STRONG LAWS OF LARGE NUMBERS FOR WEAKLY CORRELATED RANDOM VARIABLES

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Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of complex-valued random variables on a probability space (Ω, P) such that

$$\|X_n\|^2 = E[|X_n|^2] = \int_{\Omega} |X_n(\omega)|^2 \, dP(\omega) \leq 1. \tag{1}$$

We are interested primarily in second-order conditions assuring the strong law of large numbers

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} X_n = 0 \text{ a.s.} \tag{SLNN}$$

The simplest case concerns uniformly bounded random variables.

THEOREM 1. *Let* $|X_n| \leq 1$ *a.s. and suppose that*

$$\sum_{N\geq 1} \frac{1}{N} \left\| \frac{1}{N} \sum_{n\leq N} X_n \right\|^2 < \infty. \tag{2}$$

Then the SLLN *holds.*

This theorem is essentially known, various special cases having been used in [2], [1], [10, p. 31], [9, §§III.4, IV.4]. While [8] presents almost as general a theorem, apparently Theorem 1 has not appeared explicitly in print. The proof of this and our other theorems consists in showing that the SLLN holds along some subsequence $\{N_k\}$ and then applying a suitable maximal inequality to interpolate between the N_k. When the random variables are uniformly bounded, the maximal inequality is trivial. The heart of Theorem 1, then, is the following refinement of the principle of Cauchy condensation.

LEMMA 2 [2]. *Let* $\{a_n\}_{n=1}^{\infty}$ *be real numbers such that*

$$a_n \geq 0, \qquad \sum_{n\geq 1} \frac{a_n}{n} < \infty. \tag{3}$$

Then there exists an increasing sequence of integers $\{n_k\}$ *such that* $\sum_{k\geq 1} a_{n_k} < \infty$ *and* $n_{k+1}/n_k \to 1$.

We shall constantly use the following easy and well-known lemma.

LEMMA 3. *If* Y_n *are random variables such that* $\sum_{n\geq 1} \|Y_n\|^2 < \infty$, *then* $Y_n \to 0$ *a.s.*

Received June 30, 1987. Revision received February 8, 1988.
The author was partially supported by an AMS Research Fellowship.
Michigan Math. J. 35 (1988).

Proof. Because $\int \sum |Y_n|^2 \, dP = \sum \|Y_n\|^2 < \infty$, it follows that $\sum |Y_n|^2 < \infty$ a.s., whence $Y_n \to 0$ a.s. □

Proof of Theorem 1. By Lemma 2, there exists a sequence $\{N_k\}$ such that

$$\sum_{k\geq 1} \left\| \frac{1}{N_k} \sum_{n\leq N_k} X_n \right\|^2 < \infty \quad \text{and} \quad \frac{N_{k+1}}{N_k} \to 1.$$

By Lemma 3, $(1/N_k) \sum_{n\leq N_k} X_n \to 0$ a.s. On the other hand,

$$\max_{1\leq s\leq N_{k+1}-N_k} \left| \frac{1}{N_k} \sum_{N_k+1}^{N_k+s} X_n \right| \leq \frac{N_{k+1}-N_k}{N_k} \quad \text{a.s.}$$

and this too tends to 0 as $k\to\infty$. Since for $N_k \leq N < N_{k+1}$,

$$\left| \frac{1}{N} \sum_{n\leq N} X_n \right| \leq \left| \frac{1}{N_k} \sum_{n\leq N_k} X_n \right| + \max_{1\leq s<N_{k+1}-N_k} \left| \frac{1}{N_k} \sum_{N_k+1}^{N_k+s} X_n \right|,$$

the SLLN follows. □

COROLLARY 4. *Let* $|X_n| \leq 1$ *a.s. and suppose that*

$$\forall n, m, \quad \operatorname{Re} E[X_n \bar{X}_m] \leq \Phi_1(|n-m|), \tag{4}$$

where Φ_1 *satisfies*

$$\Phi_1 \geq 0, \qquad \sum_{n\geq 1} \frac{\Phi_1(n)}{n} < \infty. \tag{5}$$

Then the SLLN *holds.*

This corollary will in fact be extended below (Corollary 11).

