2q-1)!\,a_q \times a_q B_q/4q.$$

The total and the first factor are both integers by Lemma 3.1 and Lemma 3.3 (respectively). This means that $\beta(\Sigma)$ is divisible by

$$(2q-1)!\,a_q \cdot \text{denominator}(a_q B_q/4q).$$

Hence we pose the following.

DEFINITION 3.4. $\hat{\beta}(\Sigma) = \beta(\Sigma)/(2q-1)!\,a_q \cdot \text{denominator}(a_q B_q/4q) \in Z.$

4. A Generator of $\Sigma_k^n(S^1)$

As is well known, certain Brieskorn homotopy spheres naturally support semi-free S^1 actions satisfying Property (P). In this section we shall compute our $\hat{\beta}$ invariant of those examples to see that $\hat{\beta}$ can attain 1 or 2.

Let bP_{4m} $(m \geq 2)$ denote the set of diffeomorphism classes of oriented homotopy spheres bounding a parallelizable manifold of dimension $4m$. Connected sum operation makes it an abelian group. As a matter of fact it is a cyclic group of order b_m, where

(4.1) $$b_m = 2^{2m-2}(2^{2m-1}-1)\cdot \text{numerator}(4B_m/m)$$

(cf. [10, p. 531]).

We shall recall an explicit description for an element of bP_{4m}. Let δ be a fixed small number. For each integer h let M_h^{4m} (resp. Σ_h^{4m-1}) denote a manifold of dimension $4m$ (resp. $4m-1$) defined as the intersection of the algebraic set

$$\{(u, v, z_1, \dots, z_{2m-1}) \in C^{2m+1} \mid u^3 + v^{6h-1} + z_1^2 + \cdots + z_{2m-1}^2 = \delta\}$$

with the unit disk (resp. sphere) of C^{2m+1}. Clearly $\partial M_h^{4m} = \Sigma_h^{4m-1}$. The following facts are well known.

(4.2) $\operatorname{Sign} M_h^{4m} = 8h$ if we choose a suitable orientation on M_h^{4m}.

Hereafter M_h^{4m} will be oriented so that (4.2) is satisfied and Σ_h^{4m-1} will be oriented as its boundary.

(4.3) $\Sigma_h^{4m-1} \in bP_{4m}$ and Σ_1^{4m-1} is a generator of bP_{4m}.

(4.4) Σ_h^{4m-1} is diffeomorphic to S^{4m-1} if and only if h is a multiple of b_m.

These manifolds M_h^{4m} and Σ_h^{4m-1} support semi-free S^1 actions defined by rotating the last $2r$ $(r < m)$ coordinates pairwise. To be precise, an element $\exp(i\theta)$ of S^1 acts on them by

$$\begin{bmatrix} u \\ v \\ z_1 \\ \vdots \\ z_{2m-1} \end{bmatrix} \to \begin{bmatrix} I & & & \\ & D(\theta) & & \\ & & \ddots & \\ & & & D(\theta) \end{bmatrix} \begin{bmatrix} u \\ v \\ z_1 \\ \vdots \\ z_{2m-1} \end{bmatrix}, \quad D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where I is the $(2m-2r+1)\times(2m-2r+1)$ identity matrix and there are r copies of $D(\theta)$ in the diagonal. We shall denote M_h^{4m} and Σ_h^{4m-1} with this semi-free S^1 actions by $M_h^{q,r}$ and $\Sigma_h^{q,r}$ (respectively), where $q = m-r$.

These S^1 manifolds have the following properties:

(4.5) The S^1 fixed point set of $M_h^{q,r}$ (resp. $\Sigma_h^{q,r}$) is M_h^{4q} (resp. Σ_h^{4q-1}).

(4.6) The complex normal bundle of the fixed point set M_h^{4q} (resp. Σ_h^{4q-1}) to M_h^{4m} (resp. Σ_h^{4m-1}) is trivial.

Therefore (4.4), (4.5), and (4.6) tell us that $\Sigma_h^{q,r}$ belongs to $\Sigma_{2r}^{4q}(S^1)$ if and only if h is a multiple of b_q. With these understood, we have the following theorem.

THEOREM 4.7. *Suppose $q>r$. Then $\hat{\beta}(\Sigma_{b_q}^{q,r})=a_q$, where a_q is the same as in Lemma* 3.3.

Proof. To simplify notations we shall abbreviate $M_{b_q}^{q,r}$ and $\Sigma_{b_q}^{q,r}$ as M and Σ (respectively).

