# SMALL DIFFERENCES BETWEEN PRIME NUMBERS

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**1. Introduction.** Let $p_n$ denote the $n$th prime number, and let

$$E_r = \liminf_{n\to\infty} \frac{p_{n+r}-p_n}{\log p_n}. \tag{1.1}$$

The prime number theorem implies $E_r \le r$. Improving on earlier results of Erdös and of Rankin and Ricci, Bombieri and Davenport [2] showed that

$$E_r \le r - \tfrac{1}{2}. \tag{1.2}$$

They considered the integral

$$\int_0^1 |S(\alpha)|^2 T(\alpha)\, d\alpha, \tag{1.3}$$

where

$$S(\alpha) = \sum_{p\le N} \log p e(p\alpha), \qquad T(\alpha) = \sum_{n=-k}^{k} t(n)e(2n\alpha), \tag{1.4}$$

the $t(n)$ are linear functions, and evaluated it by the circle method of Hardy-Littlewood. They use the Bombieri–Vinogradoff mean-value theorem on the distribution of primes in arithmetic progressions.

Later, Huxley ([9], [10]) replaced the circle method by the "large sieve" and chose nonlinear weights $t(n)$, thereby following work of Pilt'ai. Huxley's result in [11] is

$$E_r \le \frac{2r-1}{16r}\left\{4r + (4r-1)\frac{\theta_r}{\sin\theta_r}\right\}, \tag{1.5}$$

where $\theta_r$ is the unique solution of $\theta_r + \sin\theta_r = \pi/4r$.

In particular, he obtained

$$E_1 \le 0.4425\ldots, \qquad E_2 \le 1.4105\ldots. \tag{1.6}$$

His latest result [12] is obtained by the application of a theorem of Fouvry and Iwaniec [4], a modification of Bombieri's theorem. He obtains

$$E_1 \le 0.4394\ldots.$$

The purpose of this paper is to combine the methods leading to (1.5) and (1.6) with a method developed by the author in [14]. There he considered the matrix

$$\mathcal{M} = (a_{rs}), \tag{1.7}$$

where

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$$a_{rs}=s+rP(z), \quad 1\le s\le U,$$

$$P(z)^{D-1}<r\le 2P(z)^{D-1}, \quad P(z)=\prod_{p<z} p.$$

This matrix is a union of arithmetic progressions. If $U$ is chosen appropriately, the *density* of primes in $\mathfrak{M}$ is *larger by a factor of* $e^{\gamma}$ than it is on average. It is an immediate corollary of [14] that

(1.8) $$E_r\le e^{-\gamma}r.$$

In this paper we apply the method of exponential sums to this situation. We take the range of summation over all primes from $\mathfrak{M}$ such that (1.4) is replaced by

(1.9) $$S(\alpha)=\sum_{p\in\mathfrak{M}} \log p e(p\alpha).$$

To deal with the distribution of the primes $p\in\mathfrak{M}$ on residue classes we need a new version of the Bombieri–Vinogradoff theorem in which all the moduli are multiples of one fixed modulus (Lemma 6).

Ultimately we will obtain an improvement on (1.5) and (1.6) by a factor $e^{-\gamma}$.

THEOREM. *Let $\theta_r$ be the unique solution of*

(1.10) $$\theta_r+\sin\theta_r=\pi/4r.$$

*Then*

(1.11) $$E_r\le e^{-\gamma}\frac{2r-1}{16r}\left\{4r+(4r-1)\frac{\theta_r}{\sin\theta_r}\right\}.$$

*In particular,*

(1.12) $$E_1\le 0.248\ldots, \qquad E_2\le 0.79\ldots .$$

The estimate for $E_2$ also settles affirmatively a question of Erdös, who had asked if

$$\liminf_{n\to\infty}\frac{\max(p_{n+1}-p_n,\, p_{n+2}-p_{n+1})}{\log p_n}<1.$$

There is one notable deficiency in our results. It is evident from the work of Bombieri–Davenport–Huxley that $p_{n+r}-p_n<\mu\log p_n$ for at least $c(\mu)N$ values $n\le N$, where $c(\mu)>0$, whenever $\mu$ is larger than the established upper bound for $E_r$. Thus the small values of $p_{n+r}-p_n$ actually occur in *a positive proportion* of all cases. We cannot get such a result. Our small gaps are very rare since the matrix $\mathfrak{M}$ contains only a small proportion of integers. Thus the method of Bombieri–Davenport–Huxley still gives the best results if one asks for "essential infima."

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**2. Definitions and notations.** By $z$ we denote a positive real number tending to infinity through a sequence to be specified later.

We set

$$P(z)=\prod_{p<z} p, \tag{2.1}$$

$$U=U(z)=z(\log z)^3, \tag{2.2}$$

$$D=[(\log z)^2], \tag{2.3}$$

$$l=\log(P(z)^D). \tag{2.4}$$

For technical reasons the set $\mathfrak{M}$ is defined slightly differently from (1.7):

$$\mathfrak{M}=\{n: P(z)^D<n\le 2P(z)^D,\ n\equiv s \bmod P(z) \text{ for some } s: U<s\le 2U\}. \tag{2.5}$$

The letter $p$ will always denote prime numbers. The constants $c_1, c_2, \ldots$ will always be positive, depending at most on $r$ in (1.1). Also, the constants implicit in $\ll$, $-$, and $O$-symbols will depend at most on $r$, if not indicated otherwise.

For each row

$$R_k=\{p=kP(z)+s: U<s\le 2U\} \tag{2.6}$$

we enumerate the primes $p\in R_k$ in natural order: $p_1^{(k)}, p_2^{(k)}, \ldots, p_{m(k)}^{(k)}$. Then we define the classes $C^{(i)}$ by

$$C^{(i)}=\{p_s^{(k)}\in\mathfrak{M}: s\equiv i \bmod r,\ P(z)^{D-1}<k<2P(z)^{D-1}\}. \tag{2.7}$$

We introduce a parameter $k$ satisfying

$$c_1 l<k<c_2 l \tag{2.8}$$

and the following exponential sums:

$$S^{(i)}(\alpha)=\sum_{p\in C^{(i)}} \log p\, e(p\alpha), \tag{2.9}$$

$$U(\alpha)=\sum_{n=-k}^{k} u(n)e(2n\alpha), \quad \text{and} \tag{2.10}$$

$$T(\alpha)=|U(\alpha)|^2. \tag{2.11}$$

From (2.10) and (2.11) it follows that

$$T(\alpha)=\sum_{n=-2k}^{2k} t(n)e(2n\alpha), \quad T(\alpha)\ge 0, \quad \text{for all } \alpha. \tag{2.12}$$

We also assume the real numbers $u(n)$ have been chosen such that

$$c_3 k\le |t(0)|\le c_4 k, \qquad |t(n+1)-t(n)|\le c_5. \tag{2.13}$$

We introduce

$$Z_r(2n)=\sum_{i=1}^{r}\sum_{\substack{p,p'\in C^{(i)}\\ p'-p=2n}} \log p\log p'. \tag{2.14}$$

## 3. Large sieve estimate.

LEMMA 1.

$$\sum_{i=1}^{r}\int_0^1 |S^{(i)}(\alpha)|^2 T(\alpha)\,d\alpha = t(0)Z_r(0)+2\sum_{n=1}^{2k} t(n)Z_r(2n). \tag{3.1}$$

*Proof.* This follows directly from the orthonormality of the functions $\{e(k\alpha), k\in Z\}$ and from (2.12) and (2.13). □

DEFINITION. For a fixed but arbitrarily small $\eta>0$ we set

$$Y=P(z)^{D(1/2-\eta)}. \tag{3.2}$$

LEMMA 2.

$$\sum_{i=1}^{r}\int_0^1 |S^{(i)}(\alpha)|^2 T(\alpha)\,d\alpha\cdot(P(z)^{D/2}+2Y)^2 \geq \sum_{q\leq Y}\sum_{\substack{a=1\\(a,q)=1}}^{q}\sum_{i=1}^{r}\left|S^{(i)}\left(\frac{a}{q}\right)\right|^2 T\left(\frac{a}{q}\right). \tag{3.3}$$

*Proof.* A special case of the large sieve [3] implies that

$$\sum_{q\leq Y}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\sum_{n=N_1+1}^{N_1+H} a_n e\left(\frac{na}{q}\right)\right|^2 \leq (H^{1/2}+Y)^2\int_0^1\left|\sum_{n=N_1+1}^{N_1+H} a_n e(n\alpha)\right|^2 d\alpha.$$

We apply this inequality with

$$\sum_{n=N_1+1}^{N_1+H} a_n e(n\alpha)=S^{(i)}(\alpha)U(\alpha),\quad N_1=P(z)^D,\ H=P(z)^D+2k$$

and sum over $i$. □

The next lemma already appears in Huxley [9].

