FRÉCHET ENVELOPES OF CERTAIN ALGEBRAS OF ANALYTIC FUNCTIONS

C. M. Eoff

1. Introduction. Let D denote the open unit disc in the complex plane. The Smirnov class (or Hardy algebra) N^+ consists of those analytic functions on D for which

$$\lim_{r\to 1-} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+|f(re^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+|f(e^{i\theta})|\,d\theta < +\infty,$$

where $f(e^{i\theta})$ are the boundary values of f on ∂D [2]. Although N^+ has appeared in the classical literature since 1932 (see [2, p. 31]), it was not until the early 1970s that a study of the linear topological properties was carried out by Yanagihara ([12], [13]). He showed in [13] that N^+ is an F-space (complete, metrizable linear topological space), in fact an F-algebra (multiplication is jointly continuous) with the translation-invariant metric d defined by

$$d(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log(1+|f(e^{i\theta})-g(e^{i\theta})|)\,d\theta.$$

Like the Hardy spaces H^p, for $0<p<1$, Yanagihara showed that N^+ is not locally convex but still has a rich dual space. However, in contrast to H^p, he showed that N^+ is not locally bounded (i.e., has no bounded neighborhood of zero).

The Fréchet envelope for N^+ was identified by Yanagihara [12] as F^+, those analytic functions on D for which

$$\lim_{r\to 1-} (1-r)\log^+(\max_{|z|=r}|f(z)|) = 0.$$

He showed that the topology of F^+ can be given by a family of seminorms, $(\|\cdot\|_c)_{c>0}$, defined by

$$\|f\|_c = \sum_0^{\infty} |a_n| \exp[-cn^{1/2}], \quad c>0,$$

where (a_n) are the Taylor coefficients of f. Natural generalizations of N^+ have been studied by Stoll in [11]: $(\mathrm{Log}^+ H)^\alpha$, $\alpha>1$, the Hardy-Orlicz algebra of analytic functions on D which satisfy

$$\sup_{0<r<1} \frac{1}{2\pi}\int_{-\pi}^{\pi} [\log^+(|f(re^{i\theta})|]^\alpha\,d\theta < +\infty,$$

and $(\mathrm{Log}^+ H(D))^\alpha$, $\alpha\geq 1$, the Bergman-Orlicz algebra of analytic functions on D for which

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$$\int_D (\log^+|f(z)|)^\alpha \, dA(z) < +\infty,$$

where dA is normalized area measure on D.

Stoll observed that for any α, β $(1 < \alpha < \beta < +\infty)$ and all $p > 0$,

$$H^p \subsetneqq (\mathrm{Log}^+ H)^\beta \subsetneqq (\mathrm{Log}^+ H)^\alpha \subsetneqq N^+;$$

also, that

$$\sup_{0<\gamma<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} [\log^+|f(re^{i\theta})|]^\alpha \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\log^+|f(e^{i\theta})|]^\alpha \, d\theta.$$

One could equivalently define $(\mathrm{Log}^+ H)^\alpha$, $\alpha > 1$, to consist of those $f \in N^+$ for which

$$\int_0^{2\pi} [\log^+|f(e^{i\theta})|]^\alpha \, d\theta < +\infty.$$

From this viewpoint, it would be consistent to write $(\mathrm{Log}^+ H)^\alpha = N^+$ for $\alpha = 1$. For our purposes, nothing is lost, and this is the view we shall take in the sequel since it allows us to subsume N^+ as a special case of our general results.

Stoll showed that $(\mathrm{Log}^+ H)^\alpha$, when given the metric d_α defined by

$$d_\alpha(f, g) = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} [\log(1 + |f(e^{i\theta}) - g(e^{i\theta})|]^\alpha \, d\theta \right\}^{1/\alpha},$$

is an F-algebra with separating dual (e.g., point evaluations are continuous). He obtained similar results for $(\mathrm{Log}^+ H(D))^\alpha$, $\alpha \geq 1$, given the metric ρ_α defined by

$$\rho_\alpha(f, g) = \left\{ \int_D [\log(1 + |f(z) - g(z)|)]^\alpha \, dA(z) \right\}^{1/\alpha}.$$

In connection with these spaces Stoll also studied the spaces F_β, consisting of those analytic functions on D for which

$$\lim_{r \to 1-} (1-r)^\beta \log^+(\max_{|z| \leq r} |f(z)|) = 0.$$

He proved that for each $c > 0$ and $f \in F_\beta$, $f(z) = \sum_{n=0}^\infty a_n z^n$,

$$\|f\|_c = \sum_{n=0}^\infty |a_n| \exp[-c n^{\beta/(1+\beta)}]$$

defines a seminorm on F_β, and with the topology given by this family $(\|\cdot\|_c)_{c>0}$, F_β is a Fréchet algebra. Additionally, Stoll showed that $(\mathrm{Log}^+ H)^\alpha$ is a dense linear subspace of $F_{1/\alpha}$ $(\alpha > 1)$, and that the topology given by the seminorms $(\|\cdot\|_c)_{c>0}$ is weaker than the metric topology. He indicated that analogous results hold for $(\mathrm{Log}^+ H(D))^\alpha$, $\alpha \geq 1$, and for $F_{2/\alpha}$, but included details only for the case $\alpha = 1$ since the general case follows from similar arguments. The spaces F_β have also been studied independently by Zayed ([14], [15]); many of the results in [14] parallel those of Stoll in [11], albeit in a more general setting.

