Commutator Relations and Identities in Lattice-Ordered Groups

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To Roger C. Lyndon - In Memorium

1. Introduction

There are a few occasions when a relation between the generators of a group implies that it holds throughout the group. Two classical examples are:

(1) if the generators of a group commute, the group is Abelian; and
(2) if the commutator of any pair of generators of a group commutes with all the generators of that group, then the group is nilpotent class 2.

It is well known [and trivial—see the proof of Theorem A(i)] that (1) holds for arbitrary lattice-ordered groups. In this paper, we establish

THEOREM 1. (a) *There is a lattice-ordered group G generated by elements a and b such that $[a,b]$ is in the center of G but G is not nilpotent of any class.*

(b) *Any lattice-ordered group generated by elements a and b and satisfying $[a,b,a]=[a,b,b]=1$ must be metabelian.*

Part (a) confirms a belief stated in [6]. Indeed, we will show (Theorem B) that any finite set of commutators being central is not enough to guarantee that the resulting lattice-ordered group is nilpotent. (If the number of generators exceeds 2, the resulting lattice-ordered group need not even be metabelian.) Part (b) states that although the relations $[a,b,a]=[a,b,b]=1$ on the generators are not enough to ensure nilpotency of the entire lattice-ordered group, they do ensure that the metabelian law $[[x,y],[z,t]]=1$ holds for all x, y, z, and t belonging to the lattice-ordered group. So a commutator *identity* is in fact implied by the original commutator *relations*, albeit a weaker identity than that for groups. A generalization is given in Section 3.

Theorem 1 may also be viewed as follows:

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THEOREM 2. *Let H be a lattice-ordered group and G a subgroup of H. Let G^* denote the sublattice subgroup of H generated by G. Then:*

(a) *G^* need not be nilpotent even if G is nilpotent class* 2;

(b) *G^* is metabelian if G is a* 2*-generator nilpotent class-*2 *group*; *and*

(c) *G^* need not be solvable even if G is metabelian.*

2. Notation

As usual, we write $[a,b]$ for $a^{-1}b^{-1}ab$, $[a,b,c]$ for $[[a,b],c]$, and so on. $\zeta_1(G)$ will denote the center of a group G. Since $[ab,c]=b^{-1}[a,c]b[b,c]$, a group is nilpotent class 2 if $[a,b,c]=1$ for any generators a,b,c.

A group G is said to satisfy the *n-Engel condition* if $[x,y,\dots,y]=1$ for all $x,y\in G$, where n y's occur. Any nilpotent class-n group satisfies the n-Engel condition; hence a group which satisfies no Engel condition is certainly not nilpotent of any class.

A group G with a lattice ordering on the underlying set is called a *lattice-ordered group* if $a(b\vee c)d = abd\vee acd$ and $a(b\wedge c)d = abd\wedge acd$ for all $a,b,c,d\in G$. Throughout we will adopt the abbreviations *ℓ-group* for lattice-ordered group and *ℓ-subgroup* for a subgroup that is also a sublattice. An ℓ-subgroup that is closed under arbitrary joins and meets that exist in the entire ℓ-group is called a *closed* ℓ-subgroup.

The lattice of an ℓ-group is distributive (see, e.g., [1, Proposition 1.2.14], [5, Lemma 1.11.2], or [15]); so if an ℓ-group G is generated by $a_1,\dots,a_n$, then every element of G can be written in the form $\bigvee_i \bigwedge_j w_{ij}$, where i and j range over finite index sets and each w_{ij} is an element of the group generated by $a_1,\dots,a_n$.

A homomorphism between ℓ-groups that preserves both the group and lattice operations is called an *ℓ-homomorphism;* its kernel is called an *ideal.* The ideals of an ℓ-group are precisely the normal convex ℓ-subgroups. An ℓ-group G is said to be *ℓ-solvable of length m* if there are convex ℓ-subgroups $A_0,\dots,A_m$ of G with $A_0=\{1\}$ and $A_m=G$, and if each A_i is an ideal of A_{i+1} with A_{i+1}/A_i Abelian $(0\le i\le m-1)$. Note that any ℓ-group that is ℓ-solvable of length m is solvable of length m as a group. The converse fails [14]; it is trivial to give metabelian polycyclic ℓ-groups (generated by three elements as a group) that are not ℓ-solvable of any length—the derived group need not be convex.