LEMMA 3. *Let $a,q$ be a pair of relatively prime integers, $l$ an integer. Put*

$$q'=(q,P(z)),\qquad q''=q/q'. \tag{3.4}$$

*Define $\omega_{a,q}=\omega_{a,q}(l)$ as follows:*

$$\omega_{a,q}=0 \quad \text{if } (q',q'')>1 \tag{3.5}$$

*and*

$$\omega_{a,q}=e\left(\frac{ayq'+al}{q}\right)\quad \text{if } (q',q'')=1, \tag{3.6}$$

*where $y$ denotes the solution of the congruence $yq'\equiv -l \pmod{q''}$. Then*

$$\sum_{\substack{(m,q)=1\\ m\equiv l \pmod{q'}}} e\left(\frac{am}{q}\right)=\mu(q'')\omega_{a,q}(l). \tag{3.7}$$

DEFINITION. Let

(3.8)
$$\mathcal{Q}=\Big\{q: q=q'q'' \text{ with } q'|P(z);\ (q'',P(z))=1;\ \mu^2(q'')=1;\ q'\geq U,\ q''\leq \frac{Y}{P(z)}\Big\}.$$

For all $q\in\mathcal{Q}$ and all pairs $m,s$ such that $m\equiv s \bmod q'$, we define

$$\rho(n;q,m,s,i)=\begin{cases}\log n-\dfrac{1}{r\varphi(P(z))\varphi(q'')} & \text{if } n\in C^{(i)},\ n\equiv s(\bmod P(z)),\ n\equiv m(\bmod q)\\ -\dfrac{1}{r\varphi(P(z))\varphi(q'')} & \text{for all other } n\text{: } P(z)^D<n\leq 2P(z)^D\end{cases}$$

We now restrict the sum on the RHS of (3.3) to $q\in\mathcal{Q}$. The subsequent calculations will make it plausible that the contribution from the other $q$-values is negligible.

We have

(3.10)
$$S^{(i)}\Big(\frac{a}{q}\Big)=\sum_{\substack{U<s\leq 2U\\ s\text{ prime}}}\ \sum_{\substack{m=1,\ (m,q)=1\\ m\equiv s(\bmod q')}}^{q} e\Big(m\frac{a}{q}\Big)\sum_{P(z)^D<n\leq 2P(z)^D}\cdot\Big(\frac{1}{r\varphi(P(z))\varphi(q'')}+\rho(n;q,m,s,i)\Big).$$

The following notations are borrowed from Huxley [9] and adapted to our problem:

(3.11)
$$S_{a,q}^{(i)}=\sum_{\substack{U<s\leq 2U\\ s\text{ prime}}}\ \sum_{\substack{(m,q)=1\\ m\equiv s \bmod q'}} e\Big(m\frac{a}{q}\Big)\sum_{P(z)^D<n\leq 2P(z)^D}\rho(n;q,m,s,i),$$

(3.12)
$$S_{a,q}=\sum_{i=1}^{r} S_{a,q}^{(i)},$$

(3.13)
$$A^{(i)}=\sum_{q\in\mathcal{Q}}\frac{1}{\varphi(q'')}\sum_{\substack{U<s\leq 2U\\ s\text{ prime}}}\ \sum_{\substack{a=1\\ (a,q)=1}}^{q}\overline{\omega_{a,q}(s)}S_{a,q}^{(i)}T\Big(\frac{a}{q}\Big),$$

(3.14)
$$A=\sum_{i=1}^{r}A^{(i)}.$$

The singular series $\mathfrak{S}$ is defined by

(3.15)
$$\mathfrak{S}=\sum_{q\in\mathcal{Q}}\frac{1}{\varphi(q'')^2}\sum_{\substack{U<s_1,s_2\leq 2U\\ s_i\text{ prime};\ i=1,2}}\ \sum_{\substack{a=1\\ (a,q)=1}}^{q}\cdot\overline{\omega_{a,q}(s_1)}\,\omega_{a,q}(s_2)T\Big(\frac{a}{q}\Big).$$

Then we have the following.

LEMMA 4.

$$(3.16)\quad \sum_{q\le Y}\ \sum_{\substack{a=1\\(a,q)=1}}^{q}\ \sum_{i=1}^{r}\left|S^{(i)}\left(\frac{a}{q}\right)\right|^2 T\left(\frac{a}{q}\right)\ge \frac{P(z)^{2D}}{r\varphi(P(z))^2}\mathfrak{S}+2\frac{P(z)^D}{r\varphi(P(z))}\operatorname{Re}A.$$

*Proof.* By (3.10) and (3.11) we have

$$\left|S^{(i)}\left(\frac{a}{q}\right)\right|^2=\frac{P(z)^{2D}}{r^2\varphi(P(z))^2\varphi(q'')^2}\sum_{\substack{U<s_1,s_2\le 2U\\ s_i \text{ prime};\ i=1,2}}\ \sum_{\substack{(m_i,q)=1\\ m_i\equiv s_i \bmod q';\ i=1,2}} e\left(\frac{(m_2-m_1)a}{q}\right)$$
$$+2\frac{P(z)^D}{r\varphi(P(z))\varphi(q'')}\operatorname{Re}\left(\sum_{\substack{U<s_1\le 2U\\ s_1 \text{ prime}}}\ \sum_{\substack{(m,q)=1\\ m\equiv s_1 \bmod q}} e\left(-m\frac{a}{q}\right)S_{a,q}^{(i)}\right)$$
$$+|S_{a,q}^{(i)}|^2.$$

We omit the term $|S_{a,q}^{(i)}|^2$. The result now follows by summation over $q, a, r$ and by Lemma 3 and definitions (3.12)-(3.15). □

## 4. The error term.

DEFINITIONS. For each triple $(s, m, q'')$ of positive integers with $(s, P(z)) = (m, q'') = (q'', P(z)) = 1$ we define

$$(4.1)\quad E_{s,m,q''}=\sum_{\substack{P(z)^D<p\le 2P(z)^D\\ p\equiv s \bmod P(z),\ p\equiv m \bmod q''}}\log p-\frac{P(z)^D}{\varphi(P(z))\varphi(q'')}$$

and

$$(4.2)\quad E_{q''}=\max_{(s,P(z))=1}\ \max_{(m,q'')=1}|E_{s,m,q''}|.$$

LEMMA 5. *We have*

$$(4.3)\quad |A|\ll |t(0)|kU^4\sum_{q\in\mathcal{Q}}\frac{q}{\varphi(q'')}d(q'')E_{q''}.$$

*Proof.* From (3.13) and (3.14) we have

$$A=\sum_{q\in\mathcal{Q}}\frac{1}{\varphi(q'')}\sum_{\substack{U<s_1\le 2U\\ s_1 \text{ prime}}}\ \sum_{\substack{a=1\\(a,q)=1}}^{q}\overline{\omega_{a,q}(s_1)}\sum_{\substack{U<s_2\le 2U\\ s_2 \text{ prime}}}\ \sum_{\substack{(m,q)=1\\ m\equiv s_2 \bmod q'}} e\left(m\frac{a}{q}\right)$$
$$\otimes\sum_{P(z)^D<n\le 2P(z)^D}\rho(n;q,m,s_2)T\left(\frac{a}{q}\right).$$

Now $\sum_{P(z)^D<n\le 2P(z)^D}\rho(n;q,m,s_2)=E_{s_2,m,q}$ and we obtain, by definitions (3.6) and (4.2),

$$(4.4)\quad |A|\le\sum_{q\in\mathcal{Q}}\frac{1}{\varphi(q'')}\sum_{\substack{U<s_i\le 2U\\ s_i \text{ prime};\ i=1,2}}\ \sum_{\substack{(m,q)=1\\ m\equiv s_2 \bmod q'}}\ \sum_{n'=-2k}^{2k}|t(n)|$$
$$\otimes\left|\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{a(y(s_1)q'+s_1-m+2n')}{q}\right)\right|E_{q''}.$$

