LIPSCHITZ SPACES AND SPACES OF HARMONIC FUNCTIONS IN THE UNIT DISC

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1. Introduction and the main result. In a series of papers [6; 7; 8] Shields and Williams studied the class $h_\infty(\psi)$ consisting of those functions u harmonic in the unit circle for which

$$u(z)=O\left(\psi\left(\frac{1}{1-r}\right)\right), \quad r=|z|\to 1^-, \tag{1}$$

where $\psi(x)$, $x\geq 1$, is a positive real function that grows more slowly than some power of x. In the present paper we solve Problem C of [7] by showing that each $h_\infty(\psi)$ is isomorphic, via a multiplier transform, to some space of functions continuous on the unit circle satisfying a modulus of continuity condition. Of course, our solution generalizes the classical theorems of Hardy and Littlewood and of Zygmund (see [2, Chap. 5, §§1, 2]), and is motivated by them.

Before stating our main result we recall some definitions and facts.

Moduli of continuity. For a complex-valued function h, defined on the real line, let $\Delta_t^n h$ (n is a positive integer, t is a real number) denote the nth difference with step t:

$$(\Delta_t^1 h)(\theta)=\Delta_t^1 h(\theta)=h(\theta+t)-h(\theta) \quad (\theta \text{ real}),$$
$$\Delta_t^n h=\Delta_t^1\Delta_t^{n-1}h, \quad n\geq 2.$$

If g is a complex-valued function defined on the unit circle T, then $\Delta_t^n g$ is defined by

$$\Delta_t^n g(e^{i\theta})=\Delta_t^n h(\theta), \qquad h(\theta)=g(e^{i\theta}).$$

For fixed n and t, Δ_t^n is a linear operator which preserves $C(T)$, the space of continuous functions on T. Furthermore,

$$\|\Delta_t^n g\|\leq 2^n\|g\|, \quad g\in C(T),$$

where $\|\cdot\|=\|\cdot\|_\infty$ stands for the maximum norm in $C(T)$. The modulus of continuity of order n is defined by

$$\omega_n(g,t)=\sup\{\|\Delta_s^n g\|: |s|<t\}, \quad t>0, \ g\in C(T).$$

Lipschitz spaces. For the sake of convenience we assume that all harmonic functions under consideration vanish at the origin. Similarly, if $g\in C(T)$ we assume that $\hat{g}(0)=0$, where $\hat{g}$ is the Fourier transform of g. Let $h(\Delta)$ be the class of all complex-valued functions harmonic in the unit disc Δ, and let $hC(\Delta)$ be the subspace of $h(\Delta)$ consisting of functions continuous on the closed disc. It is well known that the map $u\to u^*$, $u\in hC(\Delta)$, where u^* is the boundary function

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of u, is a linear isometry of the space $hC(\Delta)$ (endowed with the maximum-norm) onto $C(T)$. Thus any subclass of $C(T)$ may be regarded as a subclass of $hC(\Delta)$, and conversely.

Let ϕ be a positive function on $(0,1]$ and let n be a positive integer. The Lipschitz space $\mathrm{Lip}_n\,\phi$ consists of those $u\in hC(\Delta)=C(T)$ for which

(2) $$\omega_n(u^*,t)=O(\phi(t)),\quad t\to 0,$$

and is normed by

$$\|u\|_{\phi,n}=\sup\{\omega_n(u^*,t)/\phi(t)\colon 0<t\le 1\}.$$

The spaces $h_{\infty,n}(\psi)$. Let ψ be a positive function on $[1,\infty)$. For $u\in h(\Delta)$ let

$$\|u\|_\psi=\sup\{M(u,r)/\psi(1/(1-r))\colon 0<r<1\},$$

where

$$M(u,r)=M_\infty(u,r)=\max\{|u(z)|\colon |z|=r\}.$$

We define

$$h_{\infty,n}(\psi)=\{u\in h(\Delta)\colon \|D^nu\|_\psi<\infty\},\quad n=0,1,2,\dots,$$

where

$$(D^nu)(re^{i\theta})=\frac{\partial^n u}{\partial\theta^n}(re^{i\theta})=\sum_{j=-\infty}^{\infty}(ij)^n\hat{u}(j)r^{|j|}e^{ij\theta},\quad re^{i\theta}\in\Delta.$$

It is clear that the space $h_{\infty,n}(\psi)$, normed by $\|D^nu\|_\psi$, is isometric to $h_\infty(\psi)=h_{\infty,0}(\psi)$ via the multiplier transform D^n.

