On Isometric Isomorphisms of the Bloch Space on the Unit Ball of $\mathbf{C}^n$

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1. Introduction

Let B^n denote the unit ball in $\mathbf{C}^n$, $B^n = \{z \in \mathbf{C}^n : |z| < 1\}$. Let S be the boundary of B^n, $S = \{z \in \mathbf{C}^n : |z| = 1\}$. The Bloch space $\mathcal{B}(B^n)$ on B^n is defined as follows:

$$\mathcal{B}(B^n) = \left\{ f : f \text{ is holomorphic on } B^n,\ f(0) = 0,\ \sup_{\substack{z \in B^n \\ \xi \in S}} \frac{|D_\xi f(z)|}{H(z, \xi)} \equiv \|f\| < +\infty \right\},$$

where

$$D_\xi f(z) \equiv \sum_{j=1}^{n} \xi_j \frac{\partial f(z)}{\partial z_j}$$

and $H(z, \xi)$ is the length of the vector $\xi \in T_z(B^n)$ in the Kobayashi metric (see [7]). We will give the explicit form of $H(z, \xi)$ later. With the norm $\|\cdot\|$, $\mathcal{B}(B^n)$ is a Banach space. Let $\mathcal{B}_0(B^n)$ be the closed subspace of $\mathcal{B}(B^n)$ spanned by polynomials or, equivalently,

$$\mathcal{B}_0(B^n) = \left\{ f : f \in \mathcal{B}(B^n),\ \lim_{|z| \to 1} \sup_{\xi \in S} \frac{D_\xi f(z)}{H(z, \xi)} = 0 \right\}.$$

(That these are equivalent is elementary. See [11] for a proof.)

If $n = 1$ and $D = B^1$, then the definitions of $\mathcal{B}(D)$ and $\mathcal{B}_0(D)$ reduce to the more classical

$$\mathcal{B}(D) = \left\{ f : f \text{ is holomorphic on } D,\ f(0) = 0,\ \sup_{z \in B^n} (1 - |z|^2) |f'(z)| < +\infty \right\}$$

and

$$\mathcal{B}_0(D) = \left\{ f : f \in \mathcal{B}(D),\ \lim_{|z| \to 1} (1 - |z|^2) |f'(z)| = 0 \right\}.$$

For general properties of $\mathcal{B}(D)$ and $\mathcal{B}(B^n)$, see [2] and [11]. In [3], the linear isometries of $\mathcal{B}_0(D)$ and the isometric isomorphisms of $\mathcal{B}(D)$ are characterized.

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In this paper we characterize the isometric isomorphisms of $\mathcal{B}_0(B^n)$. The principal result is that an isometric isomorphism is given, up to suitable normalizations, by composition with a Möbius transformation. A similar result should hold in the strongly pseudoconvex case, but we are unable to prove it at this time. We shall comment on this at the end of the paper.

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2. Notation and Technical Lemmas

Let us introduce some notation. If $z, \xi \in \mathbf{C}^n$, let

$$\langle z, \xi \rangle = \sum_{j=1}^{n} z_j \bar{\xi}_j \quad \text{and} \quad |z| = \langle z, z \rangle^{1/2}.$$

If in addition $z \neq 0$ then we set

$$P_z \xi = \frac{\langle \xi, z \rangle}{|z|^2} z \quad \text{and} \quad Q_z \xi = \xi - P_z \xi.$$

In terms of this notation, $H(z, \xi)$ can be explicitly given (see [10]) by:

$$H(z, \xi) = \begin{cases} [(1-|z|^2)^{-2}|P_z\xi|^2 + (1-|z|^2)^{-1}|Q_z\xi|^2]^{1/2} & \text{if } z \in B^n, z \neq 0, \\ |\xi| & \text{if } z = 0. \end{cases}$$

Let $C(B^n \times S)$ denote the continuous functions on $B^n \times S$. Let $C_0(B^n \times S)$ denote the continuous functions on $\overline{B^n} \times S$ which vanish on $\partial B^n \times S = S \times S$. If $g(z, \xi) \in C(B^n \times S)$, we will sometimes extend $g(z, \xi)$ to be a continuous function on $B^n \times (\mathbf{C}^n \setminus \{0\})$ which is homogeneous of order 0 with respect to ξ; that is, if $\xi \neq 0$ then define $g(z, \xi)$ to be $g(z, \xi/|\xi|)$.