The sum

$$c_q(y(s_1)q'+s_1-m+2n') = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{a(y(s_1)q'+s_1-m+2n')}{q}\right)$$

is Ramanujan's sum. The formula [7, §16.6]

$$c_q(u) = \sum_{d\mid(q,u)} d\mu\left(\frac{q}{d}\right) \tag{4.5}$$

gives

$$\sum_{\substack{m \bmod q \\ m\equiv s_2 \bmod q'}} |c_q(y(s_1)q'+s_1-m+2n')| \le \sum_{\substack{d'\mid 2n'+s_1-s_2 \\ d'\mid q'}} d' \sum_{\substack{m \bmod q \\ m\equiv s_2 \bmod q'}} |c_{q''}(2n'-m)|$$

$$\ll U^2 q' \sum_{\substack{m \bmod q \\ m\equiv s_2 \bmod q'}} |c_{q''}(2n'-m)|.$$

For the estimate of the last sum we follow Huxley [9, p. 374]. For fixed $n'$ and each $m$ we have

$$2n'-m \equiv u \bmod q'', \quad \text{where } 1\le u\le q'',$$

such that distinct $m$ belong to distinct $u$. Thus

$$\sum_{\substack{m \bmod q \\ m\equiv s_2 \bmod q'}} |c_{q''}(2n'-m)| \le \sum_{u=1}^{q''} |c_{q''}(u)| \le \sum_{u=1}^{q''} \sum_{d\mid(q'',u)} d < q''d(q'').$$

Therefore

$$\sum_{\substack{m \bmod q \\ m\equiv s_2 \bmod q'}} |c_q(y(s_1)q'+s_1-m+2n')| \le U^2 q'q''d(q'').$$

From (4.4) we obtain

$$|A| \ll \sum_{q\in\mathcal{Q}} \frac{1}{\varphi(q'')} \Bigg( \sum_{\substack{U<s_i\le 2U \\ s_i \text{ prime};\ i=1,2}} 1 \Bigg) \left( \sum_{n'=-2k}^{2k} |t(n)| \right) U^2 q'q''d(q'')E_{q''}$$

and thus Lemma 5. □

The next lemma is a generalization of the Bombieri–Vinogradoff mean-value theorem [1].

LEMMA 6. *Let*

$$\psi(x,q,a) = \sum_{\substack{n\le x \\ n\equiv a \bmod q}} \Lambda(n).$$

*Let* $\delta_1, \delta_2, \delta_3$ *be arbitrarily small positive constants. Let* $Y>1$, *a positive integer*

$$R < \exp\left(\frac{\log Y}{(\log\log Y)^{1+\delta_1}}\right),$$

$Q\ge 1$, *and* $L=\log YQ$. *Assume that*

$$L(s,\chi)\neq 0 \quad \textit{for } \operatorname{Re} s > 1-\frac{\delta_2}{\log(R(|t|+1))} \tag{4.6}$$

*for all* $\chi \bmod M$ *with* $M \leq R^{1+\delta_3}$. *Then*

$$\sum_{\substack{q\leq Q\\(q,R)=1}} \max_{X\leq Y} \max_{(a,qR)=1} \left|\psi(X,qR,a)-\frac{X}{\varphi(qR)}\right| \ll_B \frac{Y}{\varphi(R)}(\log y)^{-B}+Y^{1/2}\frac{R^2}{\varphi(R)}QL^5, \tag{4.7}$$

*where* $B>0$ *is arbitrarily large.*

*Proof.* We will use the theorem of Vaughan [16]: Set

$$T(Y,Q)=\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\max_{X\leq Y}|\psi(X,\chi)|,$$

where $\sum^*$ denotes summation over the primitive characters $\chi \bmod q$.

Suppose that $Q\geq 1$, $Y\geq 2$, and $L=\log YQ$. Then

$$T(Y,Q)\ll L^4(Y+Y^{5/6}Q+Y^{1/2}Q^2). \tag{4.8}$$

We also use Theorem 7 of Gallagher [5]: Let $X/Q\leq h\leq x$ and $\exp(\log^{1/2}x)\leq Q\leq x^b$ with fixed $b>0$, and assume $L(s,\chi)\neq 0$ for $\operatorname{Re} s>1-\delta_4/\log(Q(|t|+1))$ for all $\chi \bmod M$ with $M\leq Q$. Then

$$\sum_{1<k\leq Q}\sum_{\chi}{}^{*}\left|\sum_{x}^{x+h}\chi(p)\log p\right|\ll h\exp\left(-\delta_5\frac{\log x}{\log Q}\right). \tag{4.9}$$

Obviously we can replace $\sum_x^{x+h}\chi(p)\log p$ with $\sum_x^{x+h}\chi(n)\Lambda(n)$. We write

$$\begin{aligned}
U(R,Q,Y)&=\sum_{\substack{q\leq Q\\(q,R)=1}}\max_{X\leq Y}\max_{(a,qR)=1}\left|\psi(X,qR,a)-\frac{X}{\varphi(qR)}\right|\\
&\leq \sum_{\substack{q\leq Q\\(q,R)=1}}\frac{1}{\varphi(qR)}\sum_{\substack{\chi \bmod qR\\ \chi\neq\chi_0}}\max_{x\leq Y}|\psi(X,\chi)|\\
&\quad+O\left(\frac{(\log Q)}{\varphi(R)}Y\exp(-(\log X)^{1/2})\right)\\
&\ll \log Q\sum_{d|R}\sum_{\substack{1<q\leq Q\\(q,R)=1}}\frac{1}{\varphi(qR)}\sum_{\chi \bmod qd}{}^{*}(\max|\psi(X,\chi)|+O(\log QR))\\
&\quad+O\left(\frac{(\log Q)}{\varphi(R)}Y\exp(-(\log X)^{1/2})\right).
\end{aligned}$$

Thus

$$\begin{aligned}
U(R,Q,Y)\ll \log Q\sum_{d|R}\frac{\varphi(d)}{\varphi(R)}\sum_{\substack{1<q\leq Q\\(q,R)=1}}\frac{1}{\varphi(dq)}\\
\otimes \sum_{\chi \bmod dq}{}^{*}\max_{X\leq Y}|\psi(X,\chi)|+O(Q(\log QR)^3)\\
+O\left(\frac{(\log Q)}{\varphi(R)}Y\exp(-(\log x)^{1/2})\right).
\end{aligned}$$

From (4.8) we get, by partial summation,

$$\sum_{\substack{Q_0<q\le Q\\(q,R)=1}} \frac{1}{\varphi(dq)} \sum_{\chi \bmod dq}^{*} \max_{X\le Y} |\psi(X,\chi)| \ll \frac{L^4}{d}\left(\frac{Y}{Q_0}+Y^{5/6}d\log Q+Y^{1/2}d^2Q\right).$$

We apply this estimate with $Q_0=R^{\delta_3}$ and obtain

$$\begin{aligned} U(R,Q,Y) \ll \log Q &\sum_{\substack{q\le R^{\delta_3}\\(q,R)=1}} \frac{1}{\varphi(qR)} \sum_{\substack{\chi \bmod dq\\ \chi\ne\chi_0}}^{*} \max_{X\le Y}|\psi(X,\chi)| \\ &+\frac{L^5}{\varphi(R)}(YR^{-\delta_3/2}+Y^{5/6}R\log Q+Y^{1/2}R^2Q)+QL^3 \\ &+\frac{\log Q}{\varphi(R)}Y\exp(-(\log X)^{1/2}). \end{aligned}$$

Since $d\,|\,R$ and $(q,R)=1$, the numbers of the form $dq$ are distinct. Thus

$$\sum_{d|R}\sum_{\substack{q\le R^{\delta_3}\\(q,R)=1}} \frac{1}{\varphi(qR)} \sum_{\substack{\chi \bmod dq\\ \chi\ne\chi_0}}^{*} \max_{X\le Y}|\psi(X,\chi)| \le \frac{1}{\varphi(R)} \sum_{1<r\le Q_2} \sum_{\chi \bmod r}^{*} \max_{X\le Y}|\psi(X,\chi)|,$$

where $Q_2=\exp((1+\delta_3)(\log Y)(\log\log Y)^{-1-\delta_1})$.

