CYCLIC VECTORS FOR MULTIPLICATION OPERATORS

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A vector f in a (complex) linear topological space is said to be a *cyclic* vector for a continuous linear transformation T on E if the set $\mathcal{P}(T)f$ is dense in E. Here $\mathcal{P}(T)=\{p(T): p\in\mathcal{P}\}$, and $\mathcal{P}$ denotes the set of polynomials. We prove two theorems about cyclic vectors in spaces of functions, first for measurable functions, then for analytic functions. The corollary to the first theorem generalizes a theorem of Bram about cyclic vectors of normal operators on Hilbert space.

If μ is a finite measure, let X denote the closed unit ball of $L^\infty(\mu)$ and let X_p denote X with the metric of $L^p(\mu)$. Recall that a residual subset of a complete metric space is a subset whose complement is a set of the first category.

THEOREM 1. *Let μ be a compactly supported Borel measure in the complex plane. If $0<p<\infty$, then:*

(a) *X_p is a complete metric space*;

(b) *the set of cyclic vectors in $L^p(\mu)$ for the operator of multiplication by z is a residual set*; *and*

(c) *the subset of cyclic vectors that lie in X is a residual subset of X_p.*

Proof. Choose $p\in(0,\infty)$ and keep it fixed throughout the proof.

(a) If $\{f_n\}\subset X$ and if $f_n\to f$ in $L^p(\mu)$, then f_n converges to f in measure, and a subsequence converges pointwise almost everywhere; thus $f\in X$ and so X_p is complete.

(b) Let $\{U_n\}$ be a countable basis for the open sets in $L^p(\mu)$. Let T denote the operator of multiplication by z. For each n, let V_n denote the set of vectors f for which there exists a polynomial p such that $p(T)f\in U_n$. Then V_n is an open set, and $\bigcap V_n$ is the set of cyclic vectors for T. Thus (b) will be proved if we show that V_n is dense in $L^p(\mu)$.

Let $K=\operatorname{supp}(\mu)$. If F is a closed set, if κ_F is its characteristic function, and if h is any function, then h_F denotes the function $\kappa_F h$. Finally, let $d(f,g)$ denote the distance from f to g in $L^p(\mu)$.

Claim 1. Let $\epsilon>0$ and $h\in L^p(\mu)$ be given. Then there exists $\delta>0$ such that if $F\subset K$ and $\mu(K\setminus F)<\delta$, then $d(h,h_F)<\epsilon$. Indeed, if $p\le 1$ then

$$d(h,h_F)=\int |h-h_F|^p\,d\mu=\int_{K\setminus F}|h|^p\,d\mu,$$

and the result follows by absolute continuity. The proof is similar when $p>1$.

Claim 2. If $\delta>0$ is given, then there exists a closed set F with empty interior and connected complement, such that $F\subset K$ and $\mu(K\setminus F)<\delta$. Indeed, let $\{I_n\}$ be an enumeration of the open subintervals with rational endpoints on the real axis,

Received November 30, 1987. Revision received March 1988.
Research supported in part by the National Science Foundation.
Michigan Math. J. 35 (1988).

and let Y denote the imaginary axis. Inside each interval I_n choose an open subinterval J_n such that $\mu(J_n \times Y) < \delta/2^n$. The set $F = K \setminus \bigcup (J_n \times Y)$ satisfies our requirements. (To see that the complement of F is connected, note that if w is a point in the complement then a small horizontal segment from w stays in the complement. This will put us inside one of the sets $J_n \times Y$, and so w can be connected to ∞ in the complement of F.)

Claim 3. Let F be as in Claim 2, and let $\mu_F = \mu | F$, the restriction of μ to F. Then $\mathcal{P}$ is dense in $L^p(\mu_F)$. Indeed, the continuous functions are dense in $L^p(\mu_F)$ (from Lusin's theorem), and the polynomials are uniformly dense in $C(F)$ by a theorem of Lavrentiev ([6, Thm. 17, p. 25]; see Carleson [3] or Conway [4, Thm. VI.8.13] for a more functional analytic proof).

We are now ready to show that V_n is dense. Let $g \in L^p(\mu)$ and let $\alpha > 0$ be given; we must show that V_n contains an element within distance α of g. Since the nonvanishing functions are dense in $L^p(\mu)$, we may assume that g never vanishes.

