A Fixed-Point Free Ergodic Flow on the 3-Sphere

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The 3-sphere S^3 supports a fixed-point free flow φ because the Euler characteristic of S^3 is zero; the same is true for any 3-manifold. However, it is not known what other properties may be imposed on φ. For example, the Seifert problem asks whether there exists a smooth flow on S^3 with neither fixed points nor periodic orbits. The problem is open. It is not even known whether S^3 supports a minimal flow—one whose only compact invariant sets are S^3 and $\emptyset$.

Ergodicity is a kind of measure-theoretic minimality—the only measurable invariant sets are of full measure or zero measure. In [6], Katok constructed examples of ergodic diffeomorphisms of surfaces. Here we point out how to "Birkhoff-suspend" them and induce flows on S^3 (or any lens space) that are smooth, ergodic respecting Lebesgue measure, *and have no fixed points*. Katok [7] has already given an example of such a flow on $\mathbf{P}^3$, but, being the geodesic flow of a semi-Finsler, it is somewhat difficult to picture. In contrast, the topology of our construction is fairly natural. It was used earlier by D. Fried (unpublished) to construct a smooth flow on S^3 for which the Lebesgue measure class is ergodic. (The measure which is invariant under Fried's flow might not be Lebesgue measure—it might not have a smooth positive density. However, its only measurable invariant sets have full or zero Lebesgue measure.) In our example, simultaneous smoothness of the flow and of the invariant ergodic measure requires some care.

It is not known if every 3-manifold supports a fixed-point free ergodic flow. This question may be related to the fact that Katok's construction takes place in the isotopy class of the identity. Whether ergodic diffeomorphisms exist in every isotopy class is not known.

Another issue is analyticity. It is not known if there is an analytic fixed point free ergodic flow on S^3. However, Gerber [5] has shown that analytic examples like Katok's do exist.

Also, if fixed points are permitted, the situation is entirely understood. Anosov and Katok [1] proved that ergodic flows exist on all manifolds M^m for $m \geq 3$, and Blohin [2] has constructed them on all surfaces except the sphere, projective plane, and Klein bottle, where they are impossible.

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The concept of a *cross-section* Σ to a flow φ is common in dynamics. It is a co-dimension 1 compact submanifold transverse to φ, such that every flow-line leaving Σ returns to Σ in finite bounded time. The *first return map*

$$f:\Sigma\to\Sigma$$
$$x\mapsto f(x)=\varphi_t(x),\quad t=t(x)>0,$$

is a diffeomorphism. It determines φ dynamically and vice-versa. The flow φ is said to *suspend* f.

The concept of *Birkhoff-section* is less widely appreciated than it should be; work of Fried [4] and Christy [3] are good places to read about it. The idea is to replace the cross-section Σ by a compact manifold-with-boundary V such that

(a) ∂V is φ-invariant,
(b) $\operatorname{int}(V)$ is a cross-section for φ off ∂V, and
(c) the angle between $\dot\varphi(x)$ and T_xV tends to 0 at the same rate that $x\to\partial V$.

It is much more likely that a flow has a Birkhoff-section than it has a cross-section. For example, no flow on S^3 has a cross-section Σ, because Σ would separate S^3, but many flows on S^3 have Birkhoff-sections.

A C^∞ function is *flat* at a point p if f and all its derivatives vanish at p. It is *boundary-flat* if it is flat at each point in the boundary of its domain of definition.

LEMMA 1. *If f_0 and f_1 are diffeomorphisms of the unit disc D^2 such that $f_0-\mathrm{id}$ and $f_1-\mathrm{id}$ are boundary-flat at ∂D^2, then there is an isotopy f_t from f_0 to f_1 such that the vector field Y on $D^2\times[0,1]$ generating it satisfies*

$$Y-\partial/\partial t\ \textit{ is boundary-flat at }\ \partial(D^2\times[0,1]).$$

Proof. It suffices to assume f_0 is the identity map. Let F extend f_1 to $\mathbf{R}^2$ such that F fixes all points off D^2. Then F is a C^∞ diffeomorphism. Let R be a square of radius r containing D^2 in its interior. According to [10] there is a deformation retraction of the space of all diffeomorphisms fixing a neighborhood of ∂R. This means that there is a universal isotopy F_t from the identity map to $F=F_1$. We may assume $F_t\equiv$ identity for t near 0, $F_t\equiv F$ for t near 1, and $F_t(z)\equiv z$ for $|z|\geq r$.