Define

$$\mathfrak{D} = \{F \in C(B^n \times S) : \exists f \in \mathcal{B}(B^n) \text{ such that } F(z, \xi) = D_\xi f(z) \cdot H(z, \xi)^{-1}\}.$$

Then $\mathfrak{D}$ is a closed subspace of $C(B^n \times S)$ and the mapping $A: \mathcal{B}(B^n) \to \mathfrak{D}$ defined by $(Af)(z) = D_\xi f(z) H(z, \xi)^{-1}$ is a linear isometry of $\mathcal{B}(B^n)$ onto $\mathfrak{D}$ [where $\mathfrak{D}$ has the norm inherited from $C(B^n \times S)$]. Let $\mathfrak{D}_0$ be the image of $\mathcal{B}_0(B^n)$ under A.

LEMMA 1. *Every $(z_0, \xi_0) \in B^n \times S$ is a peak point of $\mathfrak{D}_0$.*

Proof. For $\varphi \in \mathrm{Aut}(B^n)$, define $\tilde{\varphi}: B^n \times S \to B^n \times S$ by

$$(z, \xi) \to \left(\varphi(z), \frac{\varphi_*(z)(\xi)}{|\varphi_*(z)(\xi)|} \right).$$

Clearly, $\tilde{\varphi}$ is a bijection since $\tilde{\varphi}^{-1} = (\varphi^{-1})^{\sim}$. It is easy to check that $(Af) \circ \tilde{\varphi} = A(f \circ \varphi)$ for each $f \in \mathcal{B}(B^n)$. Thus, if $\tilde{\varphi}(z_0, \xi_0)$ is a peak point of Af then (z_0, ξ_0) is a peak point of $A(f \circ \varphi)$.

Given $(z_0, \xi_0) \in B^n \times S$, choose $\varphi \in \mathrm{Aut}(B^n)$ such that $\varphi(z_0) = 0$. Then $\tilde{\varphi}(z_0, \xi_0) = (0, \tilde{\xi}_0)$ for some $\tilde{\xi}_0 \in S$. Define $f(z) = \langle z, \tilde{\xi}_0 \rangle$. We have

$$F(z, \xi) \equiv Af(z, \xi) = \frac{\langle \xi, \tilde{\xi}_0 \rangle}{H(z, \xi)}.$$

It is not difficult to verify that $F(0, \tilde{\xi}_0) = 1$ and that $|F(z, \xi)| < 1$ if $(z, \xi) \neq (0, \xi_0)$. Thus $(0, \tilde{\xi}_0)$ is a peak point of $F = Af$. Therefore, (z_0, ξ_0) is a peak point of $A(f \circ \varphi)$. Clearly, $f \circ \varphi \in \mathcal{B}_0(B^n)$ and $A(f \circ \varphi) \in \mathfrak{D}_0$. □

LEMMA 2. *If $z^j \in B^n$, $\xi^j \in \mathbf{C}^n \setminus \{0\}$, $j = 1, 2, 3$, and $D_{\xi^1} f(z^1) + D_{\xi^2} f(z^2) + D_{\xi^3} f(z^3) = 0$ for every polynomial f, then $z^1 = z^2 = z^3$ and $\xi^1 + \xi^2 + \xi^3 = 0$.*

Proof. Elementary. □

LEMMA 3. *Let $T: \mathfrak{D}_0 \to \mathfrak{D}_0$ be an isometric isomorphism, and let $\lambda: B^n \times S \to S$ be a mapping such that*

$$TF(z, \xi) = F(z, \lambda(z, \xi)) \tag{2.0}$$

for every $F \in \mathfrak{D}_0$. Then there is a constant μ of modulus 1 such that $\lambda(z, \xi) = \mu \xi$ for all ξ.

