Bounded Group Actions on Trees and Hyperbolic and Lyndon Length Functions

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Introduction

In [6] it is shown that if a group G acting as a group of isometries on a tree T has a normal subgroup K with bounded action, then the factor group G/K acts on a quotient tree T/K. This paper considers possibilities for K and relationships between the two actions.

A group action gives an associated hyperbolic length function, defined in [1] and [5], and also for each point of the tree a Lyndon length function, defined originally in [4]. In Section 1 it is shown that the hyperbolic length function on G is determined by that on G/K, and in Section 2 it is shown that the Lyndon length functions on G are determined up to equivalence by those on G/K. In Theorem 2.2 it is established that the normal subgroups K with bounded action are contained in subgroups of G defined by Lyndon length functions, and that there is a maximal K. In Section 3, under the assumption that not every element of G has a fixed point, it is shown that (with K maximal) G/K is isomorphic to a subgroup of the additive reals or has a trivial centre.

1. Bounded Actions and Hyperbolic Lengths

Let a group G act as a group of isometries on a metric tree (or **R**-tree) T, equipped with a metric d. The following notation and properties are recalled from [6], where more detail may be found.

A metric tree T has the property that, for any two points $u, v \in T$, there is a unique isometry $\alpha: [0, r] \to T$ with $\alpha(0) = u$ and $\alpha(r) = v$, where $r = d(u, v)$. The image $\alpha([0, r])$ is denoted by $[u, v]$ and is called a *segment* of T.

For each $u \in T$, a Lyndon length function $\ell_u: G \to \mathbf{R}$ is defined by $\ell_u(x) = d(u, xu)$. The set N of non-Archimedean elements of G consists of elements x such that $\ell_u(x^2) \le \ell_u(x)$ for some (and hence all) $u \in T$. It is shown in [6, Prop. 2.2] that $x \in N$ if and only if it fixes some point of T.

A subgroup K of G has *bounded action* on T if, for some (and hence each) $u \in T$, the set of lengths $\{\ell_u(x); x \in K\}$ is bounded. Theorem 3.2 of [6] states that K has bounded action if and only if it fixes some point of T. If a normal

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subgroup K of G has bounded action on T then the factor group G/K acts on the quotient tree T/K, as described in Section 4 of [6].

LEMMA 1.1. *If a normal subgroup K of G fixes a point u of T, then K fixes xu for each $x \in G$.*

Proof. If $a \in K$ then (since K is a normal subgroup) $ax = xa'$ for some $a' \in K$. The point u is fixed by K and so $a'u = u$. Thus $axu = xa'u = xu$, showing that xu is fixed by each $a \in K$. □

Each element $x \notin N$ has a unique *axis* in T—that is, an isometric image of $\mathbf{R}$ such that if $\alpha: \mathbf{R} \to T$ is the isometry then $\alpha^{-1}x\alpha$ is a translation of $\mathbf{R}$.

LEMMA 1.2. *If a normal subgroup K of G has bounded action on T, then K fixes each axis of T pointwise.*

Proof. Each $x \notin N$ has an axis in T. The existence of axes is established in [1, Thm. 6.6] and [5, Thm. II.2.3], where it is shown that for any $u \in T$ the axis for x is contained in the union of segments $\bigcup_{n \in \mathbf{Z}} [x^n u, x^{n+1} u]$. (In fact, in [5, proof of Lemma II.2.4] it is shown that the axis $= \bigcup_{n \in \mathbf{Z}} [x^n v, x^{n+1} v]$, where $[u, xu] \cap [u, x^{-1} u] = [u, v]$.)

If K has bounded action then it fixes some point u of T by Theorem 3.2 of [6]. By Lemma 1.1, the points $x^n u$ are fixed by K for each $n \in \mathbf{Z}$. Since the points of a segment are uniquely determined by the end points, it follows that each of the points of $[x^n u, x^{n+1} u]$ is fixed by K. The axis for x is therefore fixed pointwise by K. □

The *hyperbolic length function* $L: G \to \mathbf{R}$ of an action of G on T is defined by

$$L(x) = \min_{u \in T} \ell_u(x) = \min_{u \in T} d(u, xu).$$

If $x \in N$, then x has a fixed point in T and so $L(x) = 0$. If $x \notin N$, then by [1, Thm. 6.6] and [5, Thm. II.2.3] $L(x)$ is the translation length on the axis for x. In either case, it is also shown in [1, Cor. 6.13] that, for any $u \in T$, $L(x) = \max(0, \ell_u(x^2) - \ell_u(x))$.