Given $r\le Q_2$ and $\chi \bmod r$, choose $X_0=X_0(\chi)$ so that $X_0\le Y$ and

$$|\psi(X_0,\chi)|=\max_{X\le Y}|\psi(X,\chi)|.$$

Let $K=[Q_2^{10}]$, $Z=Y/K$ and choose $k=[X_0/Z]$. Then $0\le k\le K$ and $|X_0-kZ|<Z$. Thus

$$\begin{aligned} \max_{X\le Y}|\psi(X,\chi)| &= |\psi(kZ,\chi)|+O(Z\log Y) \\ &\le \sum_{l=1}^{k} |\psi(lZ,\chi)-\psi((l-1)Z,\chi)|+O(Z\log Y). \end{aligned}$$

Clearly, if $k\ge 1$ then the term with $l=1$ is also $\ll Z\log Y$. Therefore the sum in question is

$$\ll \frac{1}{\varphi(R)}\left(\sum_{l=2}^{K}\sum_{1<r\le Q_2}\sum_{\chi \bmod r}^{*} |\psi(lZ,\chi)-\psi((l-1)Z,\chi)|+ZQ_2^2\log y\right),$$

and by (4.9) this is

$$\ll \frac{1}{\varphi(R)}\left(KZ\exp\left(-\delta_5\frac{\log Z}{\log Q_2}\right)+ZQ_2^2\log Y\right) \ll \frac{1}{\varphi(R)}Y(\log Y)^{-B}$$

as required. □

We now want to apply Lemma 6 with $R=P(z)$. We call moduli $R$ for which (4.6) is satisfied *admissible moduli*. The definition depends on the constants $\delta_2,\delta_3>0$. If we are only interested in $E_1$ then we can immediately assume that $P(z)$ is admissible for all $z$. This follows from the work of Heath-Brown [8]; if there are infinitely many real primitive $\chi \bmod q$, such that $L(\beta_0, \chi)=0$ for $\beta_0 \geq 1-1/(3\log q)$, then there exist infinitely many prime twins. The author owes this observation to Prof. C. Pomerance. For $r>1$ we need the following.

LEMMA 7. *For given $\delta_3>0$ there is a constant $\delta_2>0$ such that in terms of $\delta_2, \delta_3$ there exist arbitrarily large values of $z$ for which $P(z)$ is admissible.*

*Proof.* The idea already has been used in two papers of the author ([13], [14]). By Page's theorem (see [15, Satz 6.9b]), for sufficiently small values of $2\delta_2$ there is at most one exceptional character $\chi^*$ of modulus $M \leq R^{1+\delta_3}$ and an exceptional zero $\beta$ such that $\beta>1-2\delta_2/\log R$.

For a given $z$, we find a $z \geq z_1$ with admissible $P(z)$ as follows: If $P(z_1)$ in terms of $2\delta_2, \delta_3$ is admissible then set $z=z_1$. Otherwise there is an exceptional character $\chi^*$ of modulus $M \leq P(z_1)^{1+\delta_3}$ with an exceptional real zero $\beta$ such that $\beta> 1-2\delta_2/\log P(z_1)$. Now we take $z \geq z_1$ such that

$$\frac{\delta_2}{\log P(z)}<1-\beta<\frac{2\delta_2}{\log P(z)}.$$

Then by the second inequality $\chi^*$ is still an exceptional character. Thus for all other $\chi \bmod M$ with $M \leq P(z)^{1+\delta_3}$ we have $\beta \leq 1-2\delta_2/\log P(z)$. Thus $P(z)$ is admissible in terms of $\delta_2, \delta_3$. □

We now assume that $\delta_2>0$, $\delta_3>0$ are fixed. *In the sequel we always assume that $z \to \infty$ through a sequence of values for which $P(z)$ is admissible in terms of $\delta_2, \delta_3$.*

LEMMA 8. *We have*

$$|A| \ll_B |t(0)| k U^4 P(z)^D l^{-B} \tag{4.10}$$

*for arbitrarily large $B>0$.*

*Proof.* By Cauchy's inequality we obtain, from Lemma 5,

$$|A| \ll |t(0)| k U^4 \sum_{\substack{q' \geq U \\ q' \mid P(z)}} q' \Bigg( \sum_{\substack{q'' \leq Y/P(z) \\ (q'', P(z))=1}} \frac{q''^2 d(q'')^2}{\varphi(q'')^3} \Bigg)^{1/2} \cdot \Bigg( \sum_{\substack{q'' \leq Y/P(z) \\ (q'', P(z))=1}} \varphi(q'')(E_{q''})^2 \Bigg)^{1/2}.$$

We have the trivial estimate

$$E_{q''} \ll \frac{P(z)^D l}{\varphi(P(z))\varphi(q'')}.$$

Together with Lemma 6, we obtain the desired result. □

## 5. The singular series.

DEFINITION. Set

$$a(n)=\sum_{\substack{U<s_i \leq 2U \\ s_i \text{ prime};\ i=1,2 \\ s_2-s_1=2n}} 1.$$

LEMMA 9. *Assume we have* $-2k \le a < b \le 2k$, $b-a \ge k/\log z$, *and a sequence* $c(n)$, $a < n \le b$, *of real numbers such that* $|c(n+1)-c(n)| \le c_6$. *Then we have*

$$\sum_{a<n\le b} c(n)a(n) = 2\frac{U}{\log^2 U}\left(1+O\left(\frac{1}{\log u}\right)\right)\left(\sum_{a<n\le b} c(n)+O\left(\frac{c_6 k^2}{(\log z)^3}\right)\right). \tag{5.1}$$

*Proof.* We partition the interval $(a, b]$ into subintervals $I_l$ of lengths $|I_l|$ with

$$\frac{1}{2}\frac{b-a}{(\log z)^3} < |I_l| \le \frac{b-a}{(\log z)^3} \quad \text{and} \quad (a,b] = \bigcup_{l=1}^{R} I_l,$$

where $I_l = (\alpha_l, \alpha_{l+1}]$.

We may assume that the $\alpha_l$ are integers after changing $a$ and $b$ into integers, if necessary, and that $a \ge 0$. Then

$$\begin{aligned}\sum_{a<n\le b} c(n)a(n) &= \sum_{l=1}^{R}\left(\sum_{n\in I_l} c(n)a(n)\right) = \sum_{l=1}^{R}\left(c(\alpha_l)+O\left(\frac{c_6 k}{(\log z)^3}\right)\right)\sum_{n\in I_l} a(n)\\ &= \sum_{l=1}^{R}\left(c(\alpha_l)+O\left(\frac{c_6 k}{(\log z)^3}\right)\right)\\ &\quad\cdot\left(\sum_{\substack{U<s_1\le 2U\\ s_1 \text{ prime}}}\ \sum_{\substack{s_1+2\alpha_l<s_2\le \min(s_1+2\alpha_{l+1},2U)\\ s_2 \text{ prime}}} 1\right).\end{aligned}$$

By the prime number theorem it follows that

$$\begin{aligned}\sum_{a<n\le b} c(n)a(n) &= \sum_{l=1}^{R}\left(c(\alpha_l)+O\left(\frac{c_6 k}{(\log z)^3}\right)\right)2\frac{\alpha_{l+1}-\alpha_l}{\log U}\frac{U}{\log U}\left(1+O\left(\frac{1}{\log U}\right)\right)\\ &= 2\frac{U}{\log^2 U}\left(1+O\left(\frac{1}{\log U}\right)\right)\sum_{l=1}^{R}\left(c(\alpha_l)+O\left(\frac{c_6 k}{(\log z)^3}\right)\right)(\alpha_{l+1}-\alpha_l)\\ &= 2\frac{U}{\log^2 U}\left(\sum_{a<n\le b} c(n)+O\left(\frac{c_6 k^2}{(\log z)^3}\right)\right)\left(1+O\left(\frac{1}{\log U}\right)\right),\end{aligned}$$

which is (5.1). □

LEMMA 10. *We have*

$$\mathcal{S} = t(0)\varphi(P(z))\frac{U\log Y}{\log U}\left(1+O\left(\frac{c_7}{\log z}\right)\right)+2\sum_{\substack{n=-2k\\ n\ne 0}}^{2k} t(n)P(z)\frac{U}{\log^2 U}. \tag{5.2}$$

(*Here* $c_7$ *depends on* $c_3, c_4, c_5$ *in* (2.13).)