Proof. Let $f(z) = z_1$ and $F = Af$. Then

$$F(z, \xi) = \frac{\xi_1}{H(z, \xi)};$$

$$TF(z, \xi) = F(z, \lambda(z, \xi)) = \frac{\lambda_1(z, \xi)}{H(z, \lambda(z, \xi))}.$$

Thus

$$\frac{\lambda_1(z, \xi)}{H(z, \lambda(z, \xi))} = \frac{D_\xi g(z)}{H(z, \xi)}$$

for some g in $\mathcal{B}_0$. Letting $z = 0$ gives $D_\xi g(0) = \lambda_1(0, \xi)$. Hence $\lambda_1(0, \xi)$ is a linear function of ξ. The same argument applies to $\lambda_j(0, \xi)$, $j = 1, 2, \ldots, n$. Thus $\lambda(0, \xi)$ is linear. Hence $\lambda(0, \xi)$ is unitary since it preserves the unit sphere $\{|\xi| = 1\}$.

Let $K: \mathcal{B}_0(B^n) \to \mathcal{B}_0(B^n)$ be defined by $K = A^{-1}TA$. Thus

$$\frac{D_\xi(Kf)}{H(z, \xi)} = \frac{D_{\lambda(z, \xi)} f}{H(z, \lambda(z, \xi))}, \quad f \in \mathcal{B}_0(B^n).$$

It follows that

$$D_\xi(K(fg)) = gD_\xi(Kf) + fD_\xi(Kg) \tag{2.1}$$

for polynomials f and g. Now take $f(z) = g(z) = z_1$. Then (2.1) leads to

$$D_\xi(Kf^2) = 2z_1 D_\xi(Kf). \tag{2.2}$$

Let $(e_1, \ldots, e_n)$ denote the standard basis of $\mathbf{C}^n$. Let $h(z) = Kf^2 - 2z_1 Kf$. Now (2.2) leads to

(2.3) $$D_{e_1}h(z) = -2Kf;$$

(2.4) $$D_{e_j}h(z) = 0, \quad j = 2, \dots, n.$$

By (2.4), $h(z)$ is a function of z_1 alone; by (2.3), so is Kf. It follows that Kf^2 is a function of z_1 alone. A similar argument shows that Kf^m is a function of z_1 alone for each positive integer m.

Now this means that K, restricted to functions depending only on z_1, can be used to define an isometry on the space $\mathcal{B}_0(D)$. To this we can apply the one-variable result of Cima and Wogen [3] and conclude, comparing the form of these isometries to the form of K, that

$$Kz_1 = \mu_1 z_1$$

for some $\mu_1 \in \mathbf{C}$ with $|\mu_1| = 1$. Of course, a similar argument is true for the other variables:

$$Kz_j = \mu_j z_j, \quad j = 1, \dots, n.$$

Now applying (2.1) to $f = z_1$ and $g = z_2$ gives

$$D_\xi(K(z_1 z_2)) = z_2 D_\xi(\mu_1 z_1) + z_1 D_\xi(\mu_2 z_2).$$

Calculating $\partial^2 K(z_1 z_2)/\partial z_1 \partial z_2$ in two ways tells us that $\mu_1 = \mu_2$. We similarly conclude that all the μ_j's are equal. Thus $\mu_1 = \mu_2 = \cdots = \mu_j \equiv \mu$. The set $S = \{f \in \mathcal{B}_0(B^n) : Kf = \mu f\}$ contains $f(z) = z_j$, $j = 1, \dots, n$. It is an algebra by (2.1). Thus S contains all polynomials and hence $S = \mathcal{B}_0(B^n)$. Therefore, $\lambda(z, \xi) = \mu \xi$. □