The hypotheses on ψ,ϕ. Bernstein [1] introduced the notion of almost increasing and almost decreasing functions. A real function φ is almost increasing if there is a positive constant C such that $x<y$ implies $\varphi(x)\le C\varphi(y)$. An almost decreasing function is defined similarly. Throughout the paper we assume that ψ is almost increasing and positive on $[1,\infty)$, and satisfies the following condition:

(U) There is a constant $C<\infty$ such that $\psi(2x)\le C\psi(x)$, $x\ge 1$.

As is remarked in [8], (U) is equivalent to the existence of a positive number α such that

(U_α) $\psi(x)/x^\alpha$ is almost decreasing for $x\ge 1$.

We also assume that ϕ is positive and almost increasing on $(0,1]$ and, for some $\beta>0$,

(U^0_β) $\phi(t)/t^\beta$ is almost decreasing for $0<t\le 1$.

Our solution of the Shields-Williams problem is as follows.

THEOREM 1. *Let n be a positive integer and assume that ψ satisfies* (U) *and that ϕ satisfies* (U^0_n). *Then the following conditions are equivalent:*

(a) $\mathrm{Lip}_n\,\phi=h_{\infty,n}(\psi)$;
(b) *there are constants α ($\alpha<n$) and C ($0<C<\infty$) such that ψ satisfies* (U_α) *and* $\phi(t)/C\le t^n\psi(1/t)\le C\phi(t)$, $0<t\le 1$.

In particular, if $\phi(t)=t^n\psi(1/t)$ and ψ satisfies (U_α), $\alpha<n$, *then* $\mathrm{Lip}_n\,\phi=h_{\infty,n}(\psi)$.

REMARK. The condition (U_n^0) is not restrictive. See Lemma 4.

It follows from Theorem 1 that each $h_\infty(\psi)$ (with ψ satisfying (U)) is naturally isomorphic to some $\mathrm{Lip}_n\,\phi$. Conversely, if ϕ is a *normal* function in the sense of Shields and Williams [6; 8], that is, if

(N) there is a positive number a such that $\phi(t)/t^a$ is almost increasing for $0<t\le 1$,

then (by Theorem 1) $\mathrm{Lip}_n\,\phi = h_{\infty,n}(\psi)$, where ψ is defined by $\psi(x)=x^n\phi(1/x)$. On the other hand, we do not know any multiplier transform which maps the space $\mathrm{Lip}_1\,\phi$, $\phi(t)=1/\log(et)$, onto one of the spaces $h_\infty(\psi)$. Further remarks are contained in §5.

The proof of the implication (b) $\Rightarrow$ (a) is based on the following lemmas which are of some independent interest.

LEMMA 1. *If* $u\in hC(\Delta)$ *then*

$$M(D^n u, r)\le C(1-r)^{-n}\omega_n(u^*,1-r),\quad 0<r<1, \tag{3}$$

where $C<\infty$ *is a constant depending only on* n $(n=1,2,\dots)$.

Since $\omega_n(u^*,1-r)\le 2^n\|u^*\|$, it follows that (3) improves the classical inequality $M(D^n u,r)\le C(1-r)^{-n}\|u^*\|$.

Our proof of Lemma 1, given in §3, differs from the standard proofs of similar results (see [2; 5]) and is independent of any pointwise estimate for the Poisson (or Cauchy) kernel.

LEMMA 2. *If* $u\in h(\Delta)$ *and*

$$\int_0^1 (1-r)^{n-1}M(D^n u,r)\,dr<\infty, \tag{4}$$

then $u\in hC(\Delta)$ *and*

$$\omega_n(u^*,t)\le C\int_{1-t}^1 (1-r)^{n-1}M(D^n u,r)\,dr,\quad 0<t<1, \tag{5}$$

where C *depends only on* n.

The proof of this lemma (§4) resembles the proof of the Hardy–Littlewood theorem [2, Thm. 5.1], but there is a difference.

In the last section we give some generalizations of Theorem 1.