If $x\vee y=x$ or $x\vee y=y$ for every pair of elements x and y of an ℓ-group G, we say that G is a *totally ordered group,* or *o-group* for short. An ℓ-group that is a subdirect product of o-groups (with $f\le h$ in $\prod G_\lambda$ if $f_\lambda\le h_\lambda$ in G_λ for all λ) is called *representable.* Any ℓ-group that is nilpotent of any class is representable ([9] or [10]).

Let $\mathbf{Z}$ be the totally ordered additive group of integers. The group $\mathbf{Z}$ Wr $\mathbf{Z}$ is the set of all pairs (f,m), where $m\in\mathbf{Z}$ and f is a function from $\mathbf{Z}$ into $\mathbf{Z}$; the group operation is defined by $(f_1,m_1)\cdot(f_2,m_2)=(h,m_1+m_2)$, where

$h(n)=f_1(n)+f_2(n+m_1)$. **Z** Wr **Z** is an ℓ-group under the ordering $(f_1, m_1)\leq (f_2, m_2)$ if $m_1<m_2$, or if $m_1=m_2$ and $f_1(n)\leq f_2(n)$ for all $n\in\mathbf{Z}$; see [5, Chapter 5]. $B=\{(f,0): f:\mathbf{Z}\to\mathbf{Z}\}$ is an Abelian ideal of **Z** Wr **Z**. Hence **Z** Wr **Z** is ℓ-solvable of length 2 (ℓ-metabelian).

A group G with a total order on the underlying set is said to be a *right ordered group* if, for any elements $a, b, c\in G$, $a\leq b$ implies $ac\leq bc$. If $(G, \leq)$ is a right ordered group, G can be embedded by the right regular representation into the ℓ-group $\mathrm{Aut}(G, \leq)$ of all order-preserving permutations of the ordered set $(G, \leq)$.

If $(G, \leq)$ is a right ordered group and $f, g\in G$, write $|f|$ for $\max\{f, f^{-1}\}$ and $f\ll g$ for $f^n\leq g$ $(n\in\mathbf{Z})$. A right ordered group in which $|[x,y]|\ll \max\{|x|, |y|\}$ for all x, y is called a *C-right ordered group*. It is easy to see that, in any C-right ordered group, if $1\leq f_j\leq g^{n_j}$ for some $n_j\in\mathbf{Z}$ $(j=1,2)$ then $f_1 f_2\leq g^n$ for some $n\in\mathbf{Z}$ [11, Lemma 7.4.2]. Moreover, if G is a torsion-free nilpotent group, then it is right orderable and G is a C-right ordered group with respect to any right order on G [11, Theorem 7.5.1].

For any undefined terms from group theory, see [13].

3. Proofs

We now prove the following, which implies (a) of Theorem 1.

THEOREM A. (i) *Any representable lattice-ordered group in which the commutator of any pair of generators commutes with every generator is nilpotent class* 2.

(ii) *There is a metabelian lattice-ordered group G generated by two elements a, b such that $[a,b,a]=[a,b,b]=1$ but G is not n-Engel for any n.*

Proof. (i) If G is an o-group, then every ℓ-group word is a group word. Hence, as noted above, if the commutator of any pair of generators of G commutes with each generator of G then G is nilpotent class 2. Consequently, the same holds for any subdirect product of o-groups.