*Proof.* From (3.15) and (4.5) we have

$$\mathcal{S} = \sum_{q\in\mathcal{Q}}\frac{1}{\varphi(q'')^2}\sum_{\substack{U<s_1,s_2\le 2U\\ s_i \text{ prime};\ i=1,2}}\ \sum_{\substack{a=1\\ (a,q)=1}}^{q}\overline{\omega_{a,q}(s_1)}\,\omega_{a,q}(s_2)T\left(\frac{a}{q}\right)=$$

$$= \sum_{q \in \mathcal{Q}} \frac{1}{\varphi(q'')^2} \sum_{\substack{U < s_1, s_2 \le 2U \\ s_i \text{ prime}; \, i=1,2}} \sum_{n=-2k}^{2k} t(n) c_{q'}(s_1 - s_2 + 2n) c_{q''}(2n).$$

The main contributions come from the terms with

(a) $n=0$ and $s_1=s_2$

or

(b) $s_1-s_2+2n=0$; $q''=1$.

The rest will be treated as error terms.

We obtain

$$\mathcal{S} = t(0)\frac{U}{\log U}\sum_{q\in\mathcal{Q}}\frac{\varphi(q')}{\varphi(q'')}\left(1+O\left(\frac{1}{\log U}\right)\right) \tag{5.3}$$

$$+\sum_{\substack{q\in\mathcal{Q}\\ q''=1}}\varphi(q')\sum_{\substack{n=-2k\\ n\neq 0}}^{2k} t(n)a(n)+O(\Sigma_1)+O(\Sigma_2),$$

where

$$\Sigma_1 = \sum_{q\in\mathcal{Q}}\frac{1}{\varphi(q'')^2}\sum_{\substack{U<s_i\le 2U\\ s_i \text{ prime};\, i=1,2}}\sum_{\substack{n=-2k\\ s_1-s_2+2n\neq 0}}^{2k} |t(n)|\,|c_{q'}(s_1-s_2+2n)|\,|c_{q''}(2n)|$$

and

$$\Sigma_2 = \sum_{\substack{q\in\mathcal{Q}\\ q''>1}}\frac{1}{\varphi(q'')^2}\sum_{\substack{U<s_i\le 2U\\ s_i \text{ prime};\, i=1,2}}\sum_{\substack{n=-2k\\ n\neq 0,\, s_1-s_2+2n=0}}^{2k} |t(n)|\,|c_{q'}(s_1-s_2+2n)|\,|c_{q''}(2n)|.$$

We write $P(z)=q's$ and obtain

$$\sum_{q\in\mathcal{Q}}\frac{\varphi(q')}{\varphi(q'')} = \sum_{\substack{q\in\mathcal{Q}\\ q'>U}}\frac{\varphi(q')}{\varphi(q'')} + O\left(\frac{1}{\varphi(P(z))}\sum_{\substack{q'\mid P(z)\\ q'\le U}} q' \sum_{q''\le Y}\frac{1}{\varphi(y'')}\right)$$

$$= \varphi(P(z)) \sum_{\substack{s\mid P(z),\, s\le P(z)/U\\ (q'',P(z))=1,\, q''\le Y/P(z)}} \frac{1}{\varphi(sq'')} + O\left(\frac{U^2\log Y}{\varphi(P(z))}\right)$$

$$= \varphi(P(z))\left(\sum_{\substack{n\le Y\\ \mu^2(n)=1}}\frac{1}{\varphi(n)} + O\left(\sum_{Y/P(z)<n\le Y}\frac{1}{\varphi(n)}\right)\right) + O\left(\frac{U^2\log Y}{\varphi(P(z))}\right)$$

and thus (see [17])

$$\sum_{q\in\mathcal{Q}}\frac{\varphi(q')}{\varphi(q'')} = \varphi(P(z))(\log Y + O(\log P(z))). \tag{5.4}$$

The second sum is handled by Lemma 9.

Because of (2.13) we get

$$\sum_{\substack{q\in\mathcal{Q}\\ q''=1}}\varphi(q')\sum_{\substack{n=-2k\\ n\neq 0}}^{2k} t(n)a(n) \tag{5.5}$$

$$= 2\frac{UP(z)}{\log^2 U}\left(1+O\left(\frac{1}{\log U}\right)\right)\left(\sum_{\substack{n=-2k\\ n\neq 0}}^{2k} t(n) + O\left(\frac{c_7 k^2}{\log^3 z}\right)\right).$$

In the estimate for $\Sigma_1$ we use $|c_{q''}(2n)| \le \varphi(q'')$ and observe that if $s_1 - s_2 + 2n \ne 0$ then $c_{q'}(s_1 - s_2 + 2n) \ll U^2$. Thus

$$\Sigma_1 \ll \left( \sum_{n=-2k}^{2k} |t(n)| \right) U^4 \sum_{q \in \mathcal{Q}} \frac{1}{\varphi(q'')} \tag{5.6}$$

$$\ll \left( \sum_{n=-2k}^{2k} |t(n)| \right) U^4 (\log Y) 3^{z/\log z}.$$

In the estimate for $\Sigma_2$ we use $|c_{q'}(s_1 - s_2 + 2n)| \le \varphi(q')$. If $(q'', 2n) = 1$, then $c_{q''}(2n) = 1$. If $(q'', 2n) > 1$, then $p = (q'', 2n)$ is prime, $z < p \le 4U$, and $|c_{q''}(2n)| \le p$. We get

$$\Sigma_2 \ll \sum_{q'' > z} \frac{1}{\varphi(q'')^2} \sum_{q' | P(z)} \varphi(q') \sum_{\substack{n=-2k \\ n \ne 0}}^{2k} |t(n)| a(n)$$

$$+ \frac{1}{z} \left( 1 + \sum_{s \ge z} \frac{1}{s^2} \right) \sum_{q' | P(z)} \varphi(q') \sum_{\substack{n=-2k \\ n \ne 0}}^{2k} |t(n)| a(n)$$

and thus

$$\Sigma_2 \ll \frac{U}{z \log^2 U} P(z) \sum_{\substack{n=-2k \\ n \ne 0}}^{2k} |t(n)|. \tag{5.7}$$

(5.3)-(5.7) now yield the proof of Lemma 10. □

**6. Conclusion.** We now summarize the results of the last sections.

LEMMA 11. *Let $U(\alpha), T(\alpha)$ be as in* (2.10)-(2.13). *Then for arbitrarily small $\epsilon > 0$ and $z \ge z_0(\epsilon)$ we have*

$$r \sum_{n=1}^{2k} t(n) Z_r(2n) > 2e^{2\gamma} P(z)^{D-1} U \sum_{n=1}^{2k} t(n) \tag{6.1}$$

$$- \frac{1}{2} \left( r - \frac{1}{2} + \epsilon \right) e^{\gamma} t(0) P(z)^{D-1} U l.$$

*Proof.* By Lemma 6 we have

$$Z_r(0) = \frac{P(z)^D}{\varphi(P(z))} l \frac{U}{\log U} \left( 1 + O\left( \frac{1}{\log U} \right) \right).$$

From Lemmas 1 and 2 we obtain

$$2 \sum_{n=1}^{2k} t(n) Z_r(2n) \ge (P(z)^{D/2} + 2Y)^{-2} \sum_{q \le Y} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{i=1}^{r} \left| S^{(i)}\left( \frac{a}{q} \right) \right|^2 T\left( \frac{a}{q} \right)$$

$$- t(0) \frac{P(z)^D}{\varphi(P(z))} l \frac{U}{\log U} \left( 1 + O\left( \frac{1}{\log U} \right) \right).$$