Choose an equivariant imbedding $\psi: S^{4q-1}\times\tilde{D}^{4r}\to\Sigma$ and glue $D^{4q}\times\tilde{D}^{4r}$ to $-M$ via ψ. Then the boundary of the resulting oriented S^1 manifold is precisely $-\Sigma(\psi)$; so we can glue $D(\Sigma,\psi)$ (see §1) along their boundary to obtain a closed oriented manifold W with a semi-free S^1 action.

Observe the following facts:

(4.8) The fixed point set of W consists of two connected components: one is $-M_{b_q}^{4q}\cup D^{4q}=F$ and the other is $\Sigma(\psi)/S^1$.

(4.9) The total Chern class of the complex normal bundle of F to W is trivial by (4.6), except for the $2q$th Chern class which is trivial for the dimensional reason—the complex dimension $2r$ of the bundle is less than $2q$.

(4.10) $L(F)[F]=\operatorname{Sign}F=\operatorname{Sign}(-M_{b_q}^{4q})+\operatorname{Sign}D^{4q}=-8b_q$ by the signature theorem, additivity of signature, and (4.2).

Putting these facts together with the G-signature formula of Sign W, we get an identity

$$\operatorname{Sign}W=-8b_q\{(t+1)/(t-1)\}^{2r}$$
$$+L(\Sigma(\psi))(te^{2x(\psi)}+1)/(te^{2x(\psi)}-1)[\Sigma(\psi)/S^1].$$

By Lemma 1.8, the coefficient of $1/(t-1)^{2r}$ in the right-hand side of this identity is given by

$$-2^{2r+3}b_q+2^{2q+2r}(2^{2q-1}-1)B_q/(2q)!\times\beta(\Sigma),$$

which must be zero. Replacing b_q by (4.1), we find that

$$\beta(\Sigma)=(2q)!\,2\cdot\text{numerator}(4B_q/q)/B_q.$$

Hence by Definition 3.4 we have

$$\hat{\beta}(\Sigma)=\frac{4q\cdot\text{numerator}(4B_q/q)}{a_qB_q\cdot\text{denominator}(a_qB_q/4q)}. \tag{4.11}$$

Here, recall that the highest power of 2 dividing the denominator of B_q/q is $2^{\mu+1}$, where 2^μ is the highest power of 2 dividing q (see [18, p. 284]). This implies

$$\begin{aligned}\text{numerator}(4B_q/q)&=a_q\cdot\text{numerator}(B_q/q),\\ \text{denominator}(a_qB_q/4q)&=4\cdot\text{denominator}(B_q/q)/a_q.\end{aligned} \tag{4.12}$$

Putting (4.12) into (4.11), we get $\hat{\beta}(\Sigma)=a_q$. □

COROLLARY 4.13. *Suppose $q > r$ and q is even. Then $\Sigma_{b_q}^{q,r}$ is one of the generators of the free part of $\Sigma_{2r}^{4q}(S^1)$. In particular, it generates the free part of $\Sigma_2^{4q}(S^1)$ because* $\operatorname{rank}_Z \Sigma_2^{4q}(S^1) = 1$.

REMARK 4.14. If we relax the first condition of Property (P) to the condition that the fixed point set belongs to bP_n, then we obtain an analogous abelian group $\tilde{\Sigma}_k^n(S^1)$ containing $\Sigma_k^n(S^1)$ as a subgroup of finite index. Hence $\hat{\beta}$ can be extended to $\tilde{\Sigma}_k^n(S^1)$ algebraically. Since $b_q \tilde{\Sigma}_k^{4q}(S^1)$ is contained in $\Sigma_k^{4q}(S^1)$, $b_q \hat{\beta}$ is an *integer-valued* homomorphism from $\tilde{\Sigma}_k^{4q}(S^1)$. On the other hand, one can prove $\hat{\beta}(b_q \Sigma_1^{q,r}) = a_q$ in a similar fashion to Theorem 4.7. Hence we have $b_q \hat{\beta}(\Sigma_1^{q,r}) = \hat{\beta}(b_q \Sigma_1^{q,r}) = a_q$. These show that $\Sigma_1^{q,r}$ is one of generators of the free part of $\tilde{\Sigma}_{2r}^{4q}(S^1)$, provided that $q > r$ and q is even.

5. Splitting of Codimension-3 Knot Exact Sequences

As first observed by Montgomery and Yang [19] and then by Levine [12] in more detail, the study of semi-free S^1 actions on oriented homotopy spheres with codimension-4 fixed point set is essentially equivalent to that of oriented knots with codimension 3. On the other hand, the set of oriented knots with codimension greater than 2 is fairly well understood in terms of an exact sequence [11]. The work of Levine [12] is done from this point of view. However, as far as the author knows, little is known about group extension of the exact sequence (cf. [12, §IV]). Our homomorphism $\hat{\beta}$ can be used to give an information concerning the splitting.