(ii) Let $a=(\hat{0}, 1)$ and $b=(\hat{b}, 1)$ be elements of the ℓ-metabelian group **Z** Wr **Z**, where $\hat{0}(n)=0$ $(n\in\mathbf{Z})$ and $\hat{b}(n)=-n$ $(n\in\mathbf{Z})$. An easy computation shows that $[a,b]=(f,0)$, where $f(n)=-1$ for all $n\in\mathbf{Z}$. Thus $[a,b]\in \zeta_1(\mathbf{Z}\ \mathrm{Wr}\ \mathbf{Z})$. Therefore, if G is the ℓ-subgroup of **Z** Wr **Z** generated by a and b, then G is ℓ-metabelian and $[a,b,a]=1=[a,b,b]$. Moreover, if m_1 and m_2 are distinct integers then

$$a^{m_1}\vee b^{m_2}=\begin{cases} a^{m_1} & \text{if } m_1>m_2,\\ b^{m_2} & \text{if } m_1<m_2;\end{cases}$$

hence

$$[a^{m_1}\vee b^{m_2}, a, a]=1=[a^{m_1}\vee b^{m_2}, a, b]=[a^{m_1}\vee b^{m_2}, b, a]=[a^{m_1}\vee b^{m_2}, b, b].$$

However,

$$a \vee b = (g, 1), \quad \text{where } g(n) = \begin{cases} -n & \text{if } n \leq 0, \\ 0 & \text{if } n > 0. \end{cases}$$

Consequently,

$$[a \vee b, a] = (h, 0), \quad \text{where } h(n) = \begin{cases} 1 & \text{if } n \leq 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

It follows that $[a \vee b, a, a, \ldots, a] \neq 1$ for any number of a's. □

Actually, if $c = [a \vee b, a, a]$ then $c \wedge a^{-1}ca = 1$, but $c \neq 1$. This shows directly that the ℓ-subgroup of $\mathbf{Z}$ Wr $\mathbf{Z}$ generated by a and b is not representable. Because nilpotent ℓ-groups are indeed representable ([9] or [10]), this big theorem and the computation are also enough to establish Theorem A(ii).

The above proof permits the following easy generalization:

THEOREM B. *Let $\{w_r(x, y) : 1 \leq r \leq R\}$ be any finite set of elements of the free lattice-ordered group on two generators x and y. There is an ℓ-metabelian lattice-ordered group G generated by a and b such that $[w_r(a, b), w_s(a, b)]$ belong to the center of G for all $r, s \in \{1, \ldots, R\}$, yet G satisfies no Engle condition.*

Proof. Let $w_r(x, y) = \bigvee_i \bigwedge_j w_{rij}(x, y)$, where $w_{rij}(x, y)$ belongs to the free group on x and y. Let M_0 and N_0 be co-prime positive integers such that $M_0/2$ exceeds the sum of the absolute values of the exponents of x in all $w_{rij}(x, y)$, and $N_0/2$ exceeds the sum of the absolute values of the exponents of y in all $w_{rij}(x, y)$. Let $a = (\hat{0}, N_0)$ and $\hat{b} = (\hat{b}, M_0)$, where $\hat{b}(n) = -n$ $(n \in \mathbf{Z})$. Note that $[a, b] \in \zeta_1(\mathbf{Z} \text{ Wr } \mathbf{Z})$ and $[a, b] \geq 1$. Since $[a, b] \in \zeta_1(G)$, every element $w_{rij}(a, b)$ can be written in the form $a^m b^n [a, b]^k$. Moreover, m equals the sum of the exponents of x in w_{rij} and n that of the exponents of y in w_{rij}; hence $2|m| < M_0$ and $2|n| < N_0$. Furthermore, for any such words, $m_1 N_0 + n_1 M_0 = m_2 N_0 + n_2 M_0$ if and only if $m_1 = m_2$ and $n_1 = n_2$, by our choice of M_0 and N_0. Therefore, if $k = \min\{k_1, k_2\}$ then

$$a^{m_1} b^{n_1} [a, b]^{k_1} \vee a^{m_2} b^{n_2} [a, b]^{k_2}$$

$$= \begin{cases} a^{m_1} b^{n_1} [a, b]^{k_1} & \text{if } m_1 N_0 + n_1 M_0 > m_2 N_0 + n_2 M_0, \\ a^{m_2} b^{n_2} [a, b]^{k_2} & \text{if } m_1 N_0 + n_1 M_0 < m_2 N_0 + n_2 M_0, \\ a^{m_1} b^{n_1} [a, b]^{k} & \text{if } m_1 = m_2 \text{ and } n_1 = n_2. \end{cases}$$