Proof. We have

$$\left\| \frac{1}{N} \sum_{1\leq n\leq N} X_n \right\|^2 = \frac{1}{N^2} \sum_{n,m\leq N} E[X_n \bar{X}_m] = \frac{1}{N^2} \sum_{n,m\leq N} \operatorname{Re} E[X_n \bar{X}_m]$$
$$\leq \frac{1}{N^2} \sum_{n,m\leq N} \Phi_1(|n-m|) \leq \frac{2}{N} \sum_{0\leq r<N} \Phi_1(r),$$

whence

$$\sum_{N\geq 1} \frac{1}{N} \left\| \frac{1}{N} \sum_{n\leq N} X_n \right\|^2 \leq 2 \sum_{N\geq 1} \frac{1}{N^2} \sum_{0\leq r<N} \Phi_1(r)$$
$$= 2 \sum_{r\geq 0} \Phi_1(r) \sum_{N>r} \frac{1}{N^2} \leq 4\Phi_1(0) + 2 \sum_{r\geq 1} \frac{\Phi_1(r)}{r}.$$

Thus (5) ⇒ (2) and the conclusion follows. □

Another simple maximal inequality will suffice for our next theorem, which extends [8]. The lemma we use for selecting a subsequence is an extension of Lemma 2.

LEMMA 5. *If* $\{a_n\}$ *satisfies* (3), $1 < p \leq \infty$, *and* $1/p + 1/p' = 1$, *then there exists an increasing sequence* $\{n_k\}$ *such that* $\{a_{n_k}\} \in l^{p'}$ *and* $\{n_{k+1}/n_k - 1\} \in l^p_*$, *where* $l^p_* = l^p$ *if* $p < \infty$ *and* $l^\infty_* = c_0$.

Proof. The case $p=\infty$ is Lemma 2, so suppose that $p<\infty$. We define $\{n_k\}$ inductively as follows. Set $n_1=1$ and, if n_k has been defined, let n_{k+1} be the smallest $n\geq n_k$ such that

$$\frac{a_n}{n}<\frac{1}{n_k}\left(\frac{n}{n_k}-1\right)^{p-1} \tag{6}$$

We have

$$\begin{aligned}\infty>\sum_k\sum_{n_k\leq n<n_{k+1}}\frac{a_n}{n}&\geq\sum_k\sum_{n_k\leq n<n_{k+1}}\frac{1}{n_k}\left(\frac{n}{n_k}-1\right)^{p-1}\\&\geq\sum_k\sum_{(n_k+n_{k+1})/2\leq n<n_{k+1}}\frac{1}{n_k}\frac{1}{2^{p-1}}\left(\frac{n_{k+1}}{n_k}-1\right)^{p-1}\\&\geq\sum_k\frac{n_{k+1}-n_k-1}{2}\frac{1}{n_k}\frac{1}{2^{p-1}}\left(\frac{n_{k+1}}{n_k}-1\right)^{p-1}\\&\geq\sum_{k:n_{k+1}\geq n_k+2}\frac{n_{k+1}-n_k}{4}\frac{1}{n_k}\frac{1}{2^{p-1}}\left(\frac{n_{k+1}}{n_k}-1\right)^{p-1}\\&=\frac{1}{2^{p+1}}\sum_{k:n_{k+1}\geq n_k+2}\left(\frac{n_{k+1}}{n_k}-1\right)^{p}.\end{aligned}$$

That is, $\{n_{k+1}/n_k-1\}_{k:n_{k+1}\geq n_k+2}\in l^p$. On the other hand,

$$\left\{\frac{n_{k+1}}{n_k}-1\right\}_{k:n_{k+1}=n_k+1}=\left\{\frac{1}{n_k}\right\}_{k:n_{k+1}=n_k+1}\in l^p$$

since $p>1$. Therefore $\{n_{k+1}/n_k-1\}_k\in l^p$.