Following [11], let $\Sigma^{n+3,n}$ $(n \geq 5)$ denote the abelian group of isotopy classes of oriented pairs (Σ^{n+3}, K), where Σ^{n+3} and K are diffeomorphic to S^{n+3} and S^n (respectively). Let $\bar{\Sigma}^{n+3,3}$ denote the abelian group defined similarly to $\Sigma^{n+3,n}$, but this time we only require that Σ^{n+3} be a homotopy $(n+3)$-sphere. The group structures on them are given by knot connected sum. Obviously we have a canonical isomorphism:

$$\bar{\Sigma}^{n+3,n} = \Sigma^{n+3,n} \oplus \Theta^{n+3}, \tag{5.1}$$

where Θ^{n+3} is the abelian group of isotopy classes of oriented homotopy $(n+3)$-spheres. We note that the normal bundle of a knot K to Σ^{n+3} is trivial (see [12, p. 171]). Levine [12] observed that taking the S^1 orbit space gives rise to an isomorphism:

$$\Psi : \Sigma_2^n(S^1) \cong \bar{\Sigma}^{n+2,n-1}. \tag{5.2}$$

In fact the orbit space again turns out to be a homotopy sphere, and the fixed point set defines a knot in the orbit space.

Recall the short exact sequence (0.2). It is a direct consequence of the exact sequence $(3)_k$ on p. 20 and $(7)_{n,k}$ on p. 44 of [11]. There, $\sigma_3(n-1, 3)$ is the suspension homomorphism from $\pi_{n-1}(G_3, SO_3)$ to $\pi_{n-1}(G, SO)$ (see [11, p. 39]). The definition of the homomorphism ∂_3' is as follows. Hereafter we put $n = 4q$. Recall that Σ_n^{4q-1} defined in Section 4 is an oriented submanifold of the unit sphere S^{4q+1} of C^{2q+1} and hence of S^{4q+2}. By (4.4), the oriented pair $(S^{4q+2}, \Sigma_h^{4q-1})$ belongs to $\Sigma^{4q+2,4q-1}$ if and only if h is a multiple of b_q. Then $\partial_3'(\gamma)$ is defined to be the oriented pair $(S^{4q+2}, \Sigma_{\gamma b_q}^{4q-1})$ for $\gamma \in Z$ (cf. [11, pp. 35, 29]).

With these understood, we will prove the following theorem.

THEOREM 5.3. *Suppose $n=4q$ and $q \geq 2$. Then there exists a homomorphism $\hat{\beta}_3: \Sigma^{4q+2,4q-1} \to Z$ such that the composition $\hat{\beta}_3 \circ \partial_3'$ is multiplication by a_q. In particular, the exact sequence* (0.2) *is split when q is even.*

The definition of $\hat{\beta}_3$ is as follows. Via the isomorphism (5.2), $\hat{\beta}$ can be regarded as a homomorphism from $\bar{\Sigma}^{4q+2,4q-1}$ to Z. Since the group Θ^{4q+2} is a torsion group in (5.1), $\hat{\beta}$ falls into a homomorphism from $\Sigma^{4q+2,4q-1}$ to Z, which is the desired $\hat{\beta}_3$.

We shall interpret $\hat{\beta}_3$ geometrically. Let Σ be an element of $\Sigma_2^{4q}(S^1)$ and let $\psi: S^{4q-1} \times \tilde{D}^4 \to \Sigma$ be an equivariant imbedding. By Definitions 1.1 and 3.4, $\hat{\beta}(\Sigma)$ can be calculated once we know the qth Pontrjagin class of $\Sigma(\psi)/S^1$.

We note that $\Sigma(\psi)/S^1$ is also obtained by doing surgery of Σ/S^1 along the knotted sphere (i.e. the fixed point set) on the framing induced from ψ by taking the orbit spaces. This means that given a knot (S^{4q+2}, K) of $\Sigma^{4q+2,4q-1}$, it suffices to calculate the qth Pontrjagin class of the manifold obtained by doing surgery of S^{4q+2} along K. Note that Theorem 1.3 ensures the value to be independent of a choice of a framing used at the surgery along K.