Let $\beta:[\frac12,1]\to[1,r]$ be a C^∞ bump function such that $\beta(t)\equiv r$ for t near $\frac12$ and $\beta(t)\equiv 1$ for t near 1. Set

$$G_t(z)=\begin{cases}F_{2t}(rz)/r & 0\leq t\leq\frac12,\\ F(\beta(t)z)/\beta(t) & \frac12\leq t\leq 1.\end{cases}$$

Then G is an isotopy from the identity to F fixing all points off D^2. Restricted to $D^2\times[0,1]$, G isotopes the identity to f_1. Clearly, $Y-\partial/\partial t$ is boundary-flat where Y is $G'+\partial/\partial t$, restricted to $D^2\times[0,1]$. □

LEMMA 2. *Suppose U is a neighborhood of the origin in $\mathbf{R}^2$, $U_* = U \setminus \{0\}$, and $f: U_* \to \mathbf{R}$ is smooth. If all the derivatives of f respecting polar coordinates (r, θ) tend to 0 as $r \to 0$, then f extends to a smooth function $\hat{f}: U \to \mathbf{R}$, flat at the origin.*

REMARK. If a function is C^∞ respecting polar coordinates, it need not be C^∞ at the origin; for example, $(r, \theta) \mapsto r$.

Proof. $\hat{f} \equiv f$ on U_* and $\hat{f}(0) = 0$. From the mean value theorem,

(1) each polar derivative of f tends to 0 faster than any power of r as $r \to 0$.

Since

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{x}{r} + \frac{\partial f}{\partial \theta}\frac{y}{r^2},$$

it follows from (1) that $\partial f/\partial x$ extends to a continuous function on U, and likewise for $\partial f/\partial y$; hence $\hat{f}$ is C^1. As a function of (r, θ),

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cos\theta + \frac{\partial f}{\partial \theta}\frac{\sin\theta}{r}.$$

According to (1) then, all polar partials of $\partial f/\partial x$ tend to 0 as $r \to 0$. From the preceding, it follows that $\partial f/\partial x$ extends to a C^1 function on U and similarly for $\partial f/\partial y$, so $\hat{f}$ is C^2. By induction, $\hat{f}$ is smooth. Clearly, it is flat at the origin. □

The 3-sphere is the union of two solid tori, glued along their boundaries so that "meridians are identified with longitudes." A slightly different gluing produces a lens space, and (since our construction works just as well there) we recall the definition.

Let p, q be relatively prime integers. Under the quotient map

$$\mathbf{R}^2 \to \mathbf{R}^2/2\pi\mathbf{Z}^2,$$

lines of slope p/q become (p, q)-curves on the 2-torus $T^2 = S^1 \times S^1$. When $(p, q) = (0, 1)$ or $(1, 0)$, we have (respectively) a *meridian* or *longitude* of T^2. The lens space $L(p, q)$ is formed by identifying (p, q)-curves of the boundary of one solid torus with meridians of the boundary of a second. Since differentiability is the main hurdle below, let us be more precise.

Fix the pair (p, q) and express $1 = mp + nq$, where m and n are integers. Then

$$A = \begin{bmatrix} q & -m \\ p & n \end{bmatrix}$$

has determinant 1. If $p = 0$ then q must be ± 1 for m and n to exist; similarly, if $q = 0$ then $p = \pm 1$. Take two open, unbounded, solid tori

$$S_j = (\mathbf{R}^2 \times S^1)_j \quad (j = 1, 2),$$

and let $(r,\theta,\phi)_j$ denote "toral coordinates" on S_j. That is, (r,θ) are polar coordinates on $\mathbf{R}^2$ and ϕ is the angular coordinate on S^1. Call $\mathbf{R}^2_* = \mathbf{R}^2\setminus\{0\}$. Identify $(\mathbf{R}^2_*\times S^1)_2$ with $(\mathbf{R}^2_*\times S^1)_1$ according to

$$i = i_{pq}: (r,\theta,\phi)_2 \mapsto (r^{-1}, q\theta - m\phi, p\theta + n\phi)_1.$$

Since A bijects the lattice $2\pi\mathbf{Z}^2$ to itself, i is a well-defined diffeomorphism.