3. Statement and Proof of Main Result

Let $\psi \in \operatorname{Aut}(B^n)$ and let Ψ be the mapping from $\mathcal{B}_0(B^n)$ to $\mathcal{B}_0(B^n)$ defined by

$$f(z) = \Psi f_0(z) = f_0(\psi(z)) - f_0(\psi(0)).$$

Let $F = Af$ and $F_0 = Af_0$. Then, using the notation of Section 2,

$$F = Af = A(f_0 \circ \psi - f_0(\psi(0))) = A(f_0 \circ \psi) = (Af_0) \circ \tilde{\psi} = F_0 \circ \tilde{\psi};$$

that is,

(3.1) $$F(z, \xi) = F_0(\psi(z), \psi_*(z)\xi).$$

It follows that

(3.2) $$\sup_{\substack{z \in B^n \\ \xi \neq 0}} |F(\zeta, \xi)| = \sup_{\substack{w \in B^n \\ \eta \neq 0}} |F_0(w, \eta)|,$$

since $\tilde{\psi}$ is a bijection. That is, $\|f\|_{\mathcal{B}_0(B^n)} = \|f_0\|_{\mathcal{B}_0(B^n)}$. Therefore, Ψ is an isometric isomorphism. The following theorem says that every isometric isomorphism is of this form.

THEOREM. *Let $U: \mathcal{B}_0(B^n) \to \mathcal{B}_0(B^n)$ be an isometric isomorphism. Then there is a $\varphi \in \mathrm{Aut}(B^n)$ and a $\mu \in \mathbf{C}$ ($|\mu|=1$) with*

$$Uf = \mu(f(\varphi(z)) - f(\varphi(0))$$

for every $f \in \mathcal{B}_0(B^n)$.

Proof. The mapping $T = AUA^{-1}$ is an isometric isomorphism of $\mathfrak{D}_0$. Let $T^*: \mathfrak{D}_0^* \to \mathfrak{D}_0^*$ denote the adjoint of T. Thus T^* is also an isometric isomorphism. If X is a Banach space, $E(X)$ will denote the set of the extreme points of the unit ball of X. Then T^* maps $E(\mathfrak{D}_0^*)$ injectively onto $E(\mathfrak{D}_0^*)$.

The same argument as in [3] and [6] shows that every $\gamma \in E(\mathfrak{D}_0^*)$ is an evaluation functional on $\mathfrak{D}_0$, $\gamma = e_{(z,\xi)}$, where $e_{(z,\xi)}$ is evaluation at (z,ξ). Conversely, for every $(z,\xi) \in B^n \times S$, $e_{(z,\xi)}$ is an element of $E(\mathfrak{D}_0^*)$, which can be seen by Lemma 1. Therefore $E(\mathfrak{D}_0^*) = \{e_{(z,\xi)}: (z,\xi) \in B^n \times S\}$. Because T^*: $E(\mathfrak{D}_0^*) \to E(\mathfrak{D}_0^*)$ is a bijection, there exists a bijection $\tau: B^n \times S \to B^n \times S$ such that

$$T^* e_{(z,\xi)} = e_{(\tau(z,\xi))}. \tag{3.3}$$

Write $\tau(z,\xi) = (\sigma(z,\xi), \lambda(z,\xi))$, where $\sigma(z,\xi) \in B^n$ and $\lambda(z,\xi) \in S$. We claim that $\sigma(z,\xi)$ is independent of ξ. Now (3.3) implies that, for every $F \in \mathfrak{D}_0$,

$$(TF)(z,\xi) = F(\sigma(z,\xi), \lambda(z,\xi)). \tag{3.4}$$

Take $(z^0,\xi^1), (z^0,\xi^2) \in B^n \times S$; we want to prove that $\sigma(\xi^0,\xi^1) = \sigma(\xi^0,\xi^2)$. If $\xi^1 = \mu\xi^2$ ($\mu \in \mathbf{C}$, $|\mu|=1$) then

$$\begin{aligned} F(\sigma(z^0,\xi^1), \lambda(z^0,\xi^1)) &= TF(z^0,\xi^1) = \mu TF(z^0,\xi^2) \\ &= \mu F(\sigma(\xi^0,\xi^2), \lambda(z^0,\xi^2)) = F(\sigma(z^0,\xi^2), \mu\lambda(z^0,\xi^2)). \end{aligned}$$

This is to say:

$$F(\sigma(z^0,\xi^1), \lambda(z^0,\xi^1)) = F(\sigma(z^0,\xi^1)), \mu\lambda(z^0,\xi^2))$$

for every $F \in \mathfrak{D}_0$; hence

$$\sigma(z^0,\xi^1) = \sigma(z^0,\xi^1) \quad \text{and} \quad \lambda(z^0,\xi^1) = \mu\lambda(z^0,\xi^2).$$