Consequently, $[w_r(a, b), w_s(a, b)] \in \zeta_1(G)$ for all $r, s \in \{1, \ldots, R\}$. However, $[a^{M_0} \vee b^{N_0}, a, a, \ldots, a] \neq 1$ for any number of a's, and this establishes the theorem. □

We next prove a result that implies Theorem 2(c). For this we observe first that Aut$(\mathbf{Q}, \leq)$, the group of automorphisms (order-preserving permutations) of the linearly ordered set $(\mathbf{Q}, \leq)$ of rationals, is an ℓ-group under composition and the pointwise ordering ($f \leq g$ if $\alpha f \leq \alpha g$ for all $\alpha \in \mathbf{Q}$). A subgroup G is called *doubly transitive* if, whenever $\alpha_i, \beta_i \in \mathbf{Q}$ with $\alpha_i < \beta_i$

$(i=1,2)$, there is $g\in G$ such that $\alpha_1 g=\alpha_2$ and $\beta_1 g=\beta_2$. As Holland has shown (see [5, p. 214]), the only ℓ-group identities that hold in a doubly transitive ℓ-group are those that hold in all ℓ-groups. Consequently, any doubly transitive ℓ-group generates the variety of all ℓ-groups (and hence is unsolvable even as a group). This leads to a simple solution to Theorem 2(c).

PROPOSITION C. *There is a 2-generator metabelian subgroup G of* $\mathrm{Aut}(\mathbf{Q}, \leq)$ *such that G^* generates the variety of all lattice-ordered groups. In particular, G^* is unsolvable even as a group.*

Proof. Since $\{m/2^n: m,n\in\mathbf{Z}, n\geq 0\}$ is a dense linearly ordered subset of $(\mathbf{Q}, \leq)$ without endpoints, it is isomorphic to $(\mathbf{Q}, \leq)$ as an ordered set. We identify $(\mathbf{Q}, \leq)$ with this ordered set. Let $a:\alpha\mapsto 2\alpha$ and $b:\alpha\mapsto\alpha+1$. Clearly $(m/2^n)a^n b^{k-m} a^{-n}=k/2^n$, so the subgroup G of $\mathrm{Aut}(\mathbf{Q}, \leq)$ generated by a and b is metabelian and transitive on $\mathbf{Q}$. Moreover, $a^n b^{k-m} a^{-n}$ moves every point of $\mathbf{Q}$ up if $k>m$. Now a fixes 0 and, for each $\alpha,\beta\in\mathbf{Q}$ with $\beta>\alpha>0$, there is a positive integer n_0 such that $\alpha a^{n_0}>\beta$. Let $g\in G$ be such that $\alpha g=\beta$ and g moves every point of $\mathbf{Q}$ up. Then $a^{n_0}\wedge g\in G^*$, $\alpha(a^{n_0}\wedge g)=\beta$, and $0(a^{n_0}\wedge g)=0$. It follows that G^* is doubly transitive. This proves the proposition. □

We now establish a generalization of Theorem 1(b).

Let $N=N_m$ be the free nilpotent class-2 group on m generators $a_1,\ldots,a_m$. Then N can be right ordered, and every such order makes N a C-right ordered group. The right regular representation $\bar{N}$ on $(N, \leq)$ for such an ordering embeds N in the ℓ-group $\mathrm{Aut}(N, \leq)$. These in turn (as $\leq$ varies over *all* right orders on N) naturally induce an embedding of N in the ℓ-group $A=\prod\{\mathrm{Aut}(N, \leq): \leq \text{ is a right order on } N\}$, ordered componentwise. Moreover, $F(N)$, the ℓ-subgroup of A generated by the image of N, is free over N; that is, if H is any nilpotent class-2 ℓ-group and $\hat{a}_1,\ldots,\hat{a}_m\in H$, then there is a unique ℓ-homomorphism of $F(N)$ into H mapping the image of a_i in $F(N)$ to $\hat{a}_i$ $(1\leq i\leq m)$. Theorem 1(a) is equivalent to the statement that the free ℓ-group over N_2 is not nilpotent (which follows from a more general result in [3]), and Theorem 1(b) is equivalent to the following theorem when $m=2$.