2. Proof of Theorem 1. Throughout this section, n will denote a positive integer. The condition (U_α), mentioned in the introduction, can be written in the form

$$\psi(y)\le C(y/x)^\alpha\psi(x),\quad y\ge x\ge 1. \tag{U_α}$$

Using this, one easily proves that if $\alpha<n$ then (U_α) implies

$$\int_x^\infty \psi(y)y^{-n-1}\,dy\le Cx^{-n}\psi(x),\quad x\ge 1, \tag{A_n}$$

where C is a constant.

REMARK. We use the letters C, c to denote positive constants which may vary from line to line.

The following lemma is proved in the same way as Lemma 2 of [8]. We sketch a proof for completeness.

LEMMA 3. *ψ satisfies* (A_n) *if and only if there is* $\alpha < n$ *such that* ψ *satisfies* (U_α).

Proof. We have to prove that (A_n) implies (U_α) for some $\alpha < n$. Let ψ satisfy (A_n) and let

$$F(x) = \int_x^\infty \psi(y) y^{-n-1}\, dy, \quad x \geq 1.$$

It is easily seen that $cF(x) \leq x^{-n}\psi(x) \leq CF(x)$, $x \geq 1$, and this shows that it suffices to find $b > 0$ such that $x^b F(x)$ is nonincreasing for $x \geq 1$. We choose b so that $F(x) \leq (1/b)\psi(x)x^{-n}$, $x \geq 1$, which can be written as $F(x) \leq -(1/b)xF'(x)$, $x \geq 1$, where F' stands for the derivative of the (absolutely continuous) function F. This implies that the derivative of the function $x^b F(x)$ is ≤ 0, and this concludes the proof of Lemma 3. □

The implication (b) ⇒ (a) of Theorem 1 is an immediate consequence of Lemmas 1, 2, and 3 together with the identity

$$\int_{1/t}^\infty \psi(y) y^{-n-1}\, dy = \int_{1-t}^1 (1-r)^{n-1} \psi(1/(1-r))\, dr, \quad 0 < t < 1.$$

In order to prove the implication (a) ⇒ (b), we need some further lemmas.

LEMMA 4. *If* $g \in C(T)$ *then the function* $\omega_n(g, t)/t^n$ *is almost decreasing for* $t > 0$.

Proof. This fact is a consequence of the known inequality

$$\omega_n(g, 2t) \leq 2^n \omega_n(g, t), \quad t > 0. \tag{6}$$

Namely, if $\lambda = 2^m$ $(m = 0, 1, 2, \ldots)$ then (6) implies $\omega_n(g, \lambda t) \leq \lambda^n \omega_n(g, t)$, $t > 0$. If $\lambda > 1$ is arbitrary, we choose an integer $m \geq 0$ so that $2^m \leq \lambda \leq 2^{m+1}$, and then $\omega_n(g, \lambda t) \leq \omega_n(g, 2^{m+1} t) \leq 2^{(m+1)n} \omega_n(g, t) \leq 2^n \lambda^n \omega_n(g, t)$.

The easiest way to prove inequality (6) is to use the identity

$$\Delta_t^n g(e^{i\theta}) = \sum_j \hat{g}(j)(e^{ijt} - 1)^n e^{ij\theta}, \tag{7}$$

g being a trigonometric polynomial. Hence

$$\begin{aligned} \Delta_{2t}^n g(e^{i\theta}) &= \sum_j \hat{g}(j)(e^{ijt} + 1)^n (e^{ijt} - 1)^n \\ &= \sum_{k=0}^n \binom{n}{k} \Delta_t^n g(e^{i(\theta + kt)}), \end{aligned}$$

and this implies (6). □

LEMMA 5. *If* $g \in C(T)$ *and* $g_k(w) = g(w^k)$, *where* $w \in T$ *and* $k = 1, 2, \ldots$, *then*

$$\omega_n(g_k, \pi/k) \geq \|g\|_\infty.$$

Proof. It follows from (7) that

$$(-1)^n g(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\Delta_t^n g)(w)\,dt, \quad w \in T.$$

(Recall that we assume $\hat{g}(0)=0$ for $g \in C(T)$.) Hence

$$\|g\|_\infty \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \|\Delta_t^n g\|_\infty\,dt \le \omega_n(g,\pi) = \omega_n(g_k, \pi/k). \qquad \square$$

LEMMA 6. *Let X be one of the spaces* $\mathrm{Lip}_n\,\phi$, $h_{\infty,n}(\psi)$.