THEOREM 1.3. *Let G act as a group of isometries on a tree T, with K a proper normal subgroup of G having bounded action. If $L: G \to \mathbf{R}$ and L': $G/K \to \mathbf{R}$ are the hyperbolic length functions associated (respectively) with the actions of G on T and of G/K on T/K, then $L(x) = L'(xK)$.*

Proof. If $x \in N$ then x has a fixed point $u \in T$, by Proposition 2.2 of [6]. The element xK then fixes $[u]$ in T/K, and so $L(x) = L'(xK) = 0$.

If $x \notin N$ then, by Lemma 1.2, K fixes the axis for x pointwise. So in T/K the element xK has an identical axis on which it acts in the same way as x. Hence $L(x) = L'(xK)$, the translation length on the axis. □

2. Bounded Actions and Lyndon Lengths

In this section, possible normal subgroups of G with bounded action are determined in relation to subgroups of G defined by Lyndon length functions.

For $\ell: G \to \mathbf{R}$ a Lyndon length function on G, define

$$A = \{a \in G;\ \ell(a) = 0\}, \qquad H = \{a \in G;\ \ell(ax) = \ell(x) \text{ for all } x \notin N\}.$$

It is an easy consequence of the axioms for a length function that A is a subgroup of G contained in H. It is shown in Theorem 1.4 of [2] (where H is denoted by T) that H is a subgroup of G contained in N. Thus $A \subseteq H \subseteq N$.

Let $2c(x,y) = \ell(x) + \ell(y) - \ell(xy^{-1})$, and let

$$b = \inf\{2c(x,y), \ell(z);\ \text{for all } x,y,z \in G \backslash H \text{ with } xy^{-1} \in G \backslash H\}.$$

It is shown in Proposition 1.3 and Theorem 1.4 of [2] that b is a bound for the lengths of the elements of H, and that a length function ℓ' on G may be formed by replacing the lengths of elements of H by any length function on H bounded by b. The length function ℓ' is said to be *equivalent* to ℓ. Thus ℓ and ℓ' are equivalent if the subgroups H for ℓ and ℓ' are identical and $\ell(x) = \ell'(x)$ for $x \notin H$. For $N \neq G$ the bound b exists (if $N = G$ then $H = N = G$). We note that if $b = 0$ then the elements of H have zero lengths, so that $A = H$ and $\ell = \ell'$.

Suppose that G acts as a group of isometries on a tree T. For each $u \in T$ the subgroups A_u and H_u are defined as above, associated with the Lyndon length function $\ell_u: G \to \mathbf{R}$. Thus $A_u \subseteq H_u \subseteq N$.

PROPOSITION 2.1. *If the point v lies on an axis in T then $A_v = H_v$.*

Proof. Suppose that v lies on the axis for x. Then, since x acts as a translation on the axis,

$$\ell_v(x^2) = d(v, x^2 v) = 2d(v, xv) = 2\ell_v(x).$$

Thus $2c(x, x^{-1}) = 2\ell_v(x) - \ell_v(x^2) = 0$, and (since $x, x^{-1}, x^2 \notin N$) the bound b on the lengths for H_v is zero. Hence $A_v = H_v$. □

THEOREM 2.2. *Let G act as a group of isometries on a tree T, with K a normal subgroup of G and $N \neq G$.*

(i) *If $K \subseteq H_u$ for some $u \in T$, then K has bounded action.*
(ii) *If K has bounded action, then $K \subseteq H_u$ for each $u \in T$.*

Proof. The lengths of the elements of H_u are bounded by b. Thus, if $K \subseteq H_u$, the lengths of the elements of K are bounded by b and so K has bounded action, proving (i).