THEOREM D. *The free lattice-ordered group over any m generator torsion-free nilpotent class-2 group is ℓ-solvable of length at most $\binom{m}{2}+1$.*

Proof. It is enough to prove that, if N is the free nilpotent class-2 group on $a_1,\ldots,a_m$ and $\leq$ is any right order on N then $\bar{N}^*$, the ℓ-subgroup of $\mathrm{Aut}(N, \leq)$ generated by $\bar{N}$, is ℓ-solvable of length at most $\binom{m}{2}+1$.

Now $\zeta_1(N)$ is a free Abelian group on $\binom{m}{2}$ generators. Consequently, it has Hirsch length $\binom{m}{2}$. Therefore, there is a minimal Archimedean class of $\zeta_1(N)$; that is, there is $d_1\in\zeta_1(N)$ such that $d_1>1$ and, for all $c\in\zeta_1(N)\backslash\{1\}$, $d_1\leq c^n$ for some integer n. Let $N(d_1)=\{g\in N: |g|\leq d_1^n \text{ for some } n\in\mathbf{Z}^+\}$.

Since every right order on N is a C-right order, $N(d_1)$ is a convex subgroup of N. Moreover, $N(d_1)$ is Abelian since if $1 \le x \le y \le d_1^n$ then $[x,y] \ll y \le d_1^n$; but $[x,y] \in \zeta_1(N)$ since N is nilpotent class 2, so $[x,y]=1$ by the choice of d_1.

If $\zeta_1(N)=[N,N] \subseteq N(d_1)$, then $N(d_1) \lhd N$ and $N/N(d_1)$ is Abelian. If $\zeta_1(N) \not\subseteq N(d_1)$, let $d_2 \in \zeta_1(N)$ be such that $d_2>1$ and, for all $c \in \zeta_1(N)$, $c \in N(d_1)$ or $d_2 \le c^n$ for some integer n. Let $N(d_2)=\{g \in N: |g| \le d_2^n$ for some integer $n\}$. As before, $N(d_1) \lhd N(d_2)$ and $N(d_2)/N(d_1)$ is Abelian. Continuing in this way we obtain, after r ($\le \binom{m}{2}$) steps, convex subgroups

$$\{1\} \subseteq N(d_1) \subseteq N(d_2) \subseteq \cdots \subseteq N(d_r) \subseteq N,$$

with $N(d_i) \lhd N(d_{i+1})$ and $N(d_r) \lhd N$ such that $N(d_1)$, $N(d_{i+1})/N(d_i)$, and $N/N(d_r)$ are Abelian $(1 \le i \le r-1)$.

Note that if $\bar{x} \le \bar{y}$ then $\alpha x \le \alpha y$ for all $\alpha \in N$. Hence $x=ex \le ey=y$. Consequently, $\bar{N}(\bar{d}_i) \subseteq \overline{N(d_i)}$ $(1 \le i \le r)$, where $\bar{N}(\bar{y})=\{\bar{x} \in \bar{N}: |\bar{x}| \le |\bar{y}|^n$ for some $n \in \mathbf{Z}^+\}$. Thus $\bar{N}(\bar{d}_1)$ is Abelian. Moreover, since $\bar{N}$ is a subgroup of the ℓ-group $\mathrm{Aut}(N, \le)$ and each $\bar{d}_i \in \zeta_1(\bar{N})$, $\bar{N}(\bar{d}_i) \lhd \bar{N}$ $(1 \le i \le r)$. Furthermore, if $\bar{x}, \bar{y} \in \bar{N}(\bar{d}_{i+1})$ then $x, y \in N(d_{i+1})$; therefore $[x,y] \le d_i^n$ for some integer n. But for all $\alpha \in N$,

$$\alpha[\bar{x},\bar{y}]=\alpha[x,y]=[x,y]\alpha \le d_i^n\alpha=\alpha\bar{d}_i^n$$