In view of the results of Stoll and Yanagihara, clearly $F_{1/\alpha}$ and $F_{2/\alpha}$ are the natural candidates for the Fréchet envelopes of $(\mathrm{Log}^+ H)^\alpha$ and $(\mathrm{Log}^+ H(D))^\alpha$, respectively. Results in Section 4 will show that this is in fact the case. Although somewhat similar to Yanagihara's argument for N^+, our method of proof is different in certain essential features. In Section 2, we recall some of the basic facts about the Fréchet envelope of a nonlocally convex F-space. Section 3 consists of technical lemmas we will need for the proofs of our main theorems.

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2. The Fréchet envelope of an F-space. For an arbitrary F-space X, with translation-invariant metric d, let V_n denote the d-ball of radius n^{-1}, $n=1,2,\ldots$. The collection $\{V_n\}$ is a countable base for the zero-neighborhoods. Let $\tilde{V}_n$ denote the absolutely convex hull of V_n and let $\|\cdot\|_n$ be the Minkowski functional of $\tilde{V}_n$. Each $\|\cdot\|_n$ is a seminorm, and the collection $\{\|\cdot\|_n\}$ generates a locally convex topology on X (possibly non-Hausdorff) that is clearly weaker than the original topology. This construction describes the Mackey topology, $m=m(X)$, the unique maximal locally convex topology on X for which X still has dual space X^* ([8, Thm. 1] or [9, §2.8]). If X is not locally convex, then because of the failure of the Hahn–Banach theorem (see [7, Chap. 4]) it can happen that $X^*=\{0\}$, as the example $L^p[0,1]$, $0<p<1$, shows; if $X^*=\{0\}$, then m is just the indiscrete topology. If X^* separates the points of X, this is necessary and sufficient for m to be Hausdorff; in this case m is metrizable and the completion of X with respect to m is a Fréchet space, called the *Fréchet envelope* of X and denoted $\hat{X}$. X and $\hat{X}$ have the same dual spaces, in the sense that every continuous linear functional on $\hat{X}$ restricts to one on X, and every continuous linear functional on X extends continuously to one on $\hat{X}$.

If an F-space X has a bounded neighborhood of zero B (i.e., locally bounded), then a base for the zero-neighborhoods can be given by the sets $n^{-1}B$, $n=1,2,\ldots$. A locally bounded F-space is called a *quasi-Banach* space. It is not difficult to check that the Fréchet envelope of a quasi-Banach space (with separating dual) is a Banach space, the Banach envelope $\hat{X}$ (see [7, Chap. 2]).

The sequence space l_p and the Hardy space H^p, for $0<p<1$, are the classical examples of non-locally convex F-spaces with separating dual spaces; both are locally bounded. For l_p, $0<p<1$, the absolutely convex hull of the unit ball of l_p is the l_1-unit ball; it follows that the Mackey topology on l_p is the l_1-topology. The l_1-closure of l_p is l_1 and thus l_1 is the Banach envelope of l_p. The situation is not so transparent for the Hardy space H^p, $0<p<1$; Duren, Romberg, and Shields identified the Banach envelope of H^p in their milestone paper of 1969 [3]. Somewhat later, Shapiro gave a different proof of this result, utilizing his convex hull characterization of the Mackey topology via a reproducing kernel [9]. The Banach envelope of H^p is a Bergman space which turns out to be isomorphic to l_1 [8]. Kalton showed that the Banach envelope of any non-locally convex quasi-Banach space X (with separating dual) must be l_1-like in character; precisely, l_1 must be finitely representable in $\hat{X}$ [7, Thm. 4.14]. Further examination of the special structure of Banach envelopes was carried out by Kalton in [6], where, for example, he constructs a non-locally convex quasi-Banach space with an unconditional basis whose Banach envelope is isomorphic to c_0. However, he shows that this case is pathological by proving that c_0 can never be the Banach envelope of a non-locally convex "natural" space (a concept which includes all the non-locally convex quasi-Banach spaces that are commonly studied in analysis).

3. Preliminaries. In this section we consider the function

$$f(z)=\exp\left[c\frac{z}{(1-z)^3}\right], \quad c>0,$$

obtaining certain estimates to be used in our arguments in Section 4. First note that for $z=re^{i\theta}$,

$$\begin{aligned}\operatorname{Re}\frac{z}{(1-z)^3}&=\frac{(r+3r^3)\cos\theta+2r^4\sin^2\theta-r^4-3r^2}{(1-2r\cos\theta+r^2)^3}\\&=g(r,\theta),\quad \text{say.}\end{aligned}$$

Fixing r, we notice that $g(r,\cdot)$ is an even function of θ; elementary calculations show that there exists $\theta(r)>0$ such that $g(r,\theta)\le 0$ for $\theta\in[\theta(r),\pi]$ and $g(r,\theta)\ge 0$ for $\theta\in[0,\theta(r)]$, and that $\theta(r)\to 0$ as $r\uparrow 1$.