LEMMA 5.4. *If K bounds an oriented parallelizable submanifold V_K in D^{4q+3} with trivial normal bundle, then*

$$\hat{\beta}_3((S^{4q+2}, K)) = \text{Sign}\, V_K q/2^{2q-1}(2^{2q-1}-1)B_q \cdot \text{denominator}(a_q B_q/4q).$$

Proof. Let $\phi: S^{4q-1} \times D^3 \to S^{4q+2}$ be a normal framing of K extending to a framing of the normal bundle of V_K in D^{4q+3}, and glue the handle $D^{4q} \times D^3$ to S^{4q+2} via ϕ. The boundary of the resulting manifold $D(\phi)$ is precisely the manifold obtained by doing surgery of S^{4q+2} along K on ϕ. We want to know $p_q(\partial D(\phi))$.

Let $i: \partial D(\phi) \to D(\phi)$ be the inclusion map. Observe that

$$i^*: H^{4q}(D(\phi); Z) \to H^{4q}(\partial D(\phi); Z) \quad (\cong Z)$$

is an isomorphism. Since the restriction of the tangent bundle of $D(\phi)$ to $\partial D(\phi)$ decomposes to the Whitney sum of the tangent bundle of $\partial D(\phi)$ and the trivial line bundle, we get $i^* p_q(D(\phi)) = p_q(\partial D(\phi))$. Thus we may calculate $p_q(D(\phi))$ instead of $p_q(\partial D(\phi))$.

We note that $D(\phi)$ contains a closed submanifold $V_K \cup D^{4q}$ attached by ϕ. Its normal bundle to $D(\phi)$ is trivial because ϕ is chosen to extend to a framing of the normal bundle of V_K to D^{4q+3}. This implies

$$j^* p_q(D(\phi)) = p_q(V_K \cup D^{4q}),$$

where j is the inclusion map from $V_K \cup D^{4q}$ to $D(\phi)$. Since

$$j^*: H^{4q}(D(\phi); Z) \to H^{4q}(V_K \cup D^{4q}; Z)$$

is an isomorphism and $V_K \cup D^{4q}$ is an oriented manifold of dimension $4q$, it suffices to compute the evaluation of $p_q(V_K \cup D^{4q})$ by the fundamental class of $V_K \cup D^{4q}$. By assumption, V is parallelizable. This means that the Pontrjagin classes of $V_K \cup D^{4q}$ are trivial except for the qth one. It follows from additivity of signature and the signature theorem that

$$\begin{aligned}\operatorname{Sign} V_K &= \operatorname{Sign}(V_K \cup D^{4q}) \\ &= 2^{2q}(2^{2q-1}-1)B_q/(2q)!\, p_q(V_K \cup D^{4q})[V_K \cup D^{4q}].\end{aligned}$$

Consequently we have verified

$$(-x(\phi)) \cup p_q(\partial D(\phi))[\partial D(\phi)] = \operatorname{Sign} V_K (2q)!/2^{2q}(2^{2q-1}-1)B_q,$$

where $x(\phi)$ is a suitable generator of $H^2(\partial D(\phi); Z)$. On the other hand, by the definition of $\hat{\beta}_3$ and Definition 3.4, we have

$$\begin{aligned}&\hat{\beta}_3((S^{4q+2}, K)) \\ &\quad = (-x(\phi)) \cup p_q(\partial D(\phi))[\partial D(\phi)]/(2q-1)!\, a_q \cdot \operatorname{denominator}(a_q B_q/4q).\end{aligned}$$

These two identities verify the lemma. □

Proof of Theorem 5.3. Recall that $\partial_3'(1) = (S^{4q+2}, \Sigma_{b_q}^{4q-1})$ and $\Sigma_{b_q}^{4q-1}$ bounds a manifold $M_{b_q}^{4q}$ sitting naturally in D^{4q+2} and hence in D^{4q+3} (see §4 for $M_{b_q}^{4q}$). As is well known, $M_{b_q}^{4q}$ is parallelizable and the normal bundle in D^{4q+3} is trivial. Since $\operatorname{Sign} M_{b_q}^{4q} = 8b_q$ by (4.2), we apply Lemma 5.4 to get

$$\begin{aligned}\hat{\beta}_3(\partial_3'(1)) &= 8qb_q/2^{2q-1}(2^{2q-1}-1)B_q \cdot \operatorname{denominator}(a_q B_q/4q) \\ &= 4q \cdot \operatorname{numerator}(4B_q/q)/B_q \cdot \operatorname{denominator}(a_q B_q/4q) \quad \text{by (4.1)} \\ &= a_q \quad \text{by (4.12)},\end{aligned}$$

which verifies the theorem. □

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