Lemmas 4, 8, and 10 give

$$2\sum_{n=1}^{2k} t(n)Z_r(2n)$$
$$\geq (P(z)^{D/2}+2Y)^{-2}$$
$$\times\left[t(0)\varphi(P(z))\frac{U\log Y}{\log U}\left(1+O\left(\frac{c_7}{\log z}\right)\right)+2\sum_{\substack{n=-2k\\ n\neq 0}}^{2k} t(n)P(z)\frac{U}{\log^2 U}\right]$$
$$\times\frac{P(z)^{2D}}{r\varphi(P(z))^2}-t(0)\frac{P(z)^D}{\varphi(P(z))}l\frac{U}{\log U}\left(1+O\left(\frac{1}{\log U}\right)\right).$$

Now we observe that

$$\sum_{\substack{n=-2k\\ n\neq 0}}^{2k} t(n)=2\sum_{n=1}^{2k} t(n) \quad\text{and}\quad \varphi(P(z))=P(z)\frac{e^{-\gamma}}{\log z}\left(1+O\left(\frac{1}{\log z}\right)\right).$$

We recall that, by (3.2), $Y=P(z)^{D(1/2-\eta)}$. By choosing $\eta$ sufficiently small we obtain (6.1). □

Lemma 11 is the analogue of Lemma 1 of Huxley [11], who considers the prime numbers of $[1,N]$ instead of those of $\mathfrak{M}$. The difference is the occurrence of the factor $e^{\gamma}$ in our paper. This will ultimately lead to an improvement on Huxley's result by a factor $e^{-\gamma}$.

The remainder of the paper is now analogous to Huxley's treatment. We need an upper bound for $Z_r(2n)$ that on average corresponds to the one given in Lemma 2 of [11]. This is accomplished by a linear upper bound sieve. We borrow notations and results from [6].

Let there be given a finite set $\mathcal{A}$ of integers; a subset $\mathcal{P}$ of the set of all primes; a real number $X>1$; and a real number $z\geq 2$. Then we define

$$S(\mathcal{A},\mathcal{P},z)=\operatorname{card}\{a\in\mathcal{A}: p\nmid a \text{ for all } p\in\mathcal{P},\ p<z\}.$$

Let $\omega$ be a multiplicative function, defined for all square free positive integers $d$, such that $\omega(p)=0$ if $p\notin\mathcal{P}$. Then we define

$$\mathcal{A}_d=\{a\in\mathcal{A}: a\equiv 0 \bmod d\},$$
$$R_d=|\mathcal{A}_d|-\frac{\omega(d)}{d}X,$$
$$W(z)=\prod_{p<z}\left(1-\frac{\omega(p)}{p}\right).$$

The functions $F$ and $f$ are defined by the system of differential-difference equations:

$$F(u)=\frac{2e^{\gamma}}{u},\ f(u)=0 \quad\text{for}\quad 0<u\leq 2;$$
$$(uF(u))'=f(u-1),\ (uf(u))'=F(u-1) \quad\text{for}\quad u\geq 2.$$

We assume that $\omega$ satisfies the conditions (the $A$'s are positive constants):

$$(\Omega_1): 0 \le \frac{\omega(p)}{p} \le 1 - \frac{1}{A_1};$$

$$(\Omega_2(1,L)): -L \le \sum_{w \le p < z} \frac{\omega(p)\log p}{p} - \log\frac{z}{w} \le A_2 \quad \text{if } 2 \le w \le z,$$

and that the $R_d$ satisfy

$$(R(1,\alpha)): \sum_{\substack{d < X^{\alpha}/(\log X)^A \\ (d,\bar{\mathcal{P}})=1}} \mu^2(d) 3^{\nu(d)} |R_d| \le A_5 \frac{X}{\log^2 X}, \quad X \ge 2.$$

The following lemma is the first part of Theorem 8.4 of [6].

LEMMA 12. *Let $\mathcal{A}$ be given with $\omega$ and $R_d$ satisfying* $(\Omega_1)$, $(\Omega_2(1,L))$, $(R(1,\alpha))$. *Then, for $z \le X$,*

$$S(\mathcal{A},\mathcal{P},z) \le XW(z)\left\{F\left(\alpha\frac{\log X}{\log z}\right) + B\frac{L}{(\log X)^{1/14}}\right\},$$

*where $B$ depends only on the $A_i$.*

We now use Lemma 12 to prove the following.

LEMMA 13. *Assume that the sequence $c(n)$, $-2k \le a < n \le b \le 2k$, is nonnegative and that we have $b-a \ge k/\log z$ and $|c(n+1)-c(n)| \le c_8$. Then*

$$\sum_{a<n\le b} c(n) Z_r(2n) < (8+\epsilon) e^{2\gamma} P(z)^{D-1} U\left(\sum_{a<n\le b} c(n) + O\left(\frac{c_8 k^2}{(\log z)^3}\right)\right), \tag{6.2}$$

*with $\epsilon > 0$ arbitrarily small and $z \ge z_0(\epsilon)$.*

*Proof.* Let $s_1, s_2$ be prime numbers satisfying $U < s_1 < s_2 \le 2U$. We define

$$Z(s_1,s_2) = \mathrm{card}\{k : p_i = kP(z) + s_i \in \mathfrak{M}, \text{ prime}, i = 1,2\}.$$

Define the set $\mathcal{A}$ by

$$\mathcal{A} = \mathcal{A}(s_1,s_2) = \{n = kP(z) + s_2 : p = kP(z) + s_1 \in \mathfrak{M}, \text{ prime}\}.$$

($\mathcal{P}$ is the set of all prime numbers.) Obviously $Z(s_1,s_2) \le S(\mathcal{A},\mathcal{P},Y)$.

We define $\omega$:

$$\omega(p) = \begin{cases} 0 & \text{if } p < z, \\ p/(p-1) & \text{if } p \ge z, \end{cases}$$

and set

$$X = \frac{liP(z)^D}{\varphi(P(z))}.$$

Then $(\Omega_1)$ is satisfied with $A_1 = 2$ and $(\Omega_2(1,L))$ with $L = \log z$. We have

$$\mathrm{card}\,\mathcal{A}_d = \mathrm{card}\{p \in \mathfrak{M} : p \equiv s_1 \bmod P(z),\ p \equiv s_1 - s_2 \bmod d\}.$$

Lemma 6 shows that $(R(1,\alpha))$ is satisfied with $\alpha = \frac{1}{2} - \eta$. From Lemma 12 we have

$$Z(s_1,s_2) \le \frac{liP(z)^D}{\varphi(P(z))} \prod_{z \le p < Y}\left(1 - \frac{1}{p-1}\right) e^{\gamma}(2+\epsilon).$$

Thus

$$\sum_{a<n\le b} c(n)Z_r(2n) \le \frac{P(z)^D l}{\varphi(P(z))} \prod_{z\le p<Y}\left(1-\frac{1}{p-1}\right)e^{\gamma}(2+\epsilon)\sum_{a<n\le b} c(n)a(n).$$

Lemma 13 now follows from Lemma 9. □

In [11] Huxley does not apply Lemma 11 directly. The quality of the result is enhanced by comparison of the inequality for two exponential sums.

LEMMA 14. *Assume that* $2s \ge 2k \ge j$ *and also that* $s$ *satisfies* (2.8): $c_1 l < k < c_2 l$. *The following exponential sums are given:*

$$U(x)=\sum_{-k}^{k} u(n)e(2nx), \qquad T(x)=\sum_{-2k}^{2k} t(n)e(2nx)=|U(x)|^2,$$
$$W(x)=\sum_{-s}^{s} w(n)e(2nx), \qquad V(x)=\sum_{-2s}^{2s} v(n)e(2nx)=|W(x)|^2.$$

*Let* $t(n)$ *and* $v(n)$ *satisfy conditions* (2.10)-(2.13). *Moreover, assume*

$$w(-n)=-w(n); \quad v(n)\le 0 \quad \text{for } n\ge h.$$

*Let* $j$ *be an integer for which*

$$r\sum_{h}^{j}(-v(n))(8+\epsilon)e^{\gamma} \ge \left\{\frac{1}{2}\left(r-\frac{1}{2}\right)+2\epsilon\right\}v(0)l. \tag{6.3}$$

*Assume*

$$\sum_{h}^{j} t(n) > c_9 l^2, \quad -c_{10} \le \frac{t(j)}{v(j)} \le 0;$$
$$-\frac{v(n)}{t(n)}\begin{cases} \le -v(j)/t(j) & \text{for } h\le n\le j, \\ \ge -v(j)/t(j) & \text{for } j\le n\le 2k\le 2s. \end{cases} \tag{6.4}$$

*Then* $Z_r(2n)=0$, *for all* $n$ *with* $0<n<h$, *implies*

$$8re^{\gamma}\sum_{h}^{j} t(n) > T(0)e^{\gamma} - \left\{\frac{1}{2}\left(r-\frac{1}{2}\right)+c_{11}\epsilon\right\}t(0)l, \tag{6.5}$$

*where* $c_{11}>0$ *depends only on the other* $c_i$.