Next, for $\theta\ge 0$, because

$$\cos\theta\le 1-\frac{\theta^2}{2}+\frac{\theta^3}{6} \quad\text{and}\quad 1-2r\cos\theta+r^2\ge(1-r)^2+\frac{2\theta^2}{\pi^2}$$

for $r\ge\frac{1}{2}$, we observe that

$$\begin{aligned}g(r,\theta)&\le\frac{r(1-r)^3-\frac{1}{2}r(1-r)(4r^2+r+1)\theta^2+\frac{1}{6}\theta^3}{((1-r)^2+(2/\pi^2)\theta^2)^3}\\&\le\frac{(1-r)^3+3(1-r)\theta^2+\theta^3}{((1-r)^2+(2/\pi^2)\theta^2)^3}\end{aligned}$$

for $r\in[\frac{1}{2},1)$ and $\theta\in[0,\pi]$.

LEMMA 3.1. *Let $f(z)=\exp[cz(1-z)^{-3}]$ and $f_R(z)=f(Rz)$, where $R\in[\frac{1}{2},1)$ and $c\in(0,1)$. For $\alpha\ge 1$, there exists a constant $M=M(\alpha)$ such that*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[\log^+|f_R(e^{i\theta})|]^\alpha\,d\theta\le Mc^\alpha(1-R)^{1-3\alpha}.$$

Proof. Applying our earlier observations, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}[\log^+|f_R(e^{i\theta})|]^\alpha\,d\theta&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\log^+[\exp cg(R,\theta)]\}^\alpha\,d\theta\\&=\frac{c^\alpha}{\pi}\int_0^{\theta(R)}[g(R,\theta)]^\alpha\,d\theta\\&\le\frac{c^\alpha}{\pi}\int_0^{\pi}\left\{\frac{(1-R)^3+3(1-R)\theta^2+\theta^3}{[(1-R)^2+(2/\pi^2)\theta^2]^3}\right\}^\alpha d\theta\\&=\frac{c^\alpha}{\pi}\int_0^{\pi}\left[\frac{\rho^3+3\rho\theta^2+\theta^3}{(\rho^2+(2/\pi^2)\theta^2)^3}\right]^\alpha d\theta\quad(\text{where }\rho=1-R)\end{aligned}$$

$$\leq c^\alpha \rho^{1-3\alpha}\frac{1}{\pi}\int_0^\infty \left[\frac{1+3t^2+t^3}{(1+(2/\pi^2)t^2)^3}\right]^\alpha dt$$
$$= c^\alpha (1-R)^{1-3\alpha} M,$$

where

$$M = M(\alpha) = \frac{1}{\pi}\int_0^\infty \left[\frac{1+3t^2+t^3}{(1+(2/\pi^2)t)^3}\right]^\alpha dt. \qquad \square$$

LEMMA 3.2. *Let* $f(z) = \exp[c(z(1-z)^{-3})]$ *and* $f_R(z) = f(Rz)$, *where* $0<c<1$ *and* $\frac{1}{2}<R<1$. *For* $\alpha \geq 1$, *there is a constant* $M = M(\alpha)$ *such that*

$$\int_D [\log^+|f_R(z)|]^\alpha \, dA(z) \leq c^\alpha M(1-R)^{2-3\alpha}.$$

Proof.

$$\int_D [\log^+|f_R(z)|]^\alpha \, dA(z) = \frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi}\left[\log^+\exp\left(\operatorname{Re}\frac{cRre^{i\theta}}{(1-Rre^{i\theta})^3}\right)\right]^\alpha r\, d\theta\, dr$$
$$= \frac{2}{\pi R^2}\int_0^R\int_0^\pi \left[\log^+\exp\left(\operatorname{Re}\frac{cue^{i\theta}}{(1-ue^{i\theta})^3}\right)\right]^\alpha u\, d\theta\, du$$
$$= \frac{2c^\alpha}{\pi R^2}\int_0^{1/2}\int_0^{\theta(u)} [g(u,\theta)]^\alpha u\, d\theta\, du$$
$$+ \frac{2c^\alpha}{\pi R^2}\int_{1/2}^R\int_{1/2}^{\theta(u)} [g(u,\theta)]^\alpha u\, d\theta\, du.$$

Now,

$$\frac{2c^\alpha}{\pi R^2}\int_0^{1/2}\int_0^{\theta(u)} [g(u,\theta)]^\alpha u\, du\, d\theta \leq \frac{8c^\alpha}{\pi}\int_0^{1/2}\int_0^{\theta(u)} [g(u,\theta)]^\alpha u\, d\theta\, du$$
$$\leq M_1 = M_1(\alpha)$$

and

$$\frac{2c^\alpha}{\pi R^2}\int_{1/2}^R\int_0^{\theta(u)} [g(u,\theta)]^\alpha u\, d\theta\, du \leq \frac{8c^\alpha}{\pi}\int_{1/2}^R\int_0^\pi \left\{\frac{(1-u)^3+3(1-u)\theta^2+\theta^3}{[(1-u)^2+(2/\pi^2)\theta^2]^3}\right\}^\alpha d\theta\, du$$
$$= \frac{8c^\alpha}{\pi}\int_{1-R}^{1/2}\int_0^\pi \left[\frac{\rho^3+3\rho\theta^2+\theta^3}{(\rho^2+(2/\pi^2)\theta^2)^3}\right]^\alpha d\theta\, d\rho$$