The lens space of type (p,q) is

$$L(p,q) = S_1 \cup_i S_2.$$

Call $T_j = (D^2\times S^1)_j$, $j=1,2$. Under i, meridians of ∂T_2 are identified with (p,q)-curves of ∂T_1. By [9, p. 234] it is easy to see that $L(0,1)\approx S^2\times S^1$, $L(1,q)\approx S^3$ for all $q\in\mathbf{Z}$, and $L(p,q)\approx L(p,kp+q)$ for all $k\in\mathbf{Z}$.

In what follows, we assume that $q\neq 0$. This is no loss of generality, since if $q=0$ then p must be ± 1 and $L(1,0)\approx S^3$; but S^3 can also be expressed as $L(1,1), L(1,2),\dots$.

Let $K: D^2\to D^2$ be the diffeomorphism that Katok constructs in [6]. Respecting Lebesgue measure λ on D^2, K is ergodic. Also, $K-\mathrm{id}$ is boundary-flat at ∂D^2. By Lemma 1, there is a smooth vector field Y on $\mathbf{R}^2\times S^1$ generating a flow ψ such that

(2) the first return map of ψ on $D^2\times 0$ is K, and

(3) $Y\equiv \partial/\partial\phi$ off $D^2\times S^1$.

Under ψ, λ suspends to a smooth ψ-invariant measure α on $\mathbf{R}^2\times S^1$ and

(4) $\alpha\equiv r\,dr\,d\theta\,d\phi$ off $D^2\times S^1$.

Now consider the lens space $L=L(p,q)$. It is the union of the two open solid tori S_1 and S_2 identified by $i=i_{pq}$. Define a map $\pi: L\to L$ as follows. Let $\pi_0: (1,\infty)\to(0,\infty)$ be a diffeomorphism such that

$$\pi_0(r) \equiv \begin{cases} \sqrt{(1-r^{-2})} & \text{for } r \text{ near } 1,\\ r & \text{for } r \text{ large.}\end{cases}$$

Then, using the S_2 toral coordinates, set

$$\pi(x) = \begin{cases} x & \text{if } x\notin S_1\cap S_2,\\ (\pi_0(r),\theta,\phi)_2 & \text{if } x=(r,\theta,\phi)_2\in S_2 \text{ and } 1<r<\infty,\\ (0,\theta,\phi)_2 & \text{if } x=(r,\theta,\phi)_2\in S_2 \text{ and } 0\le r\le 1.\end{cases}$$

Under π, the meridian discs of T_2 are crushed to their centers on the core circle $C_2=(0\times S^1)_2$, and the rest of L is sent diffeomorphically onto $L\setminus C_2$.

On the first open solid torus S_1, put the vector field Y, the flow ψ, and the invariant measure α considered above. *We claim that $\varphi=\pi_*\psi$ is a smooth flow on L which is ergodic respecting the smooth measure $\beta=\pi_*\alpha$, the density of β being everywhere positive.* Then, by Moser's theorem [8], there is a diffeomorphism $h: L\to L$ carrying β onto a constant multiple of Lebesgue measure λ on L. The h-conjugate of φ is smooth (if φ is), is λ-ergodic, and has no fixed points (if φ doesn't). In effect, φ Birkhoff-suspends K where the set $\pi((D^2\times 0)_1)$ is the Birkhoff-section. Since $q\neq 0$, it is clear that φ has no fixed points. It is also clear that φ is ergodic with respect to β. Except at C_2, smoothness of φ and β is immediate.

It remains to verify that φ and β are smooth at C_2, and that β has everywhere positive density.

