INTEGRAL DOMAINS IN WHICH EACH t-IDEAL IS DIVISORIAL

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Introduction. Let D be a commutative integral domain, and let $\mathfrak{F}(D)$ be the set of nonzero fractional ideals of D. We recall the v- and t-operations. For each $A \in \mathfrak{F}(D)$, $A_v = (A^{-1})^{-1}$ = the intersection of the principal fractional ideals of D which contain A, and $A_t = \bigcup \{J_v : J$ is a finitely generated subideal of $A\}$. If $A = A_v$ (resp. $A = A_t$) then A is said to be a v-ideal or *divisorial* (resp. a *t-ideal*). The v-ideal A has *finite type* if $A = J_v$ for some finitely generated $J \in \mathfrak{F}(D)$. The fractional ideal A is *quasi-finite* if $A^{-1} = J^{-1}$ for some finitely generated fractional subideal J of A. The v- and t-operations are examples of star operations; the reader is referred to [7, §§32, 34] or to [15] for the properties of star operations (which we shall use freely).

We define a domain D to be a *TV-domain* if each t-ideal of D is divisorial (equivalently, if the v- and t-operations on D are the same). Our study of *TV*-domains is motivated by [12], where Heinzer studies domains *all* of whose nonzero ideals are divisorial. The class of *TV*-domains is, of course, much larger. It includes the class of *Mori domains,* domains satisfying the ascending chain condition on divisorial ideals. Hence Noetherian domains and Krull domains are *TV*-domains.

In the first section, we study the properties of *TV*-domains, generalizing many (but not all) of the results of [12]. Before describing the main result, we recall that each t-ideal A of D is contained in a t-ideal M maximal among t-ideals containing A, and this *maximal t-ideal* is prime [15, pp. 30–31]. We prove (Theorem 1.3) that every (proper) t-ideal of a *TV*-domain D is contained in only finitely many maximal t-ideals. We also show that every maximal t-ideal of a *TV*-domain D has the form $(a):b$ for some $a, b \in D$.

In the second section we give several characterizations of Krull domains. In particular, we give a proof more direct than (say) that given in [5, pp. 12–16] of the fact that (a) a completely integrally closed Mori domain is a Krull domain. In fact we prove that (a′) a completely integrally closed *TV*-domain is a Krull domain. This is done as follows. First, define a fractional ideal A of D to be *v-invertible* (resp. *t-invertible*) if $(AA^{-1})_v = D$ (resp. $(AA^{-1})_t = D$). In [18, Thm. 0] it was proved, as a consequence of (a), that (b) D is a Krull domain if and only if every $A \in \mathfrak{F}(D)$ is t-invertible. Here we prove (b) independently of (a) and then use (b) to prove (a′), yielding a more satisfying approach to a well-known result.

The third section characterizes Prüfer v-multiplication domains among *TV*-domains, and we show that this class of rings is a certain subclass of the class of

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rings of Krull type studied by Griffin [10]. The requisite definitions are reviewed in that section.

Finally, in the last section, we present several examples and pose some questions. Among the examples is a Noetherian domain with two maximal t-ideals containing a common prime t-ideal, thwarting our hope of generalizing [12, Thm. 2.4]. We also observe that pseudo-valuation domains are TV-domains.

1. Properties of TV-domains. We begin with a generalization of [12, Lemma 2.1].

LEMMA 1.1. *Let D be an integral domain, and let $\{A_\alpha\}$ be a set of fractional ideals of D for which $\bigcap_\alpha (A_\alpha)_v \neq 0$. Then $(\bigcap_\alpha (A_\alpha)_v)^{-1} = (\sum_\alpha A_\alpha^{-1})_v$. In particular, if the A_α are divisorial then $(\bigcap_\alpha A_\alpha)^{-1} = (\sum_\alpha A_\alpha^{-1})_v$.*

Proof. Since $\bigcap_\alpha (A_\alpha)_v \neq 0$, $\sum_\alpha A_\alpha^{-1}$ is a fractional ideal of D. For any set $\{B_\beta\}$ of fractional ideals for which $\sum_\beta B_\beta$ is a fractional ideal, the formula $(\sum_\beta B_\beta)^{-1} = \bigcap_\beta B_\beta^{-1}$ is valid. Application of this formula to the set $\{A_\alpha^{-1}\}$ yields $(\sum_\alpha A_\alpha^{-1})^{-1} = \bigcap_\alpha (A_\alpha)_v$. The result follows by inverting both sides. □

LEMMA 1.2 (cf. [12, Lemma 2.3]). *Let D be a TV-domain, let A be a t-ideal of D, and let M be a maximal t-ideal of D containing A. If $\{B_\alpha\}$ is the set of t-ideals of D which contain A but are not contained in M, then $\bigcap_\alpha B_\alpha \not\subseteq M$.*