(i) *X is a Banach space.*

(ii) *If $\{u_m\}$ is a norm convergent sequence in X, $u_m \to u$, then $u_m(z)\to u(z)$ uniformly on each compact subset of Δ.*

Proof. In the case of $h_{\infty,n}(\psi)$ the result is a consequence of Proposition 1 of [7]. If $X=\mathrm{Lip}_n\,\phi$ then, by Lemmas 5 ($k=1$) and 4,

$$\|u\|_X \ge \omega_n(u^*,1)/\phi(1) \ge c\|u^*\|_\infty = \|u\|_\infty, \quad u \in X.$$

This shows that X is continuously embedded into $hC(\Delta)=C(T)$, and this implies (ii). Also, it is easy to deduce the completeness of $\mathrm{Lip}_n\,\phi$ from the completeness of $C(T)$ and the embedding $\mathrm{Lip}_n\,\phi \subset C(T)$.

LEMMA 7. *Let ψ, ϕ satisfy* (U) *and* (U_n^0), *respectively. Let $u_k(z)=z^k$, where $|z|\le 1$ and $k=1,2,\ldots$. Then*

$$\|D^n u_k\|_\psi \asymp k^n/\psi(k), \quad k=1,2,\ldots, \tag{8}$$

$$\|u_k\|_{\phi,n} \asymp 1/\phi(1/k), \quad k=1,2,\ldots. \tag{9}$$

REMARK. For two nonnegative functions F_1, F_2 defined on a set S we write $F_1(s)\asymp F_2(s)$, $s\in S$, if there are constants C and c such that $cF_1(s)\le F_2(s)\le CF_1(s)$, $s \in S$.

Proof of Lemma 7. The proof of (8) is contained in [8, p. 22]. To prove (9) we use the equality

$$(\Delta_t^n u_k^*)(w) = w^k(e^{ikt}-1)^n, \quad w \in T.$$

Hence

$$\omega_n(u_k^*, t) = 2^n \sup\{|\sin(ks/2)|^n : 0 < s \le t\}, \quad t > 0,$$

and therefore

$$\|u_k\|_{\phi,n} = 2^n \sup\{|\sin(ks/2)|^n/\phi(s) : s \le t \le 1, 0 < s \le 1\}.$$

Since $\phi(t)\ge \phi(s)/C$ for $0<s\le t\le 1$, we have

$$\|u_k\|_{\phi,n} \le C\sup\{|\sin(ks/2)|^n/\phi(s) : 0 < s \le 1\},$$

where C is independent of k. If $1/k \le s \le 1$, then

$$|\sin(ks/2)|^n/\phi(s) \le C/\phi(1/k)$$

because $1/\phi$ is almost decreasing. If $0 < s \le 1/k$, then

$$|\sin(ks/2)|^n/\phi(s) \le 2^{-n}k^n s^n/\phi(s) \le Ck^n(1/k)^n/\phi(1/k)$$

because $s^n/\phi(s)$ is almost increasing. Thus $\|u_k\|_{\phi,n} \le C/\phi(1/k)$. The inequality $\|u_k\|_{\phi,n} \ge c/\phi(1/k)$ is simple, and the lemma is proved. □

Now we are ready to prove the implication (a) ⇒ (b) of Theorem 1. Let $\mathrm{Lip}_n\,\phi = h_{\infty,n}(\psi)$. It follows from Lemma 6 and the closed graph theorem that $\|D^n u_k\|_\psi \asymp \|u_k\|_{\phi,n}$, $k \ge 1$, where u_k are as in Lemma 7. Hence, by Lemma 7,

$$\phi(1/k) \asymp (1/k)^n \psi(k), \quad k = 1, 2, \dots .$$

This yields, by the properties of ψ and ϕ,

$$\phi(t) \asymp t^n \psi(1/t), \quad 0 < t \le 1, \tag{10}$$

and this is part of (b).