because d_i, $[x,y] \in \zeta_1(N)$. Hence $\bar{N}(\bar{d}_{i+1})/\bar{N}(\bar{d}_i)$ is Abelian. Since $\bar{N}$ is contained in the normalizer of the ℓ-subgroup of $\mathrm{Aut}(N, \le)$ generated by $\bar{N}(\bar{d}_i)$, $\bar{N}$ is contained in the normalizer of $\bar{N}^*(\bar{d}_i)$, the convex ℓ-subgroup of $\bar{N}^*$ generated by $\bar{N}(\bar{d}_i)$. Thus $\bar{N}$ normalizes $\bar{N}_i^*$, the closed convex ℓ-subgroup of $\bar{N}^*$ generated by $\bar{N}(\bar{d}_i)$. It follows easily (see, e.g., [12, Lemma 10.12.2]) that $\bar{N}_i^* \lhd \bar{N}^*$ $(1 \le i \le r)$. So the theorem will be proved once we establish that $\bar{N}_1^*$, $\bar{N}_{i+1}^*/\bar{N}_i^*$ $(1 \le i \le r-1)$, and $\bar{N}^*/\bar{N}_r^*$ are Abelian.

By [2] (or easy verification), $\bar{N}_1^*$ is Abelian if $\bar{N}^*(\bar{d}_1)$ is; and to prove that $\bar{N}^*(\bar{d}_1)$ is Abelian it is enough to show that: if, for some positive integer p, $\bigwedge_{j \in J}(\bar{w}_j \vee 1) \le \bar{d}_1^p$ and $\bigwedge_{k \in K}(\bar{v}_k \vee 1) \le \bar{d}_1^p$, then $\bigwedge_{j \in J}(\bar{w}_j \vee 1)$ and $\bigwedge_{k \in K}(\bar{v}_k \vee 1)$ commute.

Fix $\alpha \in G$ and let $J(\alpha)=\{j \in J: \alpha w_j \le \alpha$ or $\alpha w_j \alpha^{-1} \in N(d_1)\}$ and $K(\alpha)=\{k \in K: \alpha v_k \le \alpha$ or $\alpha v_k \alpha^{-1} \in N(d_1)\}$. We first show that, if $\alpha\bar{d}_1^m<\beta<\alpha\bar{d}_1^{m+1}$ for some $m \in \mathbf{Z}$, then $J(\beta)=J(\alpha)$ and $K(\beta)=K(\alpha)$. For if $j \in J(\alpha)$, then either $\beta w_j \le \beta$ or $\beta<\beta w_j<\alpha\bar{d}_1^{m+1}w_j=\alpha w_j d_1^{m+1}$. Now either $\alpha w_j \le \alpha$, in which case $\alpha w_j d_1^{m+1} \le \alpha d_1^{m+1}<\beta d_1$; or $\alpha w_j \alpha^{-1} \in N(d_1)$, whence $\alpha w_j d_1^{m+1} \le \alpha w_j \alpha^{-1} \cdot \alpha d_1^{m+1}<d_1^n \alpha d_1^{m+1}=\alpha d_1^{n+m+1}<\beta d_1^{n+1}$. Hence $\beta w_j \le \beta$ or $\beta w_j \beta^{-1} \in N(d_i)$. Consequently, $J(\alpha) \subseteq J(\beta)$. Because $\beta\bar{d}_1^{-m-1}<\alpha<\beta\bar{d}_1^{-m}$, the same argument shows that $J(\beta) \subseteq J(\alpha)$, whence $J(\beta)=J(\alpha)$. *Mutatis mutandis,* $K(\beta)=K(\alpha)$.