(where $\rho = 1-u$)

$$= c^\alpha \int_{1-R}^{1/2} \rho^{1-3\alpha}\left\{\frac{8}{\pi}\int_0^\infty \left[\frac{1+3t^2+t^3}{(1+(2/\pi^2)t^2)^3}\right]^\alpha dt\right\} d\rho$$
$$= c^\alpha M_2 \int_{1-R}^{1/2} \rho^{1-3\alpha}\, d\rho \quad (M_2 = M_2(\alpha))$$
$$= c^\alpha M_2 (3\alpha-2)^{-1}[(1-R)^{2-3\alpha} - 2^{3\alpha-2}]$$
$$\leq c^\alpha M_2 (1-R)^{2-3\alpha}.$$

Consequently,

$$\int_D [\log^+|f_R(z)|]^\alpha \, dA(z) \le M_1 + c^\alpha M_2 (1-R)^{2-3\alpha}$$
$$\le c^\alpha M(1-R)^{2-3\alpha},$$

with $M = M(\alpha)$. □

LEMMA 3.3. *For* $f(z)=\exp[cz(1-z)^{-3}]$ *and* $f(z)=\sum_{n=0}^{\infty} a_n(c)z^n$, $0<c<1$, *we have that* $\log|a_n(c)| \ge 4\cdot 3^{-3/4}c^{1/4}n^{3/4} - \sqrt{3cn} - \gamma$, *with* $\gamma \le A + B\log n$, A, B *constants independent of* c, n.

Proof.

$$\begin{aligned} f(z) &= \exp\left[c\frac{z}{(1-z)^3}\right] \\ &= 1+\sum_{k=1}^{\infty}\frac{c^k}{k!}\frac{z^k}{(1-z)^{3k}} \\ &= 1+\sum_{k=1}^{\infty}\frac{c^k}{k!}\sum_{\lambda=0}^{\infty}\binom{3k+\lambda-1}{\lambda}z^{k+\lambda} \\ &= 1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{c^k}{k!}\binom{n-2k-1}{n-k}z^n \\ &= 1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{c^k}{k!}\binom{n+2k}{n-k}\left(\frac{3k}{n+2k}\right)z^n, \end{aligned}$$

so that $a_0(c)=1$ and

$$a_n(c)=\sum_{k=1}^{n}\frac{c^k}{k!}\binom{n+2k}{n-k}\frac{3k}{n+2k}$$

for $n=1,2,3,\ldots$. Thus

$$a_n(c)\ge\frac{c^k}{k!}\binom{n+2k}{n-k}\frac{3k}{n+2k}$$

for $k=1,2,3,\ldots,n$ and $n=1,2,3,\ldots$.

For $1\le r<n$, we apply Stirling's formula and observe that

$$\binom{n}{r}\ge\frac{n^n}{(n-r)^{n-r}r^r}\frac{1}{\sqrt{2\pi n}}\exp\left[-\frac{1}{12(n-r)}-\frac{1}{12r}\right].$$

Also,

$$\begin{aligned} \log\frac{n^n}{(n-r)^{n-r}r^r} &= \log\frac{n^{n-r}n^r}{(n-r)^{n-r}r^r} \\ &= (n-r)\log\left(\frac{n}{n-r}\right)+r\log\left(\frac{n}{r}\right) \\ &\ge (n-r)\left(\frac{r}{n}\right)+r\log\frac{n}{r} \\ &= r-\frac{r^2}{n}+r\log\frac{n}{r} \end{aligned}$$

(using the inequality $-\log(1-u)\ge u$ for $u<1$). Consequently,

$$\log\binom{n}{r} \geq r - \frac{r^2}{n} + r\log\left(\frac{n}{r}\right) - \gamma_1(n,r),$$

where

$$\gamma_1(n,r) = \frac{1}{12r} + \frac{1}{12(n-r)} + \frac{1}{2}\log(\pi n)$$
$$\leq \frac{1}{6} + \frac{1}{2}\log(2\pi n);$$

thus,

$$\log\binom{n+2k}{n-k} = \log\binom{n+2k}{3k}$$
$$\geq 3k - \frac{9k^2}{n+2k} + 3k\log\left(\frac{n+2k}{3k}\right) - \gamma_2(n,k),$$

$$\gamma_2(n,k) = \gamma_2(n) \leq \frac{1}{6} + \frac{1}{2}\log[2\pi(n+2k)]$$
$$\leq \frac{1}{6} + \frac{1}{2}\log(6\pi n).$$

Since $\log k! \leq k\log k - k + \log\sqrt{2\pi k} + 1/12k$, we have

$$\log\left[\frac{c^k}{k!}\binom{n+2k}{n-k}\frac{3k}{n+2k}\right] \geq 4k + 3k\log\left(\frac{n+2k}{3k}\right) - \frac{9k^2}{n+2k}$$
$$-k\log k + k\log c - \gamma_3(n)$$
$$= 4k + 4k\left(\log\frac{3^{-3/4}c^{1/4}n^{3/4}}{k}\right) + 3k\log\left(1+\frac{2k}{n}\right)$$
$$-\frac{9k^2}{n+2k} - \gamma_3(n),$$

with

$$\gamma_3(n) = \log\left(\frac{n+2k}{3k}\right) + \log\sqrt{2\pi k} + \frac{1}{12k} + \gamma_2(n)$$
$$\leq \left(\tfrac{1}{4} + \log\sqrt{12}\,\pi\right) + 2\log n$$
$$= A + B\log n.$$