In order to prove that (a) implies (U_α) for some $\alpha < n$, we define the functions U_k $(k = 1, 2, \dots)$ by

$$U_k(z) = k^{-n}\psi(k)z^k + \sum_{j=2}^{\infty} (jk)^{-n}(\psi(jk) - \psi((j-1)k))z^{jk}, \quad z \in \Delta.$$

Assume, without loss of generality, that ψ is nondecreasing. By direct differentiation we have, for $0 < r < 1$,

$$\begin{aligned} M(D^n U_k, r) &\le \psi(k)r^k + \sum_{j=2}^{\infty} (\psi(jk) - \psi((j-1)k))r^{jk} \\ &\le \psi(k)r^k + \sum_{j=2}^{\infty} \sum_{p=(j-1)k}^{jk-1} (\psi(p+1) - \psi(p))r^{p+1} \\ &= \psi(k)r^k + \sum_{j=k}^{\infty} (\psi(p+1) - \psi(p))r^{p+1} \\ &= (1-r) \sum_{p=k}^{\infty} \psi(p)r^p. \end{aligned}$$

By applying Lemma 1(iii) of [8] we find

$$M(D^n U_k, r) \le C\psi(1/(1-r)), \quad 0 < r < 1,$$

where C is independent of k, r. This means that $\{U_k\}$ is a norm bounded sequence in $h_{\infty,n}(\psi)$. Now we use the inclusion $h_{\infty,n}(\psi) \subset \mathrm{Lip}_n\,\phi$ to conclude that U_k are continuous on the closed disc and

$$\omega_n(U_k^*, t) \le C\phi(t), \quad 0 < t \le 1, \tag{11}$$

where C is independent of t, k.

On the other hand, by Lemmas 4 and 5,

$$\begin{aligned} C\omega_n(U_k^*, 1/k) \ge \omega_n(U_k^*, \pi/k) \ge \|U_k^*\|_\infty \\ = k^{-n}\psi(k) + k^{-n} \sum_{j=2}^{\infty} j^{-n}(\psi(jk) - \psi((j-1)k)) \\ = k^{-n} \sum_{j=1}^{\infty} (j^{-n} - (j+1)^{-n})\psi(jk), \quad k = 1, 2, \dots . \end{aligned}$$

Hence, by (11), (10), and (U),

$$k^{-n}\sum_{j=1}^{\infty} j^{-n-1}\psi((j+1)k)\le C\phi(1/k)\le Ck^{-n}\psi(k)$$

and therefore

$$\int_k^{\infty} y^{-n-1}\psi(y)\,dy = k^{-n}\int_1^{\infty} y^{-n-1}\psi(yk)\,dy$$
$$\le k^{-n}\sum_{j=1}^{\infty} j^{-n-1}\psi((j+1)k)\le C(k+1)^{-n}\psi(k),\quad k=1,2,\dots .$$

It is easily verified that this implies (A_n). Thus ψ satisfies (U_α) for some $\alpha<n$ (by Lemma 3), and this concludes the proof of Theorem 1. □

3. Proof of Lemma 1. Without loss of generality, we can assume that u is harmonic in $|z|<R$ for some $R>1$. For fixed r $(0<r<1)$ let

$$h(\theta)=u_r(\theta)=u(re^{i\theta})\quad (\theta \text{ real}).$$

By induction,

$$(\Delta_t^n h)(\theta)=\int_{tE} h^{(n)}(\theta+x_1+\cdots+x_n)\,dx_1\cdots dx_n, \tag{12}$$

where tE $(t>0)$ is the n-dimensional cube $[0,t]^n$. Hence

$$h^{(n)}(\theta)t^n=(D^n u)(re^{i\theta})t^n$$
$$=(\Delta_t^n h)(\theta)-\int_{tE}(h^{(n)}(\theta+x_1+\cdots+x_n)-h^{(n)}(\theta))\,dx_1\cdots dx_n.$$

Since

$$|h^{(n)}(\theta+x)-h^{(n)}(\theta)|=\left|\int_0^x h^{(n+1)}(\theta+y)\,dy\right|\le M(D^{n+1}u,r)x,$$

$x=x_1+\cdots+x_n$, we obtain

$$M(D^n u,r)t^n\le \|\Delta_t^n u_r\|_\infty+\int_{tE} M(D^{n+1}u,r)(x_1+\cdots+x_n)\,dx_1\cdots dx_n$$
$$=\|\Delta_t^n u_r\|_\infty+(n/2)M(D^{n+1}u,r)t^{n+1},\quad 0\le r\le 1,\ t>0.$$