Observe that if $j \in J(\alpha)$ then $\alpha(\bar{w}_j \vee 1)\alpha^{-1} \in N(d_1)$, and similarly if $k \in K(\alpha)$. Since $N(d_1)$ is Abelian,

$$\alpha\Big(\bigwedge_{j\in J(\alpha)}(\bar{w}_j\vee 1)\Big)\Big(\bigwedge_{k\in K(\alpha)}(\bar{v}_k\vee 1)\Big)=\alpha\Big(\bigwedge_{k\in K(\alpha)}(\bar{v}_k\vee 1)\Big)\Big(\bigwedge_{j\in J(\alpha)}(\bar{w}_j\vee 1)\Big).$$

Moreover, by the previous paragraph,

$$\alpha\Big(\bigwedge_{j\in J}(\bar{w}_j\vee 1)\Big)\Big(\bigwedge_{k\in K}(\bar{v}_k\vee 1)\Big)=\alpha\Big(\bigwedge_{j\in J(\alpha)}(\bar{w}_j\vee 1)\Big)\Big(\bigwedge_{k\in K(\alpha)}(\bar{v}_k\vee 1)\Big)$$

and

$$\alpha\Big(\bigwedge_{k\in K}(\bar{v}_k\vee 1)\Big)\Big(\bigwedge_{j\in J}(\bar{w}_j\vee 1)\Big)=\alpha\Big(\bigwedge_{k\in K(\alpha)}(\bar{v}_k\vee 1)\Big)\Big(\bigwedge_{j\in J(\alpha)}(\bar{w}_j\vee 1)\Big)$$

[since $\alpha\le\beta_J=\alpha(\bigwedge_{j\in J(\alpha)}(\bar{w}_j\vee 1))\le\alpha\bar{d}_1^p$ and $\alpha\le\beta_K=\alpha(\bigwedge_{k\in K(\alpha)}(\bar{v}_k\vee 1))\le\alpha\bar{d}_1^p$, so $J(\beta_K)=J(\alpha)$ and $K(\beta_J)=K(\alpha)$]. Thus

$$\alpha\Big(\bigwedge_{j\in J}(\bar{w}_j\vee 1)\Big)\Big(\bigwedge_{k\in K}(\bar{v}_k\vee 1)\Big)=\alpha\Big(\bigwedge_{k\in K}(\bar{v}_k\vee 1)\Big)\Big(\bigwedge_{j\in J}(\bar{w}_j\vee 1)\Big).$$

Since this holds for all $\alpha\in N$, $\bigwedge_{j\in J}(\bar{w}_j\vee 1)$ and $\bigwedge_{k\in K}(\bar{v}_k\vee 1)$ commute. Thus $\bar{N}_1^*$ is Abelian.

Next let $a=\bigwedge_{j\in J}(\bar{w}_j\vee 1)$ and $b=\bigwedge_{k\in K}(\bar{v}_k\vee 1)$, and suppose that $a,b\le\bar{d}_{i+1}^p$. The above argument shows that, for each $\alpha\in N$, $\alpha c\le\alpha\bar{d}_i^{p(\alpha)}$ for some positive integer $p(\alpha)$, where $c=|[a^{-1},b^{-1}]|$. Hence $c=\bigvee_{q=1}^{\infty}(c\wedge\bar{d}_i^q)\in\bar{N}_i^*$. Thus, by [2] again, $\bar{N}_{i+1}^*/\bar{N}_i^*$ is Abelian $(1\le i\le r-1)$. Similarly $\bar{N}^*/\bar{N}_r^*$ is Abelian, and the proof is complete.

Otherwise, the last paragraph can be replaced by the use of [5, Corollary 8.3.7] to note that $\bar{N}^*/\bar{N}_i^*$ acts faithfully on the $\bar{N}_i^*$-orbits of N, since $\bar{N}_i^*$ is a closed ideal of $\bar{N}^*$; therefore $\bar{N}_{i+1}^*/\bar{N}_i^*$ is Abelian by the argument given for $\bar{N}_1^*$, or by induction on the Hirsch length of N. □