Let K have bounded action with $a \in K$ and $x \notin N$. Then, to show $K \subseteq H_u$ and establish (ii), we need to prove that $\ell_u(ax) = \ell_u(x)$ for any $u \in T$. In

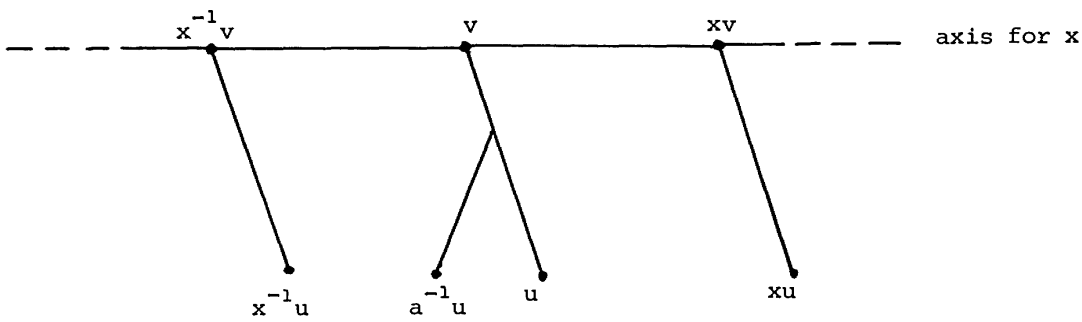

Figure 1

the proofs of Theorem 6.6 of [1] and Lemma II.2.4 of [5] it is shown that the segment $[u, xu]$ intersects the axis for x in the segment $[v, xv]$, where $[u, xu] \cap [u, x^{-1}u] = [u, v]$. By Lemma 1.2, K fixes v and xv; thus the isometry a sends the segment $[a^{-1}u, v]$ to $[u, v]$ and hence $d(a^{-1}u, v) = d(u, v)$ (see Figure 1). Thus

$$\begin{aligned} \ell_u(ax) &= d(u, axu) = d(a^{-1}u, xu) \\ &= d(a^{-1}u, v) + d(v, xv) + d(xv, xu) \\ &= d(u, v) + d(v, xv) + d(xv, xu) \\ &= d(u, xu) = \ell_u(x). \end{aligned}$$ □

COROLLARY 2.3. *Let G act as a group of isometries on a tree T, with K a proper normal subgroup of G. If K has bounded action then G/K acts on a tree T/K such that, for each $u \in T$, the Lyndon length function ℓ_u is equivalent to an extension of a length function ℓ_1 on K by $\ell_{[u]}$ on G/K.*

Proof. Define $\ell' \colon G \to \mathbf{R}$ by $\ell'(x) = \ell_{[u]}(xK)$. The length function ℓ' is then an extension of $\ell_1 = 0$ on K by $\ell_{[u]}$ on G/K. By Theorem 2.2, $K \subseteq H_u$; by [2, Thm. 1.4], $\ell_u(ax) = \ell_u(x)$ for each $a \in H_u$ and $x \notin H_u$. Thus $\ell_u(ax) = \ell_u(x)$ for each $a \in K$ and $x \notin H_u$, so $d(u, axu) = d(u, xu)$. Thus, in the notation of [6],

$$\begin{aligned} \ell'(x) &= \ell_{[u]}(xK) = d'([u], xK[u]) = d'([u], [xu]) \\ &= \inf_{a \in K} d(u, axu) = d(u, xu) = \ell_u(x). \end{aligned}$$

Since ℓ_u and ℓ' agree outside H_u, they are equivalent. □

COROLLARY 2.4. *If G acts as a group of isometries on a tree T with $N \neq G$, then there is a maximal normal subgroup of G having bounded action on T, namely $K = \text{core}\, H_u$, for any $u \in T$. Moreover, under the action on T/K the group G/K has no nontrivial normal subgroup with bounded action.*

Proof. By Theorem 2.2, the normal subgroups of G with bounded action are those normal subgroups contained in H_u. The maximal normal subgroup with this property is the core H_u, which is $\bigcap_{x \in G} x^{-1} H_u x$.