The function $\Delta_t^n u$ defined by $(\Delta_t^n u)(re^{i\theta})=(\Delta_t^n u_r)(\theta)$ is harmonic in $|z|\le 1$, and therefore

$$\|\Delta_t^n u_r\|_\infty\le\|\Delta_t^n u^*\|_\infty\le\omega_n(u^*,t),\quad t>0.$$

These inequalities together with the familiar estimate

$$M(D^{n+1}u,r)\le C(1-r)^{-1}M(D^n u,(1+r)/2),\quad 0<r<1$$

(see [2, p. 80]), yield

$$M(D^n u,r)\le t^{-n}\omega_n(u^*,t)+Kt(1-r)^{-1}M(D^n u,(1+r)/2),\quad 0<r<1,\ t>0, \tag{13}$$

where $K<\infty$ depends only on n. Let

$$A(r)=(1-r)^n M(D^n u,r),\quad 0<r<1.$$

It follows from (13) that

$$A(r) \le t^{-n}(1-r)^n\omega(t) + 2^n Kt(1-r)^{-1}A((1+r)/2),$$

where $\omega(t) = \omega_n(u^*, t)$. Let m be the smallest integer such that $2^n K \le (1/4)2^m$ and take $t = a(1-r)$, $a = 2^{-m}$. Then we have

$$A(r) \le a^{-m}\omega(1-r) + (1/4)A((1+r)/2), \quad 0 < r < 1.$$

Since u is harmonic in the closed disc, A is bounded near 1 and therefore $A \in L^1(0,1)$. Thus we have, for $0 < \rho < 1$,

$$\begin{aligned}\int_\rho^1 A(r)\,dr &\le a^{-m}\int_0^{1-\rho}\omega(t)\,dt + (1/4)\int_\rho^1 A((1+r)/2)\,dr \\ &= a^{-m}\int_0^{1-\rho}\omega(t)\,dt + (1/2)\int_{(1+\rho)/2}^1 A(r)\,dr \\ &\le a^{-m}\int_0^{1-\rho}\omega(t)\,dt + (1/2)\int_\rho^1 A(r)\,dr.\end{aligned}$$

Hence

$$(1/2)\int_\rho^1 A(r)\,dr \le a^{-m}\int_0^{1-\rho}\omega(t)\,dt \le a^{-m}(1-\rho)\omega(1-\rho).$$

Since

$$M(D^n u, \rho)(1-\rho)^{n+1} \le (n+1)\int_\rho^1 A(r)\,dr, \quad 0 < \rho < 1,$$

the lemma is proved. □

4. Proof of Lemma 2. We assume, without loss of generality, that u is a real-valued function harmonic in Δ, $u(0) = 0$. Then u is the real part of an analytic function f with $f(0) = 0$. We have, by Taylor's formula,

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(rz)}{k!} z^k(1-r)^k + \frac{1}{n!}\int_r^1 (1-s)^n z^{n+1} f^{(n+1)}(sz)\,ds, \qquad (14)$$
$$|z| < 1, \ 0 < r < 1.$$

Denoting the sum by $f_{r,n}$ we have

$$|f(z) - f_{r,n}(z)| \le \frac{1}{n!}\int_r^1 (1-s)^n M(f^{(n+1)}, s)\,ds, \quad |z| < 1.$$

Now we use the familiar estimate

$$M(f^{(n+1)}, s) \le C(1-s)^{-1}M(D^n u, (1+s)/2), \quad 0 < s < 1, \qquad (15)$$

(see Remark 1 below) to conclude that (4) implies

$$\|f - f_{r,n}\|_\infty \to 0 \quad (r \to 1^-).$$

(Here, as usual, $\|F\|_\infty = \sup_{|z|<1}|F(z)|$.) Since $f_{r,n}$ ($r < 1$) is continuous in $|z| \le 1$, we see that (4) implies the continuity of f, and consequently of u, in $|z| \le 1$.