4. Concluding Remarks

1. Theorem D gives an upper bound on the ℓ-solvable length of the free ℓ-group over a finitely generated torsion-free nilpotent class-2 group. In general, the ℓ-solvable length can exceed half of the number of generators, as we now show. Totally order N_m by

$$a_1^{n_{11}}a_2^{n_{22}}\cdots a_m^{n_{mm}}\prod_{i<j}[a_i,a_j]^{n_{ij}}>1$$

if $n_{11}>0$, or $n_{11}=0$ and $n_{22}>0$, or $n_{11}=0=n_{22}$ and $n_{12}>0$, or $n_{11}=n_{22}=n_{12}=0$ and $n_{33}>0,\dots$, or $n_{ij}=0$ $(1\le i\le j<m)$ and $n_{mm}>0$, or $n_{ij}=0$ $(1\le i\le j<m)$ and $n_{mm}=0$ and $n_{1m}>0,\dots$, or $n_{ij}=0$ $(1\le i\le j<m)$ and $n_{mm}=n_{1m}=\cdots=n_{m-2,m}=0$ and $n_{m-1,m}>0$. Then N_m is an o-group and so (with respect to this order) $\bar{N}_m^*\cong N_m$, which has ℓ-solvable length $(m+1)/2$ if m is odd and $(m/2)+1$ if m is even.

What is the ℓ-solvable length of $F(N_m)$?

2. If N_ω is the free nilpotent class-2 group on a countably infinite set of generators, then it follows from our lower bound that $F(N_\omega)$ is not ℓ-solvable even though N_ω is a torsion-free nilpotent class-2 group; that is, *the free lattice-ordered group over a torsion-free nilpotent class-2 group can fail to be ℓ-solvable.*

Is it unsolvable as a group?

3. We now show that $F(N_2)$ actually generates the entire variety of ℓ-metabelian ℓ-groups. Define a right total order on N_2 by $a_1^{n_1}a_2^{n_2}[a_1, a_2]^{n_{12}} > 1$ if $n_1 > 0$, or $n_1 = 0 < n_{12}$, or $n_1 = n_{12} = 0 < n_2$. Let $\bar{N}^*$ be the ℓ-subgroup of $\mathrm{Aut}(N_2, \leq)$ generated by $\bar{N}$, the right regular action of N_2 on itself. Then, if $\alpha = a_1^{n_1}a_2^{n_2}[a_1, a_2]^{n_{12}}$, $\alpha < \alpha\bar{a}_2$ if and only if $n_1 \geq 0$ whereas $\alpha < \alpha\bar{a}_1^{-1}\bar{a}_2^{-1}\bar{a}_1$ if and only if $n_1 < 1$. Consequently, if $g = (\bar{a}_2 \wedge \bar{a}_1^{-1}\bar{a}_2^{-1}\bar{a}_1) \vee 1$ then $\mathrm{supp}(g) = \{\alpha \in N : \alpha g \neq \alpha\} = \{\alpha_2^{n_2}[a_1, a_2]^{n_{12}} : n_2, n_{12} \in \mathbf{Z}\}$. Hence $\bar{a}_1^{-1}g\bar{a}_1 \wedge g = 1$ and the ℓ-subgroup of $\bar{N}^*$ generated by $\{\bar{a}_1, g\}$ is ℓ-isomorphic to $\mathbf{Z} \mathrm{wr} \mathbf{Z}$. Because $\mathbf{Z} \mathrm{wr} \mathbf{Z}$ generates the variety of all ℓ-metabelian ℓ-groups [7] and $F(N_2)$ is ℓ-metabelian, $F(N_2)$ indeed generates the variety of all ℓ-metabelian ℓ-groups.

If we alter the above ordering by replacing $n_{12} > 0$ with $n_{12} < 0$ in the definition of the ordering, then the map $a_1 \to (\hat{0}, 1)$, $a_2 \to (\hat{b}, 1)$ defines an ℓ-isomorphism between $\bar{N}^*$ and the ℓ-group used in the proof of Theorem A(ii).

4. What results hold for finitely generated torsion-free nilpotent groups of nilpotency class exceeding 2? If $d_1 > 1$ is an element of the smallest convex subgroup of the center, then $N(d_1)$ is Abelian. However, even if the group is nilpotent class 3, $N(d_2)/N(d_1)$ need not be Abelian ($d_2 > 1$ is an element of the next smallest convex subgroup of the center).

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