Now suppose that ξ^1, ξ^2 are linearly independent; then $\xi^1 + \xi^2 \neq 0$. Let $\xi^3 = |\xi^1+\xi^2|^{-1}(\xi^1+\xi^2)$, $f = A^{-1}F$, $g = A^{-1}(TF)$, and

$$(\tilde{z}^j, \tilde{\xi}^j) = (\sigma(z^0,\xi^j), \lambda(z^0,\xi^j)), \quad j=1,2,3.$$

Then we have:

$$H(z^0,\xi^j)TF(z^0,\xi^j) = D_{\xi^1} g(\xi_0), \quad j=1,2,3; \tag{3.6}$$

$$H(z^0,\xi^3)TF(z^0,\xi^3) = |\xi^1+\xi^2|^{-1}[H(z^0,\xi^1)TF(z^0,\xi^1) + H(z^0,\xi^2)TF(z^0,\xi^2)]$$

(3.7) and

$$H(z^0,\xi^3)F(\tilde{z}^3,\tilde{\xi}^3) = |\xi^1+\xi^2|^{-1}[H(z^0,\xi^1)F(\tilde{z}^1,\tilde{\xi}^1) + H(z^0,\xi^2)F(\tilde{z}^2,\tilde{\xi}^2)];$$

$$H(z^0,\xi^3)[H(\tilde{z}^3,\tilde{\xi}^3))^{-1}D_{\tilde{\xi}^3}f(\tilde{z}^3)$$
(3.8)
$$=|\xi^1+\xi^2|^{-1}\sum_{j=1}^{2}[H(z^0,\xi^j)(H(\tilde{z}^j,\tilde{\xi}^j))^{-1}D_{\tilde{\xi}^j}f(\tilde{z}^j)].$$

Let

$$\eta^3=-H(z^0,\xi^3)(H(\tilde{z}^3,\tilde{\xi}^3))^{-1}\tilde{\xi}^3;$$
$$\eta^j=|\xi^1+\xi^2|^{-1}H(z^0,\xi^j)(H(\tilde{z}^j,\tilde{\xi}^j))^{-1}\tilde{\xi}^j,\quad j=1,2.$$

Then (3.8) becomes

$$\sum_{j=1}^{3}D_{\eta^j}f(\tilde{z}^j)=0,\quad \forall f\in \mathcal{B}_0(B^n). \tag{3.9}$$

By Lemma 2,

$$\tilde{z}^1=\tilde{z}^2=\tilde{z}^3\quad\text{and}\quad \eta^1+\eta^2+\eta^3=0.$$

In particular, $\sigma(z^0,\xi^1)=\sigma(z^0,\xi^2)$. Therefore $\sigma(z,\xi)$ depends only on z, $\sigma(z,\xi)=\sigma(z)$. Because T^{-1} is also an isometric isomorphism, the same argument applies to T^{-1}; thus $\sigma(z)$ is a bijection from B^n to B^n. We now prove that $\sigma(z)$ is holomorphic.

Because (z,ξ) and $(\sigma(z),\lambda(z,\xi))$ are in one-to-one correspondence, there is a $\tilde{\lambda}(z,\xi)$ such that $\lambda(z,\tilde{\lambda}(z,\xi))=\xi$; hence (3.4) can be written as

$$TF(z,\tilde{\xi})=F(\sigma(z),\xi),\quad \tilde{\xi}=\tilde{\lambda}(z,\xi). \tag{3.10}$$

Write $F=Af$, $TF=Ag$, and $g=Uf$, where $f,g\in\mathcal{B}_0(B^n)$; then (3.10) becomes

$$H((z,\tilde{\xi}))^{-1}D_{\tilde{\xi}}g(z)=(H(\sigma(z),\xi))^{-1}D_{\xi}f(\sigma(z)). \tag{3.11}$$