Proof. The set $\{B_\alpha\}$ contains D and is therefore nonempty. For any $B \in \{B_\alpha\}$ we have $(M+B)_t = D$, since $B \not\subseteq M$ and M is a maximal t-ideal. It follows that $D = B^{-1} \cap M^{-1}$. Since M is a t-ideal, it is divisorial, and we may choose $x \in M^{-1} - D$. Hence $x \notin B^{-1}$ for each $B \in \{B_\alpha\}$. We claim that $x \notin (\sum_\alpha B_\alpha^{-1})_v = (\sum_\alpha B_\alpha^{-1})_t$. Otherwise, $x \in F_v$ for some finitely generated D-submodule F of $\sum B_\alpha^{-1}$, whence $x \in (\sum_{i=1}^n B_i^{-1})_v$ for some finite subset $\{B_1, \ldots, B_n\}$ of $\{B_\alpha\}$. By Lemma 1.1 this implies $x \in (\bigcap_{i=1}^n B_i)^{-1}$. However, $\bigcap B_i \in \{B_\alpha\}$, a contradiction. Hence $x \notin (\sum B_\alpha^{-1})_v = (\bigcap B_\alpha)^{-1}$, and $M^{-1} \not\subseteq (\bigcap B_\alpha)^{-1}$. It follows that $\bigcap B_\alpha \not\subseteq M$. □

We are now ready for the main result on TV-domains.

THEOREM 1.3 (cf. [12, Thm. 2.5]). *If A is a proper t-ideal of the TV-domain D, then A is contained in only finitely many maximal t-ideals of D.*

Proof. Let $\{M_\alpha\}$ denote the set of maximal t-ideals which contain A. For each α let $T_\alpha = \bigcap_{\beta \neq \alpha} M_\beta$. By the preceding lemma, $T_\alpha \not\subseteq M_\alpha$. Hence $A \subseteq \sum T_\alpha \equiv T$, and $T \not\subseteq M_\alpha$ for each α. It follows that $T_t = D$. Thus $1 \in T_t$ whence $1 \in (\sum_{i=1}^n T_i)_t$ for some finite subset $\{T_1, \ldots, T_n\}$ of $\{T_\alpha\}$. Denote the corresponding subset of $\{M_\alpha\}$ by $\{M_1, \ldots, M_n\}$. If $M_\alpha \notin \{M_1, \ldots, M_n\}$, then $M_\alpha \supseteq \sum_{i=1}^n T_i$, contradicting the facts that M_α is a proper t-ideal and $(\sum_{i=1}^n T_i)_t = D$. □

Recall that a *Mori domain* is an integral domain D which satisfies ACC on divisorial ideals. According to Querré [19, Thm. 1] this is equivalent to the requirement that every nonzero fractional ideal A of D be quasi-finite. It follows that in a Mori domain the t- and v-operations are the same, so that a Mori domain is also a TV-domain. As an application of Theorem 1.3, we therefore have the following result, which has been proved by other methods in [2], [4], [14], and [20].

PROPOSITION 1.4. *If D is a Mori domain, then every nonzero element of D is contained in only finitely many maximal divisorial ideals of D.*

REMARK 1.5. Actually, [14, Thm. 2.1] is the stronger result that a nonzero element of a Mori domain D is contained in only finitely many divisorial prime ideals of D (maximal or not). This stronger result does not generalize to TV-domains. To see this recall from [12, Lemma 5.2] that a valuation domain has all its nonzero ideals divisorial if and only if its maximal ideal is principal. This same property characterizes TV-domains among valuation domains, since every nonzero ideal of a valuation domain is a t-ideal. Thus if V is a valuation domain with principal maximal ideal which also contains an infinite ascending chain of prime ideals, then V is a TV-domain containing elements which lie in infinitely many divisorial primes.

PROPOSITION 1.6 (cf. [12, Lemma 2.2]). *In a TV-domain D, every maximal t-ideal has the form $(a):b$ for suitable $a, b \in D$.*

Proof. Let M be a maximal t-ideal of D, and pick $x \in M^{-1}-D$. It is easy to show that $M^{-1}=(D+xD)_v$, whence $M=(D+xD)^{-1}=D\cap(1/x)D=(a):b$, where $x=b/a$. □

REMARK 1.7. In a Mori domain *every* divisorial prime ideal has the form $(a):b$ [14, Cor. 2.5]. However, in a valuation domain, ideals of this form are principal. Thus if V is a Mori domain of Krull dimension >1 and with principal maximal ideal M, then V is a TV-domain, but M is the *only* prime t-ideal of V of the form $(a):b$.