For $[3^{-3/4}c^{1/4}n^{3/4}] \geq 1$, set $k = [3^{-3/4}c^{1/4}n^{3/4}] = 3^{-3/4}c^{1/4}n^{3/4} - \delta$, with $\delta = \delta(n,c)$, $0 \leq \delta < 1$. Consequently, we obtain

$$\log a_n(c) \geq 4\cdot 3^{-3/4}c^{1/4}n^{3/4} - \sqrt{3cn} - \gamma_3(n),$$
$$\gamma_3(n) \leq A + B\log n.$$

for the appropriately chosen constants A, B independent of c and n. This completes the proof. □

4. The Fréchet envelopes of $(\mathrm{Log}^+ H)^\alpha$ and $(\mathrm{Log}^+ H(D))^\alpha$, $\alpha \geq 1$. Let X be an F-space with Fréchet envelope $(\hat{X}, \tau)$; recall that τ is weaker than the metric topology on X and is the strongest locally convex topology on X such that X still has dual space X^*. Now if (Y, μ) is a Fréchet space, X is a dense linear subspace of Y such that $\tau \subseteq \mu$, and μ is weaker than the metric topology, then necessarily $(\hat{X}, \tau) = (Y, \mu)$.

The aim of this section is to show that the topology of $F_{1/\alpha}$ (resp., $F_{2/\alpha}$) is stronger than that of the Fréchet envelope of $(\mathrm{Log}^+ H)^\alpha$ (resp., $(\mathrm{Log}^+ H(D))^\alpha$) for $\alpha \geq 1$. In view of our earlier remarks and Stoll's results, this will prove that $F_{1/\alpha}$ (resp., $F_{2/\alpha}$) is the Fréchet envelope of $(\mathrm{Log}^+ H)^\alpha$ (resp., $(\mathrm{Log}^+ H(D))]^\alpha$), $\alpha \geq 1$. Of course the case $\alpha = 1$ has been done by Yanagihara in [12], as previously mentioned; it will follow as a special case of our results. For completeness, we include statements of the results of Stoll ($\alpha > 1$) and Yanagihara ($\alpha = 1$).

THEOREM A ([11], [12], [13]). *Let* $\alpha \geq 1$, $f \in (\mathrm{Log}^+ H)^\alpha$, *and* $f_r(z) = f(rz)$, $0 < r < 1$. *Then*

(i) $\lim_{r \uparrow 1} d_\alpha(f_r, f) = 0$;
(ii) $(\mathrm{Log}^+ H)^\alpha$ *is a dense subspace of* $F_{1/\alpha}$; *and*
(iii) *the topology in* $F_{1/\alpha}$, *defined by the family of seminorms* $(\|\cdot\|_c)_{c>0}$, *is weaker than the metric topology in* $(\mathrm{Log}^+ H)^\alpha$.

THEOREM B ([11]). *Let* $\alpha \geq 1$, $f \in (\mathrm{Log}^+ H(D))^\alpha$, *and* $f_r(z) = f(rz)$, $0 < r < 1$. *Then*

(i) $\lim_{r \uparrow 1} \rho_\alpha(f_r, f) = 0$;
(ii) $(\mathrm{Log}^+ H(D))^\alpha$ *is a dense subspace of* $F_{2/\alpha}$; *and*
(iii) *the topology in* $F_{2/\alpha}$, *defined by the family of seminorms* $(\|\cdot\|_c)_{c>0}$, *is weaker than the metric topology in* $(\mathrm{Log}^+ H(D))^\alpha$.

Now for $X = (\mathrm{Log}^+ H)^\alpha$ or $(\mathrm{Log}^+ H(D))^\alpha$, $\alpha \geq 1$, the metric $d = d_\alpha$ or ρ_α is rotation-invariant; that is, $d(f_\theta, 0) = d(f, 0)$ where $f_\theta(z) = f(e^{i\theta} z)$. Recall the construction of the Fréchet envelope $\hat{X}$ as described in the introduction. It is easy to see that the Minkowski functional of the convex hull of a d-ball must be rotation-invariant. Thus the topology of $\hat{X}$ can always be given by a family of rotation-invariant seminorms. We will exploit this via the next useful proposition (suggested by N. Kalton).

PROPOSITION 4.1. *Let* $X = (\mathrm{Log}^+ H)^\alpha$ *or* $(\mathrm{Log}^+ H(D))^\alpha$, $\alpha \geq 1$, *and let* $\|\!|\cdot|\!\|$ *be any continuous, rotation-invariant seminorm on X. Let* $e_n(z) = z^{(n)}$, $w_n = \|\!|e_n|\!\|$, *and* $f \in X$, *and let* $f(z) = \sum_{n=0}^\infty a_n z^n$ *be the Taylor series expansion of f. Then the following hold:*

(i) $\|\!|f|\!\| \leq \sum_{n=0}^\infty |a_n| w_n$;
(ii) $|a_n| w_n \leq \|\!|f|\!\|$, $n = 0, 1, 2, \ldots$.