Express α in the S_1 coordinates as

$$\alpha = a_1\, dr_1\, d\theta_1\, d\phi_1.$$

By (4) we know that $a_1 - r_1 \equiv 0$ for $r_1 \geq 1$. Re-express α in the S_2 coordinates as

$$\alpha = a_2\, dr_2\, d\theta_2\, d\phi_2.$$

Since $r_1 = r_2^{-1}$, $\theta_1 = q\theta_2 - m\phi_2$, and $\phi_1 = p\theta_2 + n\phi_2$, we see that

$$a_2 = a_1\left|\frac{\partial(r_1,\theta_1,\phi_1)}{\partial(r_2,\theta_2,\phi_2)}\right| = a_1 r_2^{-2}.$$

Then,

(5) $a_2 \equiv r_2^{-3}$ for $r_2 \leq 1$.

Express $\beta = \pi_*\alpha$ in the S_2 coordinates as

$$\beta = b\, dr_2\, d\theta_2\, d\phi_2.$$

Then

$$b = (a_2\circ\pi^{-1})\,\mathrm{Jac}(\pi^{-1}) = a_2(R,\theta,\phi) r R^3,$$

where $R = \pi_0^{-1}(r) = 1/\sqrt{(1-r^2)}$ for r small. By (5) and the fact that (even at $r=0$) R is a smooth nonvanishing function of r, it follows that all the polar derivatives of

$$b - r = (a_2(R,\theta,\phi) - R^{-3})(rR^3)$$

tend to 0 as $r\to 0$. By Lemma 2, $b((r,\theta,\phi)_2) - r_2$ extends to a smooth function on T_2, flat at C_2. By flatness, $(b-r_2)/r_2$ is also smooth at C_2. Expressing β as

$$\beta = \left[\frac{b-r_2}{r_2} + 1\right] r_2\, dr_2\, d\theta_2\, d\phi_2 = \delta\lambda_2,$$

where λ_2 is Lebesgue measure on T_2, shows that β is a smooth measure with everywhere positive density δ.

The vector field Y generating ψ is expressed in the S_1 coordinates as

$$Y_1 = R_1\frac{\partial}{\partial r_1} + \Theta_1\frac{\partial}{\partial\theta_1} + \Phi_1\frac{\partial}{\partial\phi_1}.$$

Off T_1, we know that $R_1 \equiv \Theta_1 \equiv 0$ and $\Phi_1 \equiv 1$. The vector field $X = \pi_* Y$ generating $\varphi = \pi_*\psi$ is expressed in the S_2 coordinates as

$$X = R_2\frac{\partial}{\partial r_2} + \Theta_2\frac{\partial}{\partial\theta_2} + \Phi_2\frac{\partial}{\partial\phi_2},$$

where

$$\begin{bmatrix} R_2 \\ \Theta_2 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} D(\pi \circ i^{-1}) \end{bmatrix} \begin{bmatrix} R_1 \\ \Theta_1 \\ \Phi_1 \end{bmatrix},$$

and all functions on the right-hand side are evaluated at

$$i \circ \pi^{-1}((r, \theta, \phi)_2) = (\sqrt{(1-r^2)}, q\theta - m\phi, p\theta + n\phi)_1$$

with r small. Thus,

$$R_2 = \frac{-\sqrt{(1-r^2)}}{r} R_1, \qquad \Theta_2 = n\Theta_1 + m\Phi_1, \qquad \Phi_2 = -p\Theta_1 + q\Phi_1$$

for small r. All the polar derivatives of $i \circ \pi^{-1}$ exist and are continuous for r small and nonnegative. Since R_1 is boundary-flat at ∂T_1, all the polar derivatives of the composition $R_1 \circ i \circ \pi^{-1}$ tend to 0 as $r \to 0$. By the mean value theorem and Leibniz' rule, the same is true of R_2. Similarly, all the polar derivatives of Θ_1 and $\Phi_1 - 1$ tend to 0 as $r \to 0$. From Lemma 2, we conclude that R_2, Θ_2, and Φ_2 extend smoothly to C_2. The fields $\partial/\partial r_2$, $\partial/\partial \theta_2$, and $\partial/\partial \phi_2$ are smooth, even at $r = 0$, so it follows that X and φ are smooth.

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