*Proof.* If $Z_r(2n)=0$ for $n<h$ then, by Lemma 11,

$$r\sum_{n=h}^{2s} v(n)Z_r(2n) > 2e^{2\gamma}P(z)^{D-1}U\sum_{n=1}^{2s} v(n)$$
$$-\frac{1}{2}\left(r-\frac{1}{2}+\epsilon\right)e^{\gamma}v(0)P(z)^{D-1}Ul.$$

Now $w(-n)=-w(n)$ implies $W(0)=0$ and thus $V(0)=0$. We have $2\sum_{n=1}^{2s} v(n) = -v(0)$, which yields

$$r\sum_{n=h}^{2s}(-v(n))Z_r(2n) < \frac{1}{2}\left(r-\frac{1}{2}+2\epsilon\right)e^{\gamma}v(0)P(z)^{D-1}Ul. \tag{6.6}$$

Now

$$r\sum_h^{2k} t(n)Z_r(2n) = r\left(-\frac{t(j)}{v(j)}\right)\sum_h^{2k}(-v(n))Z_r(2n) + r\sum_h^{2k}\left[t(n)-\left(\frac{t(j)}{v(j)}\right)v(n)\right]Z_r(2n).$$

By (6.6) and Lemma 13 we get

$$\begin{aligned} r\sum_h^{2k} t(n)Z_r(2n) &\le r\left(-\frac{t(j)}{v(j)}\right)\left\{\frac{1}{2}\left(r-\frac{1}{2}+2\epsilon\right)e^{\gamma}v(0)P(z)^{D-1}Ul\right\} \\ &\quad + r\left(\sum_h^{2k} t(n)-\left(\frac{t(j)}{v(j)}\right)v(n)+O\left(\frac{c_5c_{10}k^2}{(\log z)^3}\right)\right)(8+\epsilon)e^{2\gamma}P(z)^{D-1}U \\ &\le r\left(-\frac{t(j)}{v(j)}\right)\sum_h^{j}(-v(n))(8+\epsilon)e^{2\gamma}P(z)^{D-1}U \\ &\quad + r\sum_h^{j}\left[t(n)-\frac{t(j)}{v(j)}v(n)\right](8+2\epsilon)e^{2\gamma}P(z)^{D-1}U \\ &\le r\left(\sum_h^{j} t(n)\right)(8+2\epsilon)e^{2\gamma}P(z)^{D-1}U. \end{aligned}$$

Now Lemma 11 gives the result (6.5) □

Suppose that

$$h > e^{-\gamma}\frac{3\pi}{32\sqrt{2}}l, \tag{6.7}$$

and assume that

$$Z_r(2n) = 0 \quad \text{for } n < h. \tag{6.8}$$

Our next task will be the construction of $j$ and of the weights $u(n)$ and $w(n)$ such that the conditions of Lemma 14 are satisfied. Finally we shall derive an inequality for $h$ in (6.7) from Lemma 14. This then will prove the theorem.

Here we closely follow Huxley [11]. The adaptation to our problem is accomplished by a simple scale change involving a factor $e^{-\gamma}$.

We write

$$\begin{aligned} n &= e^{-\gamma}\frac{xl}{2}, & 2k &= e^{-\gamma}\frac{\lambda l}{2}, \\ h &= e^{-\gamma}\frac{\mu l}{2}, & j &= e^{-\gamma}\frac{\nu l}{2}. \end{aligned} \tag{6.9}$$

We fix $\epsilon > 0$ and choose the integer $j$ such that

$$\frac{\mu+\nu}{\pi}\sin\left(\frac{\nu-\mu}{\nu+\mu}\right)\pi = \frac{\{(2r-1)+16\epsilon\}}{(8+\epsilon)r} + O(l^{-1}) \tag{6.10}$$

and

$$\nu < \tfrac{5}{3}\mu. \tag{6.11}$$

This is possible for sufficiently small $\epsilon > 0$, since for fixed $\mu$ the expression

$$\frac{\mu+\nu}{\pi}\sin\left(\frac{\nu-\mu}{\nu+\mu}\right)\pi$$

increases monotonically from 0 to

$$\frac{8}{3\pi}\mu\sin\frac{\pi}{4}=\frac{4\sqrt{2}}{3\pi}\mu$$

as $\nu$ increases from $\mu$ to $\frac{5}{3}\mu$, and since

$$\frac{4\sqrt{2}}{3\pi}\mu>\frac{\{2r-1+16\epsilon\}}{(8+\epsilon)r}+O(l^{-1})$$

because of (6.7) and (6.9).

We set $\theta=(\nu-\mu)/(\nu+\mu)\pi$ and get

$$\frac{\mu+\nu}{\pi}\sin\theta=\frac{2r-1+16\epsilon}{(8+\epsilon)r}+O(l^{-1}). \tag{6.12}$$

We now set

$$2s+1=2k+1=h+j \tag{6.13}$$

and define $\delta=2\pi/(2s+1)$,

$$\begin{aligned} w(n)&=\sin\delta n \quad \text{for } -s\le n\le s,\\ u(n)&=\tfrac{1}{2}+\tfrac{1}{2}\cos\delta n \quad \text{for } -k\le n\le k. \end{aligned} \tag{6.14}$$

We have

$$\sum_{h\le|m-n|\le j} w(n)=-\frac{\sin\delta m\sin^{\delta(j-h+1)/2}}{\sin^{\delta/2}} \quad \text{for } -s\le n\le s$$

and thus the eigenvector equation

$$\sum_{h\le|m-n|\le j} w(n)+\frac{l}{2A}w(m)=0,$$

where

$$\frac{l}{2A}=\frac{\sin\delta(j-h+1)/2}{\sin^{\delta/2}} \quad \text{or} \tag{6.15}$$

$$\frac{1}{A}=e^{-\gamma}\frac{\mu+\nu}{\pi}\sin\theta+O(l^{-1}). \tag{6.16}$$

Multiplying by $w(m)$ and adding gives

$$2\sum_{h}^{j}(-v(n))=\frac{l}{2A}v(0).$$

The condition (6.3) of Lemma 14 now follows because of (6.10) and (6.16).

We now check the condition $v(n)\le 0$ for $n\ge h$. We have

$$\begin{aligned} v(n)&=\sum_{m=-s}^{s}\sin\delta m\sin\delta(m+n)=\frac{1}{2}\sum_{m=-s}^{s}(\cos\delta n-\cos\delta(2m+n))\\ &=\frac{(2s+1-n)\cos\delta n}{2}-\frac{\sin\delta(2s+1-n)}{2\sin\delta}. \end{aligned}$$

The smallest positive solution of $\Phi-\tan\Phi=2\pi$ satisfies $\pi/2<\Phi<3\pi/4$. The expression

$$\frac{(2s+1-n)\cos\delta n}{2}-\frac{\sin\delta(2s+1-n)}{2\delta}=v(n)+O(l^{-1})$$

changes from positive to negative as $\delta n$ passes through $\Phi$ and $v(n)$ remains negative until $\delta n=2\pi$; or $n=h+j$.