Choose h, and $\epsilon > 0$, so that the open ball $B(h, 2\epsilon)$ is contained in U_n. By Claim 1 there exists $\delta > 0$ such that, if $F \subset K$ and if $\mu(K \setminus F) < \delta$, then $d(g, g_F) < \alpha$ and $d(h, h_F) < \epsilon$. We shall choose F so that $g_F \in V_n$.

First choose a closed set $J \subset K$ with $\mu(K \setminus J) < \delta/2$, such that $|g|$ is both bounded above and bounded away from zero on J. By Claim 2 there is a closed set $F \subset J$ with empty interior and connected complement, and with $\mu(J \setminus F) < \delta/2$. Then $\mathcal{P}$ is dense in $L^p(\mu_F)$, by Claim 3. Therefore, $g_F \mathcal{P}$ is dense in $L^p(\mu_F)$, since multiplication by g_F is an invertible bounded linear transformation on $L^p(\mu_F)$. Hence there is a polynomial p such that $d(h_F, pg_F) < \epsilon$. (Since the functions vanish off F, the distances in $L^p(\mu)$ and in $L^p(\mu_F)$ coincide.) Thus $d(h, pg_F) < 2\epsilon$, and so $g_F \in V_n$, which completes the proof of (b).

(c) The proof here is similar to the proof of (b), and we use the same notations. Let $W_n = X \cap V_n$. Then W_n is an open subset of X_p, and $\bigcap W_n$ is the set of cyclic vectors lying in X. We must show that W_n is dense in X_p. Let $g \in X_p$ and let $\alpha > 0$ be given. The nonvanishing functions are dense in X_p, so we may assume that g never vanishes. The proof of (b) shows that, for a suitable set F, $d(g, g_F) < \alpha$ and $g_F \in V_n$. Since $g_F \in X$, we have $g_F \in W_n$, which completes the proof. □

A vector f in a Hilbert space H is said to be a *-cyclic vector for an operator T on H if the smallest subspace containing f and invariant for both T and T^* (i.e., the smallest reducing subspace for T containing f) is all of H. (Note: subspaces are closed, and operators are linear and bounded.) A well-known theorem of Bram states that if a normal operator has a *-cyclic vector then it has a cyclic vector (see [2, Thm. 6] or [4, Thm. VI.8.14, p. 344]). We generalize this as follows.

COROLLARY. *Let T be a normal operator on Hilbert space and suppose that T has a *-cyclic vector. Then the set of noncyclic vectors is a set of the first category.*

Proof. A form of the spectral theorem for normal operators states that, in the presence of a *-cyclic vector, there is a Borel probability measure μ on the spectrum of T such that T is unitarily equivalent to the operator of multiplication by z on $L^2(\mu)$ (see [4, Thm. II.4.3, p. 74]). The result now follows from the theorem. □

Returning to the theorem for a moment, we note that even in simple cases it may not be obvious that there are any cyclic vectors, let alone how to describe them. This is illustrated by the bilateral shift operator, where the theory has been completely worked out. This is the operator of multiplication by z on $L^2(\mu)$, where μ is normalized Lebesgue measure on the unit circle $\mathbf{T}$. We denote the operator by T. The following results are known (see Helson [5, Chap. 2]). The reducing subspaces of $L^2(\mu)$ (i.e., invariant under both T and T^*) are the spaces of functions vanishing on a fixed set of positive measure. The nonreducing invariant subspaces are the spaces of the form φH^2, where $|\varphi| = 1$ almost everywhere. From this it can be shown that f is a cyclic vector if and only if both of the following two conditions are satisfied: (i) $|f| > 0$ a.e.; (ii) $\int_{\mathbf{T}} \log|f| = -\infty$.

If we replace $L^p(\mu)$ by a space of analytic functions, then the situation changes somewhat. We prove one theorem to indicate this, without striving for maximum generality.

Let E be a Banach space whose elements are analytic functions in the open unit disc $\mathbf{D}$ in the complex plane, with the usual vector space operations. If f is analytic in $\mathbf{D}$, let $f_r(z) = f(rz)$, $r < 1$. Also, $\mathbf{D}^-$ denotes the closure of $\mathbf{D}$. Finally, "analytic on $\mathbf{D}^-$" means "analytic in a neighborhood of $\mathbf{D}^-$". We make four additional assumptions about E:

(1) point evaluation at each point of $\mathbf{D}$ is a bounded linear functional on E;
(2) E contains the polynomials as a dense subset;
(3) if φ is analytic on $\mathbf{D}^-$ then $\varphi E \subset E$; and
(4) if $f \in E$ then $f_r \to f$ in norm as $r \uparrow 1$.