Proof. (i) If P is any polynomial, $P(z) = \sum_{n=0}^N b_n e_n$, we clearly have

$$\|\!|P|\!\| \leq \sum_{n=0}^{N} |b_n| w_n.$$

For $f \in X$ and $f(z) = \sum_{n=0}^\infty a_n z^n$, f is the uniform limit of the partial sums of its Taylor series on each circle $|z| \leq r < 1$, so that if $P_{N,r}(z) = \sum_{n=0}^N a_n r^n z^n$ then

$d(P_{N,r}, f_r) \to 0$ as $N \to \infty$, whereby $\|P_{N,r} - f_r\| \to 0$ as $N \to \infty$. Consequently, for each r $(0<r<1)$,

$$\|f_r\| = \lim_N \|P_{N,r}\| \le \sum_{n=0}^{\infty} r^n |a_n| w_n.$$

Now $d(f_r, f) \to 0$ as $r \uparrow 1$ (Theorems A and B), whence $\|f_r - f\| \to 0$ as $r \uparrow 1$ and so

$$\|f\| = \lim_{r\uparrow 1} \|f_r\| \le \sum_{n=0}^{\infty} |a_n| w_n.$$

(ii) As the argument for (i) demonstrates, we need only show that (ii) holds for any polynomial $P(z) = \sum_{n=0}^{N} b_n z^n$. For $z, w \in D$ and $\theta \in [-\pi, \pi]$, let $P_\theta(z) = P(e^{i\theta} z)$ and $P_z(w) = P(zw)$. Since $\|\cdot\|$ is rotation-invariant, $\|P_\theta\| = \|P\|$. For each $z \in D$, $P_z(w) = \sum_{n=0}^{N} (a_n z^n) w^n$ so that

$$a_n z^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_z(e^{i\theta}) e^{-in\theta}\, d\theta.$$

For clarity, write $F(z, \theta) = P(e^{i\theta} z)$; note that $F(\cdot, \theta) = P_\theta$. For each $z \in D$,

$$a_n e_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} F(z, \theta)\, d\theta.$$

Since

$$\begin{aligned}\left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} F(\cdot, \theta)\, d\theta \right\| &= \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} P_\theta\, d\theta \right\| \\ &\le \frac{1}{2\pi} \int_{-\pi}^{\pi} \|P_\theta\|\, d\theta \\ &= \|P\|,\end{aligned}$$

we have

$$a_n e_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} F(\cdot, \theta)\, d\theta.$$

Consequently,

$$\begin{aligned}|a_n| w_n &= \|a_n e_n\| \\ &= \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} F(\cdot, \theta)\, d\theta \right\| \\ &\le \|P\|,\end{aligned}$$

and (ii) follows. □

Proposition 4.1 and the lemmas from Section 2 furnish us with the necessary tools to prove our main results. Also, for $(\mathrm{Log}^+ H)^\alpha$, $\alpha \ge 1$, it is straightforward to show that the sets

$$B(\epsilon, a) = \left\{ g \in (\mathrm{Log}^+ H)^\alpha;\ \int_{-\pi}^{\pi} [\log^+(a|g(e^{i\theta})|)]^\alpha\, d\theta < \epsilon \right\}$$

$(a>0,\ \epsilon>0)$ form a base for zero-neighborhoods that defines a topology equivalent to the metric topology on $(\mathrm{Log}^+ H)^\alpha$. An analogous situation exists for

$(\mathrm{Log}^+ H(D))^\alpha$. This observation allows for certain simplifications in the proofs of Theorems 4.2 and 4.3. It is worth remarking that while there are some technical differences between the cases $(\mathrm{Log}^+ H)^\alpha$ and $(\mathrm{Log}^+ H(D))^\alpha$, $\alpha \geq 1$, the idea behind both proofs is the same.

THEOREM 4.2. *For* $\alpha \geq 1$, $F_{1/\alpha}$ *is the Fréchet envelope of* $(\mathrm{Log}^+ H)^\alpha$.

Proof. As noted previously, the topology for the Fréchet envelope of $(\mathrm{Log}^+ H)^\alpha$ can be given by a family $\mathfrak{F}$ of rotation-invariant seminorms. Let $|||\cdot||| \in \mathfrak{F}$ and $w_n = |||e_n|||$.

Now $|||\cdot|||$ is continuous on $(\mathrm{Log}^+ H)^\alpha$, so there is a zero-neighborhood V such that if $h \in V$ then $|||h||| \leq 1$. Notice that by Proposition 4.1(ii), for each nonnegative integer k,

$$w_k \leq \inf\{|a_k(h)|^{-1} : h \in V\},$$

where $a_k(h)$ is the kth Taylor coefficient of h. Keeping in mind the remarks made at the beginning of Section 4, we see by Proposition 4.1(i) that an appropriate estimate of the coefficients of a suitable family of test functions will yield the desired result. To this end, recall that we may take V to be of the form

$$V = \left\{ g \in (\mathrm{Log}^+ H)^\alpha : \int_{-\pi}^{\pi} [\log^+(r|g(e^{i\theta})|)]^\alpha \, d\theta < \delta \right\}$$

for some $r > 0$, $\delta > 0$. Notice that if

$$\int_{-\pi}^{\pi} [\log^+|g(e^{i\theta})|]^\alpha \, d\theta < \delta$$

then $ag \in V$, where $a = \min\{r^{-1}, 1\}$. Consider the family of analytic functions $f_k(z) = \exp[c_k r_k z(1 - r_k z)^{-3}]$ for sequences (c_k), (r_k), $0 < c_k < 1$ and $\frac{1}{2} < r_k < 1$, which are to be specified later. With $a = \min\{r^{-1}, 1\}$, as before, if