In particular, $\{n_{k+1}/n_k\}$ is bounded. Now, if we raise both sides of (6) to the power p', we obtain $a_{n_{k+1}}^{p'}<(n_{k+1}/n_k)^{p'}(n_{k+1}/n_k-1)^p$. Therefore $\{a_{n_k}\}\in l^{p'}$. □

THEOREM 6. *Assume* (1) *and*

$$\sum_{N\geq 1}\frac{1}{N}\left\|\frac{1}{N}\sum_{n\leq N}X_n\right\|<\infty. \tag{7}$$

Then the SLLN *holds.*

Proof. By Lemma 5, there exists a subsequence $\{N_k\}$ such that

$$\sum_{k\geq 1}\left\|\frac{1}{N_k}\sum_{n\leq N_k}X_n\right\|^2<\infty\quad\text{and}\quad\sum\left(\frac{N_{k+1}}{N_k}-1\right)^2<\infty.$$

Hence $(1/N_k)\sum_{n\leq N_k}X_n\to 0$ a.s. In addition,

$$\begin{aligned}\left\|\max_{1\leq s<N_{k+1}-N_k}\left|\frac{1}{N_k}\sum_{N_k+1}^{N_k+s}X_n\right|\right\|&\leq\left\|\max\frac{1}{N_k}\sum_{N_k+1}^{N_k+s}|X_n|\right\|\leq\left\|\frac{1}{N_k}\sum_{N_k+1}^{N_{k+1}}|X_n|\right\|\\&\leq\frac{1}{N_k}\sum_{N_k+1}^{N_{k+1}}\|X_n\|\leq\frac{N_{k+1}}{N_k}-1,\end{aligned}$$

whence

$$\sum_{k\geq 1}\left\|\max_{1\leq s<N_{k+1}-N_k}\left|\frac{1}{N_k}\sum_{N_k+1}^{N_k+s}X_n\right|\right\|^2<\infty$$

and

$$\max_{1\le s<N_{k+1}-N_k}\left|\frac{1}{N_k}\sum_{N_k+1}^{N_k+s}X_n\right|\to 0 \text{ a.s.}$$

The conclusion follows as before. □

The following theorem is proved in exactly the same way. It includes both Theorems 1 and 6.

THEOREM 7. *Let* $1<p\le\infty$ *and* $0<r\le q\le\infty$, *with*

$$\frac{1}{p}+\frac{r}{q}\le 1.$$

Suppose that $\{X_n\}$ *are random variables such that*

$$\|X_n\|_p\le 1 \quad \textit{and} \quad \sum_{n\ge 1}\frac{1}{N}\left\|\frac{1}{N}\sum_{n\le N}X_n\right\|_q^r<\infty.$$

Then the SLLN *holds.*

Proof. By decreasing q if necessary, we may assume that $1/p+r/q=1$, whence $q<\infty$. By Lemma 5, there exists $\{N_k\}$ such that $\{\|(1/N_k)\sum_1^{N_k}X_n\|_q^r\}\in l^{q/r}$ and $\{N_{k+1}/N_k-1\}\in l_*^p$. The SLLN along $\{N_k\}$ follows from the first of these, while the second combines with the maximal inequality

$$\left\|\max_{1\le s<N_{k+1}-N_k}\left|\frac{1}{N_k}\sum_{N_{k+1}}^{N_k+s}X_n\right|\right\|_p\le\frac{N_{k+1}}{N_k}-1$$

to yield the rest of the SLLN. □

Our final theorem depends on the following subsequence principle.

LEMMA 8. *If* $\{a_n\}$ *satisfies* (3) *and also* $a_{2n}\le C(a_{n-p}+a_{n+p})$ *for some constant* C *and all* $0\le p<n/3$, *then* $\sum_{k\ge 1}a_{2^k}<\infty$.

Proof. Slightly rearranging the order of summation, we obtain

$$\begin{aligned}\infty>\sum_{n\ge 1}\frac{a_n}{n}&\ge\frac{1}{2}\sum_{k\ge 0}\sum_{0\le p<2^k/3}\left(\frac{a_{2^k-p}}{2^k-p}+\frac{a_{2^k+p}}{2^k+p}\right)\\&\ge\frac{1}{2}\sum_{k\ge 0}\sum_{0\le p<2^k/3}\frac{a_{2^k-p}+a_{2^k+p}}{2^k(1+\frac{1}{3})}\ge\frac{1}{2C}\sum_{k\ge 0}\frac{a_{2^{k+1}}}{2^k(\frac{4}{3})}\cdot\frac{2^k}{3}\\&=\frac{1}{8C}\sum_{k\ge 1}a_{2^k}.\end{aligned}$$

□

To prove our theorem, we could now follow the lines of [3] and [5], which depend implicitly on a weak-type maximal inequality. Instead, we prefer to use the following strong-type inequality.