In general, if M is a maximal t-ideal of a domain D then MD_M need not be a t-ideal [22]. However, from Proposition 1.6 we have

COROLLARY 1.8. *If M is a maximal t-ideal of a TV-domain D, then MD_M is divisorial in D_M.*

2. Characterization of Krull domains. Recall that a fractional ideal A of a domain D is quasi-finite if $A^{-1}=J^{-1}$ (equivalently, $A_v=J_v$) for some finitely generated fractional ideal $J\subseteq A$. It is easy to show that if $A\in\mathfrak{F}(D)$ is t-invertible then A is quasi-finite [15, Thm. 8]. In fact, A is t-invertible if and only if A is v-invertible and both A and A^{-1} are quasi-finite.

We begin with yet another analogue of a theorem of Cohen.

THEOREM 2.1. *If D is a domain in which every prime t-ideal is t-invertible, then every t-ideal of D is t-invertible.* (*It follows easily that every nonzero fractional ideal of D is t-invertible.*)

Proof. We proceed contrapositively. Suppose that $\Gamma=\{A: A$ is a non-t-invertible t-ideal of $D\}$ is nonempty. If Γ is partially ordered by inclusion, and if $\{A_\alpha\}$ is a chain in Γ, then it is easy to see that $A=\bigcup A_\alpha$ is a t-ideal. Moreover, since

t-invertible ideals are quasi-finite, it is clear that A is not t-invertible. Hence Zorn's lemma applies, and Γ has maximal elements. We shall complete the proof by showing that any maximal element P of Γ is prime. Accordingly, suppose that $a, b \in D$ with $ab \in P$, $b \notin P$. Let $J = P : a$. Then J is a t-ideal properly containing P, so that J is t-invertible. Hence $P = P \cdot (JJ^{-1})_t \subseteq (PJJ^{-1})_t \subseteq P_t = P$, and $P = (PJJ^{-1})_t = (J \cdot (PJ^{-1})_t)_t$. Since P is not t-invertible, it follows that $(PJ^{-1})_t$ is not t-invertible (since any product of two t-invertible ideals is t-invertible). Therefore, $(PJ^{-1})_t$ cannot properly contain P, whence $(PJ^{-1})_t = P$. However, $aJ \subseteq P$, so that, as J is t-invertible, $a \in (PJ^{-1})_t = P$. This completes the proof. □

Before stating our main characterization of Krull domains, we record for easy reference the following lemma. The proof is straightforward and follows from [9, Prop. 4].

LEMMA 2.2. *Let A be a t-invertible ideal of the domain D. Then for every prime t-ideal P of D containing A, AD_P is principal.*

We now present our characterizations of Krull domains. Although we believe the proofs are new, many of the characterizations themselves are not. In particular, the equivalence of (1) and (4) appears in [6] and [18]; [16] contains the equivalence of (1), (3), and (4); and [17] contains the equivalence of (1) and (2). Of course, the equivalence of (1) and (5) is well known. What we offer is an efficient approach to these results.

THEOREM 2.3. *The following statements are equivalent for a domain D.*

(1) *D is a Krull domain.*
(2) *Every associated prime of D is t-invertible. (P is an associated prime of D if P is minimal over $(a) : b$ for some $a, b \in D$.)*
(3) *Every prime t-ideal of D is t-invertible.*
(4) *Every nonzero fractional ideal of D is t-invertible.*
(5) *D is a completely integrally closed Mori domain.*
(6) *D is a completely integrally closed TV-domain.*

Proof. We shall prove $(1) \Rightarrow (5) \Rightarrow (6) \Rightarrow (4) \Rightarrow (1)$ and $(4) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$. Of these $(1) \Rightarrow (5)$ is well known, and $(5) \Rightarrow (6)$ and $(4) \Rightarrow (2)$ are trivial.

$(6) \Rightarrow (4)$. Let A be a nonzero fractional ideal of D. Since D is completely integrally closed, $(AA^{-1})_v = D$ [7, Thm. 34.3]. Since D is a TV domain, $(AA^{-1})_t = D$ as well.

$(4) \Rightarrow (1)$. In general $D = \bigcap \{D_M \mid M$ is a maximal t-ideal of $D\}$ [9, Prop. 4]. It suffices to show that each nonzero element of D lies in only finitely many maximal t-ideals M and that each D_M is a PID. The first requirement is satisfied as a result of Theorem 1.3, since t-invertibility implies quasi-finiteness, so that D is a TV-domain. The second follows from Lemma 2.2.