$$\int_{-\pi}^{\pi} [\log^+|f_k(e^{i\theta})|]^\alpha \, d\theta < \delta$$

then

$$\{af_k\} \subseteq V.$$

From Lemma 3.1 we have that

$$\int_{-\pi}^{\pi} [\log^+|f_k(e^{i\theta})|]^\alpha \, d\theta \leq M c_k^\alpha (1 - r_k)^{1-3\alpha}$$

for some constant $M = M(\alpha)$. For each k, put

$$\begin{aligned} c_k &= M^{-1/\alpha} \delta^{1/\alpha} (1 - r_k)^{(3\alpha-1)/\alpha} \\ &= \lambda^{1/\alpha} (1 - r_k)^{(3\alpha-1)/\alpha}, \end{aligned}$$

with $\lambda = M^{-1}\delta$. For any choice of $r_k \uparrow 1$ and $r_k \geq \frac{1}{2}$, $af_k \in V$ for all k. In particular, set

$$\begin{aligned} r_k &= 1 - 3^{-3\alpha/(\alpha+1)} \lambda^{1/(1+\alpha)} \left(\frac{3\alpha - 1}{\alpha} \right)^{4\alpha/(1+\alpha)} k^{-\alpha/(\alpha+1)} \\ &= 1 - A_1 k^{-\alpha/(\alpha+1)}, \end{aligned}$$

so that

$$c_k = \lambda^{4/(\alpha+1)} 3^{3((1-3\alpha)/(\alpha-1))} \beta^{4((3\alpha-1)/(\alpha+1))} k^{(1-3\alpha)/(\alpha+1)}$$

and

$$\begin{aligned} 4\cdot 3^{-3/4} c_k^{1/4} k^{3/4} &= 4\cdot 3^{-3\alpha/(\alpha+1)} \lambda^{1/(\alpha+1)} \beta^{4\alpha/(\alpha+1)} k^{1/(\alpha+1)} \\ &= A_2 k^{1/(\alpha+1)} \end{aligned}$$

with $\beta = (3\alpha-1)/\alpha$. Following the notation of Lemma 3.3, if $f_k(z) = \sum_{n=0}^{\infty} b_n^{(k)} z^n$ then $b_n^{(k)} = r_k^n a_n(c_k)$. Apply Lemma 3.3 and obtain that

$$\begin{aligned} \log b_k^{(k)} &= \log r_k^k a_k(c_k) \\ &= \log a_k(c_k) + k \log r_k \\ &\geq 4\cdot 3^{-3/4} c_k^{1/4} k^{3/4} [1+o(1)] + k \log r_k \\ &= k^{1/(\alpha+1)} [A_2 + k^{\alpha/(\alpha+1)} \log(1 - A_1 k^{-\alpha/(\alpha+1)}) + o(1)]. \end{aligned}$$

Now,

$$\lim_{k\to\infty} k^{\alpha/(\alpha+1)} \log(1 - A_1 k^{-\alpha/(\alpha+1)}) = -A_1$$

and

$$A_2 - A_1 = 4\cdot 3^{-3\alpha/(\alpha+1)} \beta^{(3\alpha-1)/(\alpha+1)} \lambda^{1/(\alpha+1)} \left(1 - \frac{\beta}{4}\right) > 0,$$

because $\beta = (3\alpha-1)/\alpha = 3 - 1/\alpha$. Consequently, there exists $\eta = \eta(V, \alpha) > 0$ and k_0 so that

$$\log b_k^{(k)} \geq \eta k^{1/(\alpha+1)}$$

for all $k \geq k_0$. It follows that

$$(b_k^{(k)})^{-1} = O[\exp(-\eta k^{1/(\alpha+1)})].$$

Since $\{af_k\}_k \subseteq V$, $\|f_k\| \leq a^{-1}$ for all $k = 1, 2, \ldots$. By Proposition 4.1(ii), we have $w_k b_k^{(k)} \leq a^{-1}$, so that

$$w_k \leq a^{-1} (b_k^{(k)})^{-1} \leq C[\exp(-\eta k^{1/(\alpha+1)})]$$

for some constant $C = C(V, \alpha) > 0$.

For any $g \in (\mathrm{Log}^+ H)^\alpha$ and $g(z) = \sum_{n=0}^{\infty} \zeta_n z^n$, we have

$$\begin{aligned} \|g\| &\leq \sum_{n=0}^{\infty} |\zeta_n| w_n \\ &\leq C \sum_{n=0}^{\infty} |\zeta_n| \exp[-\eta n^{1/(\alpha+1)}] \\ &= C\|g\|_\eta, \end{aligned}$$

thereby demonstrating that the topology of $F_{1/\alpha}$ is stronger than the topology of the Fréchet envelope of $(\mathrm{Log}^+ H)^\alpha$. In view of our remarks at the beginning of Section 4, this completes the proof. □

We next consider the case $(\mathrm{Log}^+ H(D))^\alpha$, $\alpha \geq 1$. Since the idea of the proof of the next theorem is essentially the same as for Theorem 4.2, we shall keep the argument as brief as possible.