LEMMA 9. *Suppose that for all* M *and* N,

(8) $$\left\|\frac{1}{N}\sum_{n=M+1}^{M+N}X_n\right\|^2\le\Phi_2(N).$$

Then for all M and n, we have

$$\left\| \max_{1\le s<2^n} \left| \frac{1}{2^n} \sum_{k=M+1}^{M+s} X_k \right| \right\|^2 \le \frac{1}{2} \sum_{p=1}^{n} \left(\frac{3}{4}\right)^{p-1} \Phi_2(2^{n-p}). \tag{9}$$

Proof. Let $B_n(M)$ be the random variable

$$B_n(M) = \max_{1\le s<2^n} \left| \frac{1}{2^n} \sum_{M+1}^{M+s} X_k \right|$$

with $B_0(M)\equiv 0$. We are interested in $A_n = \sup_M \|B_n(M)\|^2$. For $n\ge 1$ we have

$$\max_{1\le s<2^n} \left| \sum_{M+1}^{M+s} X_k \right| \le \max\left\{ \max_{1\le s<2^{n-1}} \left| \sum_{M+1}^{M+s} X_k \right|, \left| \sum_{M+1}^{M+2^{n-1}} X_k \right| + \max_{1\le s<2^{n-1}} \left| \sum_{M+2^{n-1}+1}^{M+2^{n-1}+s} X_k \right| \right\},$$

whence

$$\begin{aligned} B_n(M)^2 &\le \max\left\{ \frac{1}{4} B_{n-1}(M)^2, \left[\frac{1}{2} \left| \frac{1}{2^{n-1}} \sum_{M+1}^{M+2^{n-1}} X_k \right| + \frac{1}{2} B_{n-1}(M+2^{n-1}) \right]^2 \right\} \\ &\le \frac{1}{4} B_{n-1}(M)^2 + \frac{1}{2} \left| \frac{1}{2^{n-1}} \sum_{M+1}^{M+2^{n-1}} X_k \right|^2 + \frac{1}{2} B_{n-1}(M+2^{n-1})^2. \end{aligned}$$

Taking expectations and then the supremum over M yields

$$A_n \le \tfrac{1}{4} A_{n-1} + \tfrac{1}{2}\Phi_2(2^{n-1}) + \tfrac{1}{2} A_{n-1} = \tfrac{1}{2}\Phi_2(2^{n-1}) + \tfrac{3}{4} A_{n-1}.$$

This establishes (9) for $n=1$ and all M, and also provides an inductive argument giving (9) for all n. □

We can now establish the following theorem, which improves [7], [4], [5], and [6, p. 307] in not requiring $\Phi_2 \downarrow 0$.

THEOREM 10. *Assume* (8) *and that* $\sum_{N\ge 1} \Phi_2(N)/N < \infty$. *Then* $\{X_n\}$ *satisfies the* SLLN.

Proof. We may assume that

$$\Phi_2(N) = \sup_M \left\| \frac{1}{N} \sum_{M+1}^{M+N} X_k \right\|^2.$$

Taking expectations of

$$\left| \sum_{M+1}^{M+2N} X_k \right|^2 \le 2 \left| \sum_{M+1}^{M+N-P} X_k \right|^2 + 2 \left| \sum_{M+N-P+1}^{M+2N} X_k \right|^2$$

yields

$$\left\| \frac{1}{2N} \sum_{M+1}^{M+2N} X_k \right\|^2 \le 2 \left\| \frac{1}{N-P} \sum_{M+1}^{M+N-P} X_k \right\|^2 + 2 \left\| \frac{1}{N+P} \sum_{M+N-P+1}^{M+2N} X_k \right\|^2$$

for $0\le P<N$, whence $\Phi_2(2N) \le 2[\Phi_2(N-P)+\Phi_2(N+P)]$. Lemma 8 therefore applies and gives $\sum_{n\ge 1}\Phi_2(2^n) < \infty$, whence $(1/2^n)\sum_{k\le 2^n} X_k \to 0$ a.s. Furthermore, Lemma 9 implies that

$$\sum_{n\geq 1}\left\|\max_{1\leq s<2^n}\left|\frac{1}{2^n}\sum_{k=2^n+1}^{2^n+s}X_k\right|\right\|^2\leq\frac{1}{2}\sum_{n\geq 1}\sum_{p=1}^{n}\left(\frac{3}{4}\right)^{p-1}\Phi_2(2^{n-p})=2\sum_{r\geq 0}\Phi_2(2^r)<\infty.$$

Hence the SLLN holds. □

The same argument that led to Corollary 4 now gives the following.