Let $f_j(z)=z_j$ and $g_j=Uf_j$; let ξ^k be defined as $\xi^k_j=\delta_{jk}$ with $\tilde{\xi}^k=\tilde{\lambda}(z,\xi^k)$. Then

$$(H(z,\tilde{\xi}^k))^{-1}\sum_{i=1}^{n}\frac{\partial g_j(z)}{\partial z_i}\tilde{\xi}^k_i=(H(\sigma(z),\xi^k))^{-1}\frac{\partial f_j(\sigma(z))}{\partial z_k}=(H(\sigma(z),\xi^k))^{-1}\delta_{jk},$$

or

$$G_0X=I. \tag{3.12}$$

Here I is the identity matrix,

$$G_0=\left(\frac{\partial g_i(z)}{\partial z_k}\right)^n_{i,k=1},\quad\text{and}\quad X=(x_{ik})^n_{i,k=1},$$

where $x_{ik}=(H(z,\tilde{\xi}^k))^{-1}H(\sigma(z),\xi^k)\tilde{\xi}^k_i$. Similarly, letting $\hat{f}_j(\zeta)=\frac{1}{2}z_j^2$, $\hat{g}_j=U\hat{f}_j$, and $\sigma(z)=(\sigma_1(z),\dots,\sigma_n(z))$, we obtain

$$\hat{G}X=D(\sigma_1(z),\dots,\sigma_n(z)). \tag{3.13}$$

Here, the right-hand side denotes the diagonal matrix with the specified diagonal elements, and

$$\hat{G} = \left(\frac{\partial \hat{g}_i(z)}{\partial z_k} \right)_{i,k=1}^{n}.$$

From (3.12) and (3.13), we obtain

$$D(\sigma_1(z), \ldots, \sigma_n(z)) = \hat{G}(z)[G_0(z)]^{-1}.$$

This clearly implies that $\sigma(z)$ is holomorphic. Hence $\sigma \in \mathrm{Aut}(B^n)$.

Now define the mapping $\Psi: \mathcal{B}_0(B^n) \to \mathcal{B}_0(B^n)$ by

$$\Psi f(z) = f(\sigma^{-1}(z)) - f(\sigma^{-1}(0)).$$

Let $L = A\Psi A^{-1}$. Then

$$LH(z, \xi) = H(\sigma^{-1}(z), \sigma_*^{-1}(z)\xi)), \quad H \in \mathfrak{D}_0. \tag{3.14}$$

Combining (3.4) and (3.14) we have

$$LTH(z, \xi) = TH(\sigma^{-1}(z), \sigma_*^{-1}(z)\xi) = H(\zeta, \lambda(\sigma^{-1}(z), \sigma_*^{-1}(z)\xi))$$

or

$$\hat{T}H(z, \xi) = H(z, \hat{\lambda}(z, \xi)) \quad (H \in \mathfrak{D}_0), \tag{3.15}$$

where $\hat{T} = LT$ and $\hat{\lambda}(z, \xi) = \lambda(\sigma^{-1}(z), \sigma_*^{-1}(z)\xi)$.

By Lemma 3, (3.15) implies that

$$LTH(z, \xi) = \hat{T}H(z, \xi) = H(z, \mu\xi) = \mu H(z, \xi)$$

and

$$\Psi Uh(z) = \mu h(z), \quad h \in \mathcal{B}_0(B^n).$$

Therefore

$$Uh(z) = \mu(h(\sigma(z)) - h(\sigma(0))), \quad h \in \mathcal{B}_0(B^n). \qquad \square$$

4. Closing Remarks

Lemma 3 is the key technical result required for our characterization of isometries of $\mathcal{B}_0(B^n)$. It is this point which we find intractible for more general domains. Implicit in the proof of Lemma 3 is our explicit knowledge that the coordinate discs are totally geodesic manifolds for the Kobayashi metric. While a great deal is known about the boundary behavior of the Kobayashi metric on strongly pseudoconvex domains, little is known about the global interior behavior. Lempert [9] has constructed totally geodesic discs in strictly convex domains, but their global properties are not well understood.

On the other hand (see [5]), generic strongly pseudoconvex domains, even those near the ball, have no automorphisms except the identity. Thus we see that the functional analytic question of characterizing isometries of the Bloch space is linked to deep questions of geometry. We hope to explore this matter in future work.

Related work on Bloch spaces, from the Kobayashi-metric point of view, appears in [8].

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