It is known that if a sequence in E converges in norm, or weakly, then the functions converge uniformly on compact subsets of $\mathbf{D}$. Also, if $\varphi E \subset E$ then the operator M_φ of multiplication by φ is a bounded linear transformation on E. Finally, if $f \in E$ is a cyclic vector for the operator M_z then $|f(z)| > 0$ for all $z \in \mathbf{D}$. See [1, pp. 271–273] for these results. Let $\sigma(T)$ denote the spectrum of the operator T.

LEMMA. *Let φ be analytic on $\mathbf{D}^-$. Then $\sigma(M_\varphi) = \varphi(\mathbf{D}^-)$.*

Proof. Let $w \notin \varphi(\mathbf{D}^-)$. The operator $M_\varphi - w$ is invertible, since $(\varphi - w)^{-1}$ is analytic on $\mathbf{D}^-$. Thus $\sigma(M_\varphi) \subset \varphi(\mathbf{D}^-)$.

For the reverse inclusion, let $\alpha \in \mathbf{D}$ and let N_α denote the set of all those functions in E that vanish at α. Then $N_\alpha \neq E$ (by (1), N_α is the kernel of a linear functional). The operator $M_\varphi - \varphi(\alpha)$ is not invertible, since its range lies in N_α. Thus $\varphi(\mathbf{D}) \subset \sigma(M_\varphi)$. This completes the proof, since the spectrum is a closed set. □

Let N denote the nonvanishing functions in E, together with the zero function:

$$N = \{0\} \cup \{f \in E : |f(z)| > 0,\ z \in \mathbf{D}\}.$$

As mentioned above, N contains the cyclic vectors for M_z. Let X denote the set of those $f \in N$ for which $|f| \leq 1$ in $\mathbf{D}$.

THEOREM 2. (a) *Both N and X are closed subsets of E.*
(b) *The set of cyclic vectors for M_z is a residual subset of N.*
(c) *The cyclic vectors in X form a residual subset of X.*

Proof. (a) To see that N is closed, we recall that norm convergence implies uniform convergence on compact sets. Now apply Hurwitz' theorem: if a sequence of analytic functions converges uniformly on compact sets and if the limit function vanishes at a point w, then either the limit is identically zero, or all but finitely many of the approximating functions vanish in each neighborhood of w.

That X is closed follows from the fact, noted above, that norm convergence implies pointwise convergence.

(b) We begin as in the proof of Theorem 1. Let $\{U_n\}$ be a countable basis for the open sets in E (E is separable since $\mathcal{P}$ is dense). Let V_n be the set of f in E for which there exists $p \in \mathcal{P}$ with $pf \in U_n$. Then V_n is open, and $\bigcap V_n$ coincides with the set of cyclic vectors. We must show that V_n is dense in N. It will be sufficient to show that the set of cyclic vectors is dense in N.

Let $f \in N$ be given, $f \neq 0$. Then f_r is close to f for r near 1, and $f_r E \subset E$. Also, $0 \notin f_r(\mathbf{D}^-)$ and so, by the lemma, multiplication by f_r is an invertible operator on E. Since $\mathcal{P}$ is a dense subset of E it follows that $f_r \mathcal{P}$ is dense in E, and thus f_r is a cyclic vector. This completes the proof of (b).

(c) The proof here is similar to the proof of (b) and we use the same notations. Let $W_n = X \cap V_n$. Then W_n is an open subset of X, and $\bigcap W_n$ is the set of cyclic vectors contained in X. We now show that this set is dense. If $f \in X$, then the proof in (b) shows that f_r is cyclic. Since $f_r \in X$, the proof is complete. □

Finally, we return to Theorem 1 to raise the following question. If μ is a positive finite Borel measure with compact support in the complex plane, then does the operator of multiplication by z on $L^p(\mu)$ necessarily have cyclic vectors that are *continuous* on the support of μ? This is true for the bilateral shift operator mentioned earlier.

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