In order to prove the inequality (5) let $u_r(\theta) = u(re^{i\theta})$ for $0 \le r \le 1$. Then (5) is equivalent to

(16) $$\|\Delta_t^n u_1\|_\infty \le C\int_{1-t}^1 (1-s)^{n-1} M(D^n u, s)\, ds, \quad 0<t<1.$$

Let $r=1-2t$, $0<t<1/4$. Then

$$\|\Delta_t^n u_1\| \le \|\Delta_t^n (u_1-u_r)\| + \|\Delta_t^n u_r\|.$$

It follows from (12) and the increasing property of $M(D^n u, r)$ that

$$\|\Delta_t^n u_r\| \le t^n M(D^n u, r) \le n\int_{1-t}^1 (1-s)^{n-1} M(D^n u, s)\, ds,$$

and therefore we have to prove that $\|\Delta_t^n(u_1-u_r)\|$ is dominated by the right-hand side of (16). Since $\|\Delta_t^n(u_1-u_r)\| \le \|\Delta_t^n(f_1-f_r)\|$, where $f_r(\theta)=f(re^{i\theta})$, it is enough to prove that

$$\|\Delta_t^n(f_1-f_r)\| \le C\int_{1-t}^1 (1-s)^{n-1} M(D^n u, s)\, ds.$$

To prove this write (14) in the form

$$f_1(\theta)-f_r(\theta) = H(\theta) + \sum_{k=1}^n h_k(\theta)(1-r)^k/k!,$$

where

$$H(\theta) = \frac{1}{n!}\int_r^1 (1-s)^n e^{i(n+1)\theta} f^{(n+1)}(se^{i\theta})\, ds,$$

$$h_k(\theta) = f^{(k)}(re^{i\theta})e^{ik\theta}.$$

We have

$$\begin{aligned}
\|\Delta_t^n H\| \le 2^n\|H\| &\le \frac{2^n}{n!}\int_r^1 (1-s)^n M(f^{(n+1)}, s)\, ds \\
&\le C\int_r^1 (1-s)^{n-1} M(D^n u, (1+s)/2)\, ds \quad \text{(by (15))} \\
&= 2^n C\int_{1-t}^1 (1-s)^{n-1} M(D^n u, s)\, ds.
\end{aligned}$$

In order to estimate $\|\Delta_t^n h_k\|$ let $m=n-k+1$ $(1\le k\le n)$ and observe that

$$\|\Delta_t^n h_k\| = \|\Delta_t^{k-1}\Delta_t^m h_k\| \le 2^{k-1}\|\Delta_t^m h_k\| \le 2^{k-1} t^m \|h_k^{(m)}\|$$

(see (12)). Now we use the inequality (see Remark 1 below)

(19) $$\|h_k^{(m)}\| \le C(1-r)^{-1} M(D^n u, (1+r)/2)$$

to obtain

$$\|\Delta_t^n h_k\| \le Ct^{n-k} M(D^n u, 1-t) \le Ct^{-k}\int_{1-t}^1 (1-s)^{n-1} M(D^n u, s)\, ds,$$

where C is independent of t. Combining all the above results yields (5) for $0<t<1/4$. If $1/4\le t\le 1$, then we use Lemma 4 to reduce (5) to the case $t<1/4$, and this completes the proof of Lemma 2. □

REMARK 1. Although the inequalities (15) and (19) are well known (see [2, Chap. 5]) we shall sketch a proof of (19). The inequality (15) is proved similarly.

Observe first that we have used (19) only for $r \geq 1/2$. Thus for our purposes it suffices to prove that

$$r^k \|h_k^{(m)}\| \leq C(1-r)^{-1} M(D^n u, r^{1/2}), \quad 0<r<1, \ m=n-k+1.$$

By using the relation $f(z) = 2\sum_1 \hat{u}(j) z^j$ and Parseval's formula,

$$r^k h_k^{(m)}(\theta) = 2k! \sum_{j=1}^{\infty} \binom{j}{k} (ij)^m \hat{u}(j) r^j e^{ij\theta} = \frac{1}{2\pi} \int_0^{2\pi} U(x) V(\theta - x)\, dx,$$

where

$$u(\theta) = (D^n u)(r^{1/2} e^{i\theta}),$$

$$V(\theta) = 2k! \sum_{j=1}^{\infty} \binom{j}{k} (ij)^{m-n} r^{j/2} e^{ij\theta}.$$