THEOREM 4.3. *For* $\alpha \ge 1$, $F_{2/\alpha}$ *is the Fréchet envelope of* $(\mathrm{Log}^+ H(D))^\alpha$.

Proof. It is enough to show that, for any continuous rotation-invariant semi-norm on $(\mathrm{Log}^+ H(D))^\alpha$ (say, $\|\cdot\|$) and $\|e_n\| = w_n$, we have

$$w_n = O[\exp(-\eta n^{2/(\alpha+2)})] \quad \text{for some } \eta > 0.$$

There is a neighborhood V of zero such that if $h \in V$ then $\|h\| \le 1$. Consider again the family $f_k(z) = \exp[c_k r_k z(1-r_k z)^{-3}]$. There are constants $a>0$ and $\delta > 0$ such that, if

$$\int_D (\log^+|f_k(z)|)^\alpha \, dA(z) \le \delta,$$

then $\{af_k\}_k \subseteq V$. From Lemma 3.2 we know that

$$\int_D (\log^+|f_k(z)|)^\alpha \, dA(z) \le c_k^\alpha M(1-r_k)^{2-3\alpha}$$

for some $M = M(\alpha)$. Set

$$\begin{aligned} c_k &= M^{-1/\alpha}\delta^{1/\alpha}(1-r_k)^{(3\alpha-2)/\alpha} \\ &= \lambda^{1/\alpha}(1-r_k)^{(3\alpha-2)/\alpha} \end{aligned}$$

with $\lambda = M^{-1}\delta$, so that $\{af_k\}_k \subseteq V$. Set

$$\begin{aligned} r_k &= 1 - 3^{-3\alpha/(\alpha+2)}\beta^{4\alpha/(\alpha+2)}\lambda^{1/(\alpha+2)}k^{-\alpha/(\alpha+2)} \\ &= 1 - A_1 k^{-\alpha/(\alpha+2)} \end{aligned}$$

with $\beta = (3\alpha-2)/\alpha$, so that

$$c_k = 3^{3((2-3\alpha)/(\alpha+2))}\beta^{4((3\alpha-2)/(\alpha+2))}\lambda^{4/(2+\alpha)}k^{(2-3\alpha)/(\alpha+2)}$$

and

$$\begin{aligned} 4\cdot 3^{-3/4}c_k^{1/4}k^{3/4} &= 4\cdot 3^{-3\alpha/(\alpha+2)}\beta^{(3\alpha-2)/(\alpha+2)}\lambda^{1/(\alpha+2)}k^{2/(\alpha+2)} \\ &= A_2 k^{2/(\alpha+2)}. \end{aligned}$$

As in Theorem 4.2, with $f_k(z) = \sum_{n=0}^\infty b_n^{(k)} z^n = \sum_{n=0}^\infty a_n(c_k) r_k^n z^n$, we apply Lemma 3.3 and obtain:

$$\begin{aligned} \log b_k^{(k)} &= \log a_k(c_k) + k \log r_k \\ &\ge 4\cdot 3^{-3/4}c_k^{1/4}k^{3/4}[1+o(1)] + k\log r_k \\ &= k^{2/(\alpha+2)}[A_2 + k^{2/(\alpha+2)}\log(1-A_1 k^{-2/(\alpha+2)}) + o(1)]. \end{aligned}$$

Since

$$A_2 - A_1 = 4\cdot 3^{-3\alpha/(\alpha+2)}\beta^{(3\alpha-2)/(\alpha+2)}\lambda^{2/(\alpha+2)}\left(1-\frac{\beta}{4}\right) > 0,$$

it follows that there is $\eta = \eta(V,\alpha) > 0$ and k_0 such that

$$\log b_k^{(k)} \ge \eta k^{2/(\alpha+2)}$$

for all $k \ge k_0$. Consequently,

$$(b_k^{(k)})^{-1} = \mathcal{O}[\exp(-\eta k^{2/(\alpha+2)})].$$

Applying Proposition 4.1(ii), we obtain

$$w_k b_k^{(k)} \leq \|f_k\| \leq a^{-1}$$

or

$$w_k \leq a^{-1}(b_k^{(k)})^{-1} \leq C \exp[-\eta k^{2/(\alpha+2)}]$$

for some constant $C = C(V, \alpha) > 0$, which completely proves the theorem. □

REMARKS. Recall the construction of the Fréchet envelope given in Section 2. By analogy, one may take $\|\cdot\|_{n,p}$ as the Minkowski functional of the absolutely p-convex hull of V_n, $0 < p < 1$. The family $\{\|\cdot\|_{n,p}\}$ generates a locally p-convex topology on X (see [7, Chap. 1]); the completion of X with respect to this topology is called the p-envelope of X and is denoted $\hat{X}_p$. Using the results proved above and [5], we show in [4] that $\hat{X}_p = \hat{X}$ for $X = (\text{Log}^+ H)^\alpha$ or $(\text{Log}^+ H(D))^\alpha$, $\alpha \geq 1$. By contrast, if X is locally bounded but not locally convex, then $\hat{X}_p$ and $\hat{X}$ can never coincide [4]. For example, see Coifman and Rochberg [1] for the q-envelopes of H_p where $0 < p < q < 1$; see Duren, Romberg and Shields [3] for the case $q = 1$.

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Department of Mathematical Sciences
University of Arkansas
Fayetteville, AR 72701