COROLLARY 11. *If* (1), (4) *and* (5) *hold, then so does the* SLLN.

In this corollary, if $\Phi_1\downarrow 0$ and (5) fails, then there are counterexamples to the SLLN. Likewise, in Theorem 10, if $\Phi_2\downarrow 0$, $N\Phi_2(N)$ is increasing and

$$\sum_{N\geq 1}\frac{\Phi_2(N)}{N}=\infty,$$

then there are counterexamples to the SLLN. See [6, p. 307] for a proof.

We are indebted to Stanislaw Szarek for the following construction, which shows that Theorem 7 is also best possible when given a uniform bound on the norm of X_n.

PROPOSITION 12. *Let* $1<p\leq\infty$, $0<r\leq q<\infty$, $1/p+r/q=1$, *and* $\Psi(t)$ *be any nonnegative function on* $\mathbf{R}^+$ *such that* $\Psi(t)=o(t^r)$ *as* $t\to 0^+$. *Then there exist random variables* $\{X_n\}$ *such that*

$$\|X_n\|_p\leq 1 \quad \textit{and} \quad \sum_{n\geq 1}\frac{1}{N}\Psi\left(\left\|\frac{1}{N}\sum_{n\leq N}X_n\right\|_q\right)<\infty,$$

yet the SLLN *fails.*

Proof. Our probability space will be Lebesgue measure on $\mathbf{R}/\mathbf{Z}$. If $\Psi'(t)=\sup_{s\leq t}\Psi(s)$, then also $\Psi'(t)=o(t^r)$. Thus, there exist $\epsilon_k'\in\,]0,2^{-1/(q-r)}[$ such that $\sum_{k\geq 1}(\epsilon_k')^q=\infty$ and $\sum_{k\geq 1}(\epsilon_k')^{q-r}\Psi'(\epsilon_k')<\infty$. Let $\epsilon_k=\epsilon_k'/4$ and choose $N_k\geq 2$ so that $\epsilon_k^{q-r}N_k\geq 2$ and $N_{k+1}-M_{k+1}+1>N_k+M_k$, where M_k is the smallest integer $\geq k(k+1)^{-1}\epsilon_k^{q-r}N_k$. Denote $B_k=[N_k-M_k+1,N_k+M_k]$. If $n\notin\bigcup_k B_k$, then set $X_n=0$. Otherwise, if $n\in B_k$, set

$$X_n=\begin{cases}\epsilon_k^{r-q}\mathbf{1}_{I_k} & \text{if } n\leq N_k,\\ -\epsilon_k^{r-q}\mathbf{1}_{I_k} & \text{if } n>N_k,\end{cases}$$

where $I_k=[\sum_{j<k}\epsilon_k^q,\sum_{j\leq k}\epsilon_k^q]$. We have $\|X_n\|_p=1$ if $n\in\bigcup_k B_k$, while $\|X_n\|_p=0$ otherwise. In addition,

$$\begin{aligned}\sum_N\frac{1}{N}\Psi\left(\left\|\frac{1}{N}\sum_{n=1}^{N}X_n\right\|_q\right)&=\sum_k\sum_{N\in B_k}\frac{1}{N}\Psi\left(\left\|\frac{1}{N}\sum_{n=1}^{N}X_n\right\|_q\right)\\&\leq\sum_k\frac{2M_k}{N_k-M_k+1}\Psi'\left(\left\|\frac{1}{N_k-M_k+1}\sum_{N_k-M_k+1}^{N_k}X_n\right\|_q\right)\\&\leq\sum_k 8\epsilon_k^{q-r}\Psi'(4\epsilon_k)<\infty.\end{aligned}$$

Thus, the conditions on $\{X_n\}$ are satisfied, yet

$$\overline{\lim}\,\frac{1}{N}\sum_1^N X_n=1 \quad\text{and}\quad \underline{\lim}\,\frac{1}{N}\sum_1^N X_n=0$$

everywhere. □

ACKNOWLEDGMENT. I am grateful to the Université de Paris-Sud (Orsay) for their hospitality while I performed this research.

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