Because of (6.11) we have that $\delta h>3\pi/4$ and thus $v(n)\leq 0$. For $n\geq 0$ we obtain

$$\begin{aligned}4t(n)&=\sum_{m=-s}^{s-n}(1+\cos\delta m)(1+\cos\delta(m+n))\\&=(2s+1-n)(1+\tfrac{1}{2}\cos\delta n)+\sum_{m=-s}^{s-n}\{\cos\delta m+\cos\delta(m+n)+\tfrac{1}{2}\cos\delta(2m+n)\}\\&=(2s+1-n)(1+\tfrac{1}{2}\cos\delta n)+\{\sin\delta(s+\tfrac{1}{2}-n)-\sin\delta(-s-\tfrac{1}{2})\\&\qquad+\sin\delta(s+\tfrac{1}{2})-\sin\delta(-s-\tfrac{1}{2}-n)\}/2\sin^{\delta/2}\\&\quad+\{\sin\delta(2s-2n+1-n)-\sin\delta(-2s-1+n)\}/4\sin\delta\\&=(2s+1-n)(1+\tfrac{1}{2}\cos\delta n)+\sin\delta n\left\{\frac{1}{\sin^{\delta/2}}-\frac{1}{2\sin\delta}\right\},\end{aligned}$$

from which we conclude $\sum_h^j t(n)>c_9 l^2$, $-c_{10}\leq t(j)/v(j)\leq 0$. Also, conditions (2.10)-(2.13) for $t(n)$ are easily verified.

To show the condition (6.4) we set $y=\delta(2s+1-n)$ and replace $\sin\delta/2$ by $\delta/2$ and $\sin\delta$ by $\delta$ with a negligible error; we must then show:

$$\text{(6.17)}\qquad \frac{d}{dy}\,\frac{y\cos y-\sin y}{y(2+\cos y)-3\sin y}>0\quad\text{for}\quad\frac{\pi}{2}<y\leq 2\pi.$$

We shall give a new proof of this, since the computation in Huxley [11] contains an error. The numerator of this derivative is $2f(y)$, with

$$f(y)=-y^2\sin y+y-y\cos y+\sin y-\sin y\cos y.$$

An elementary but lengthy computation gives

$$f^{(m)}(0)=0\ \text{ for } m\leq 6\quad\text{and}$$

$$f^{(7)}(y)=-36\cos y+13y\sin y+y^2\cos y+64-128\sin^2 y,$$

and thus $f^{(7)}(0)=28$. Moreover, we have

$$f^{(8)}(0)=0\quad\text{and}$$

$$f^{(9)}(y)=64\cos y-17y\sin y-y^2\cos y-256+512\sin^2 y.$$

For $y>0$ Taylor's formula gives

$$f(y)=\frac{f^{(7)}(0)}{7!}y^7+\frac{f^{(9)}(z)}{9!}y^9\quad\text{or}$$

$$\text{(6.18)}\qquad f(y)=\frac{y^7}{7!}\left(28+\frac{f^{(9)}(z)}{72}y^2\right)\quad\text{with } 0<z<y.$$

The simple estimates

$$f^{(9)}(z) \geq -256 \quad \text{for } 0 < z \leq \frac{\pi}{4},$$

$$f^{(9)}(z) \geq -20 \quad \text{for } \frac{\pi}{4} < z \leq \frac{\pi}{2},$$

together with (6.18), yield:

$$f(y) > 0 \quad \text{for } 0 < y \leq \frac{\pi}{2}.$$

For the range $\pi/2 < y \leq \pi$ we observe that

$$f''(y) = (y^2 - 1)\sin y + (3y - 4\sin y)(-\cos y) > 0,$$

because both products are positive in $(\pi/2, \pi]$. This implies that $f'(y) > 0$ in $(\pi/2, \pi)$, since $f'(\pi/2) = 2 - \pi/2 > 0$. Finally $f(y) > 0$ in $(\pi/2, \pi)$ since, by the discussion above, $f(\pi/2) > 0$.

For the range $[\pi, 2\pi]$ we follow Huxley [11] and use the identity

$$\begin{aligned} \frac{d}{dy}\frac{y\cos y - \sin y}{y(2+\cos y) - 3\sin y} &= \{y(2+\cos y) - 3\sin y\}\{-y\sin y\} - \{y\cos y - \sin y\}\{-y\sin y + 2 - 2\cos y\} \\ &= 2(1-\cos y)(y + \sin y) - 2y^2 \sin y \\ &= 4\sin(y/2)\{(y+\sin y)\sin y/2 - y^2 \cos y/2\}. \end{aligned}$$

In the last expression all terms are positive or zero for $\pi \leq y < 2\pi$.

All the conditions of Lemma 4 have now been verified, and we can draw the conclusion:

$$8re^{\gamma}\sum_{h}^{j} t(n) > T(0)e^{\gamma} - \left\{\frac{1}{2}\left(r - \frac{1}{2}\right) + c_{11}\epsilon\right\} t(0)l. \tag{6.5}$$

Now the $u(m)$ satisfy

$$u(m+1+j) - u(m+h) + u(m+1-h) - u(m-j) = -\frac{l}{2A}\{u(m+1) - u(m)\},$$

where $u(n)$ is interpreted as 0 if $|n| > k$. Addition gives

$$\sum_{h \leq |m-n| \leq j} u(n) + \frac{l}{2A} u(m) = 2\rho U(0) \tag{6.19}$$

for some fixed $\rho$.

Multiplying by $u(m)$ and adding gives

$$E = 2\rho U^2(0), \tag{6.20}$$

with

$$\begin{aligned} E &= \sum_{\substack{m,n=-k \\ h \leq |m-n| \leq j}}^{k} u(m)u(n) + \frac{l}{2A}\sum_{-k}^{k} u^2(n) \\ &= 2\sum_{h}^{j} t(n) + \frac{l}{2A} t(0), \end{aligned}$$

whereas simply adding (6.19) for $-k \le m \le k$ gives

$$\left(j-h+1+\frac{l}{2A}\right)U(0)=2\rho U(0)(2k+1)$$

and thus

(6.21) $$\rho=(h+j)^{-1}\left(j-h+1+\frac{l}{2A}\right).$$

(6.5) now gives

$$E>\frac{U^2(0)}{4r},$$

$$j-h+1+\frac{l}{2A}>\frac{h+j}{4r}.$$

Division by $l/2$ gives

$$2\mu<(\mu+\nu)\left(1-\frac{1}{4r}\right)+\frac{e^\gamma}{A}+O\left(\frac{1}{l}\right),$$

and by (6.16):

(6.22) $$\mu<\frac{\pi e^\gamma}{2A\sin\theta}\left(1-\frac{1}{4r}\right)+\frac{e^\gamma}{2A}+O\left(\frac{1}{l}\right).$$

Now set

(6.23) $$F_r=\frac{2r-1}{16r}\left\{4r+(4r-1)\frac{\theta_r}{\sin\theta_r}\right\}.$$

We claim that, given $\delta>0$, the inequality (6.22) implies

(6.24) $$\mu<F_r+\delta,$$

provided that $\epsilon=\epsilon(\delta)$ in (6.10) is chosen sufficiently small and $z=z(\delta)$ is sufficiently large. This clearly will conclude the proof of (1.5) because of (6.8) and (6.9).

To establish (6.24) we consider the function $\nu^*=\nu^*(\mu^*)$, defined implicitly by

(6.25) $$\frac{\mu^*+\nu^*}{\pi}\sin\left(\left(\frac{\nu^*-\mu^*}{\nu^*+\mu^*}\right)\pi\right)=\frac{e^\gamma}{A_0},$$

where $A_0=8e^\gamma r/(2r-1)$. (This is (6.10) without the $\epsilon$ terms and error terms.) For $\mu^*=F_r$ we obtain

(6.26) $$\mu^*=\frac{\pi e^\gamma}{2A_0\sin\left(\left(\frac{\nu^*-\mu^*}{\nu^*+\mu^*}\right)\pi\right)}\left(1-\frac{1}{4r}\right)+\frac{e^\gamma}{2A_0},$$

and thus, by (6.25),

(6.27) $$\mu^*\left(\frac{1}{2}+\frac{1}{8r}\right)=\nu^*\left(\frac{1}{2}-\frac{1}{8r}\right)+\frac{e^\gamma}{2A_0}.$$

From (6.25) we obtain by implicit differentiation that

$$\frac{d\nu^*}{d\mu^*}<0 \quad \text{for all } \mu^*.$$

Thus, for $\mu^*>F_r$ we have

$$\mu^* > \frac{\pi e^\gamma}{2A_0 \sin\left(\left(\frac{\nu^*-\mu^*}{\nu^*+\mu^*}\right)\pi\right)}\left(1-\frac{1}{4r}\right)+\frac{e^\gamma}{2A_0}.$$

That proves our claim that (6.22) implies (6.24) $\mu < F_r + \delta$ and thus concludes the proof of the theorem. □

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