Suppose that K fixes $v \in T$. Then, in the notation of [6],

$$\ell_{[v]}(xK) = d'([v], xK[v]) = d'([v], [xv])$$
$$= d'([xv], [v]) = \inf_{a \in K} d(xv, av) = d(xv, v) = \ell_v(x)$$

for any $x \in G$. Thus, if a normal subgroup of G/K has bounded action on T/K with the lengths of its elements under $\ell_{[v]}$ bounded by M, then its pre-image in G will have bounded action on T with the lengths of its elements under ℓ_v also bounded by M. The maximal property for K ensures that no such nontrivial normal subgroup of G/K can exist. □

The results of this section are illustrated by the following simple example. Let $G = \langle a, b, c;\, a^2 = b^2 = c^2 = acac = bcbc = 1 \rangle$; that is, G is the direct product of the free product of two groups of order 2 with another group of order 2. Let G act on the tree T shown in Figure 2. The isometries a and b are given by reflections in vertical lines through w and v, respectively. The isometry c

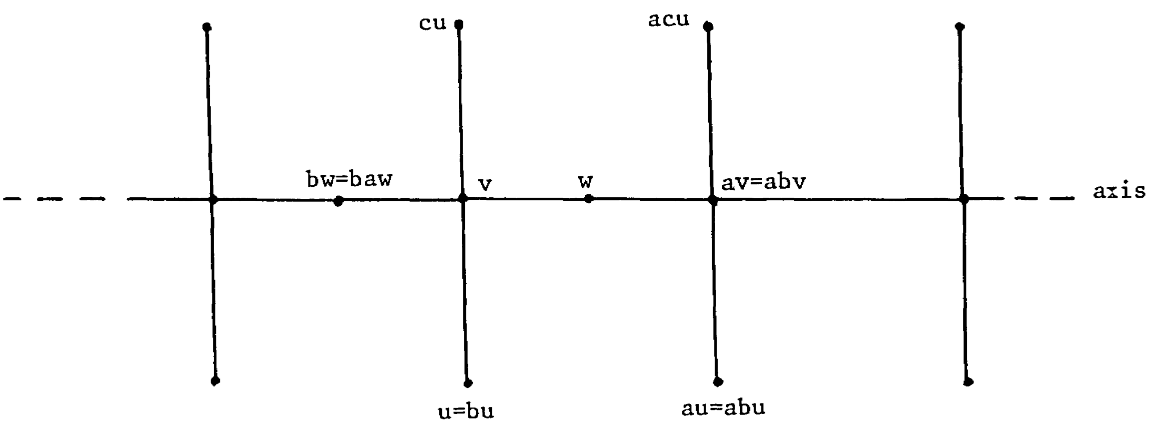

Figure 2

is given by reflection in the horizontal line, which is the axis for all Archimedean elements of G. Here $A_w = H_w = \langle a, c \rangle$, $A_v = H_v = \langle b, c \rangle$, and $H_u = \langle b, c \rangle$. The maximal normal subgroup K with bounded action is the group of order 2 generated by c. The quotient tree T/K is given by identifying the upper and lower projections from the axis. The Lyndon lengths functions ℓ_w and ℓ_v are extensions as they stand. However, $\ell_u(c) \neq 0$ with $\ell_u(b) = 0$, and ℓ_u is equivalent to an extension given by reducing the length of c to zero.

3. Centres of Group Actions

Theorem 2.2 and Corollary 2.3, relating the maximal normal subgroup of G with bounded action K to equivalence of length functions, allow the results of [2, Thm. 2.1] to be translated to results for group actions on trees.

If for an action of G on a tree T the subset N has bounded action, then $N = K = H_u$ for each $u \in T$, and we have the situation described in [6, Thm. 4.4]. In parts (i) and (ii) of [2, Thm. 2.1] there is the case of an Archimedean length function on G/N which is an abelian group. It is shown in [3, §3] that here G/N is isomorphic to a subgroup of the additive reals. We thus have the following.

THEOREM 3.1. *Let G act as a group of isometries on a tree T, with $N \neq G$. Let K be the maximal normal subgroup of G having bounded action, and let Z be the centre of G.*

(i) *If $Z \not\subseteq N$, then $K = N$ and G/N (which is isomorphic to a subgroup of the additive reals) acts on the tree T/N without fixed points.*

(ii) *If $Z \subseteq N$, which has bounded action on T, then $K = N$ and G/N (which is isomorphic to a subgroup of the additive reals or has trivial centre) acts on the tree T/N without fixed points.*

(iii) *If $Z \subseteq N$, which does not have bounded action on T, then G/K has trivial centre.*

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