Hence

$$r^k \|h_k^{(m)}\| \leq \|U\|_\infty \|V\|_1 = M(D^n u, r^{1/2}) \|V\|_1.$$

Since

$$\binom{j}{k} j^{m-n} = \binom{j}{k} j^{1-k} = j + O(1), \quad j \to \infty,$$

we have

$$V(\theta) = 2k!\, i^{1-k} r^{1/2} e^{i\theta} (1 - r^{1/2} e^{i\theta})^{-2} + O((1-r)^{-1}), \quad r \to 1^-$$

uniformly in θ. This gives

$$\|V\|_1 = \frac{1}{2\pi} \int_0^{2\pi} |V(\theta)|\, d\theta \leq C(1-r)^{-1}, \quad 0<r<1,$$

and this completes the proof. □

REMARK 2. In the case $n=1$ our proof of Lemma 2 is similar but not identical to the proof of the Hardy–Littlewood theorem [2, Thm. 5.1]. The only difference is that the second-order derivatives need not be used in the analytic case.

REMARK 3. The condition (4) is known to be independent of n. For more information see [3].

5. Some extensions of Theorem 1. An inspection of the proof of Theorem 1 shows that the following more general result is valid.

THEOREM 2. *If* ψ, ϕ *satisfy* (U) *and* (U_n^0), *respectively, then the following hold:*

(i) $\mathrm{Lip}_n \phi \subset h_{\infty,n}(\psi)$ *if and only if* $\phi(t) = O(t^n \psi(1/t))$;

(ii) $h_{\infty,n}(\psi) \subset \mathrm{Lip}_n \phi$ *if and only if* $\int_x^\infty \psi(y) y^{-n-1}\, dy = O(\phi(1/x))$, $x \to \infty$.

EXAMPLE. If $\phi(t) = 1/\log(e/t)$, $0 < t \leq 1$, then $h_{\infty,1}(\psi_1) \subset \mathrm{Lip}_1 \phi \subset h_{\infty,1}(\psi_2)$, where $\psi_1(x) = x/(\log(ex))^2$ and $\psi_2(x) = x/\log(ex)$, $x \geq 1$, and these inclusions are best possible. It would be interesting to check whether this Lipschitz space is isomorphic, via a multiplier transform, to some of the spaces $h_\infty(\psi)$.

It should be noted that condition (U_n^0) in Theorems 1 and 2 is not restrictive. Namely, if $t^n/\phi(t)$ is not bounded, then $\mathrm{Lip}_n\,\phi=\{0\}$. If $t^n/\phi(t)$ is bounded, we define ϕ_0 by

$$\phi_0(t)=t^n \inf_{0<s<t} s^{-n}\phi(s)=\inf_{0<s<1} s^{-n}\phi(st), \quad 0<t\leq 1.$$

Then ϕ_0 is positive and almost increasing on $(0,1]$, and satisfies (U_n^0). Furthermore, by using Lemma 4 one easily proves that $\mathrm{Lip}_n\,\phi=\mathrm{Lip}_n\,\phi_0$.

Some of our results can be generalized to the case of L^p spaces and, more generally, to the class of homogeneous Banach spaces (see [4, p. 14]). For example, the following theorem is a generalization of the implication (b) ⇒ (a) of Theorem 1. Here: $\omega_n(g,t)_p=\sup\{\|\Delta_s^n g\|_p: |s|<t\}$ $(t>0,\ g\in L^p(T))$ and $M_p(U,r)=\|U_r\|_p$ $(0<r<1,\ U\in h(\Delta))$, where $U_r(e^{i\theta})=U(re^{i\theta})$ and $\|\cdot\|_p$ stands for the norm in $L^p(T)$.

THEOREM 3. *Let $u\in h(\Delta)$, $p\geq 1$ and let ψ satisfy (U_α), $\alpha<n$. Then the following are equivalent:*

(a) *u is the Poisson integral of a function $g\in L^p(T)$ with*

$$\omega_n(g,t)_p=O(t^n\psi(1/t)), \quad t\to 0;$$

(b) $M_p(D^n u,r)=O(\psi(1/(1-r)))$, $r\to 1^-$.

This theorem follows immediately from the corresponding generalizations of Lemmas 1 and 2. The proofs are essentially the same as in the case $p=\infty$. We note that if f is analytic and $u=\mathrm{Re}\,f$, then (4) $(M=M_p)$ implies that f belongs to the Hardy space H^p. (This follows from (14) $(r=0)$ and (15).) Therefore, if (4) holds, then u is the Poisson integral of a function $g\in L^p(T)$ (see [2, Chap. 3]).

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