$(2) \Rightarrow (3)$. Let P be a prime t-ideal. Shrink P to a prime Q minimal over a nonzero principal ideal, so that Q is an associated prime of D. We claim that $P = Q$. If not choose $x \in P - Q$. Q, being t-invertible, is quasi-finite, whence $Q + xD$ is also quasi-finite. Hence $(Q+xD)_v \subseteq P$, and we may pick $u \in (Q+xD)^{-1} - D$. Let Q' be a minimal prime of $D:u$. Then $Q \subsetneqq Q'$ and both Q, Q' are associated, hence t-invertible, primes of D. Lemma 2.2 therefore implies that both $QD_{Q'}$ and $Q'D_{Q'}$ are principal in $D_{Q'}$, an impossibility.

(3) $\Rightarrow$ (4). This is Theorem 2.1. □

In [8] Glaz and Vasconcelos introduce the concept of an H-domain: a domain D in which every ideal A with $A^{-1} = D$ is quasi-finite. They then prove that a completely integrally closed H-domain is a Krull domain. An alternate proof can be obtained as follows. Let D be a completely integrally closed H-domain. If A is a nonzero ideal, then, by complete integral closure, $(AA^{-1})_v = D$. Hence, since D is an H-domain, AA^{-1} is quasi-finite, from which it follows that $(AA^{-1})_t = D$. Now apply (4) $\Rightarrow$ (1) of Theorem 2.3.

We close this section with more on the connection between H-domains and TV-domains.

PROPOSITION 2.4. *A domain D is an H-domain if and only if every maximal t-ideal of D is divisorial. Thus every TV-domain is an H-domain.*

Proof. Assume that D is an H-domain, and let M be a maximal t-ideal. It suffices to show that $M_v \neq D$. However, $M_v = D$ implies the existence of a finite subideal J of M with $J_v = D$. This is impossible, since we would then have $D = J_v = J_t \subseteq M$.

Conversely, assume that every maximal t-ideal of D is divisorial. Let A be an ideal of D with $A^{-1} = D$. Then $A \not\subseteq M$ for every maximal t-ideal M of D. Hence $A_t = D$, and the result follows. □

REMARK 2.5. It is not the case that every H-domain is a TV-domain; that is, divisoriality of the *maximal* t-ideals does not ensure divisoriality of *all* t-ideals. An example illustrating this may be constructed as follows. Let D be a Dedekind domain with infinitely many prime ideals. Let K be the quotient field of D, and let V be a valuation domain on K of the form $K+M$. Put $R = D+M$. By [3, Thm. 2.1] R is a Prüfer domain, and its (infinitely many) maximal t-ideals (that is, its maximal ideals) are invertible, hence divisorial. However, the prime t-ideal M is contained in all of the maximal t-ideals, so R is not a TV-domain by Theorem 1.3.

3. *TV*-Prüfer *v*-multiplication domains. We begin with some terminology. Let D be a domain. Then D is said to be a *v-domain* if every nonzero finitely generated ideal of D is v-invertible. D is a *Prüfer v-multiplication domain* (PVMD) if every nonzero finitely generated ideal of D is v-invertible with quasi-finite inverse, that is, if every finitely generated ideal of D is t-invertible. A useful characterization is as follows [9, Thm. 5]: D is a PVMD if and only if D_M is a valuation domain for each maximal t-ideal M of D. We show below that a TV-domain is a PVMD if and only if it is a v-domain. Finally, D is a *ring of Krull type* if D is a locally finite intersection of essential valuation overrings, equivalently [10] if D is a PVMD in which each nonzero element belongs to only finitely many maximal t-ideals. The ring D of Krull type is an *independent* ring of Krull type if each prime t-ideal of D lies in a unique maximal t-ideal.

We can now state our main result on TV PVMD's. The proof borrows heavily from ideas in [12].

THEOREM 3.1. *The following statements are equivalent for a domain D.*

(1) *D is a TV-domain which is also a v-domain.*
(2) *D is a TV-domain which is also a PVMD.*
(3) *D is an independent ring of Krull type whose maximal t-ideals are quasi-finite (and therefore t-invertible).*

Proof. (1) $\Rightarrow$ (2). If A is a finitely generated ideal of D, then $(AA^{-1})_v = D$. Since D is a TV-domain $(AA^{-1})_t = D$ also, so D is a PVMD.

(2) $\Rightarrow$ (3). By Theorem 1.3 and the discussion above, D is a ring of Krull type. By [13, Prop. 2.1] divisoriality of a maximal t-ideal of a PVMD is equivalent to quasi-finiteness and to t-invertibility. It remains to show that each prime t-ideal P of D is contained in a unique maximal t-ideal. Suppose, on the contrary, that $P \subseteq M \cap N$, where M, N are maximal t-ideals. Set $B = \bigcap_\alpha \{B_\alpha : B_\alpha$ is a t-ideal of D which contains P and is not contained in $M\}$. By Lemma 1.2, $B \not\subseteq M$. If $y \in B - M$ then $(P, y^2)_t \in \{B_\alpha\}$, whence $y \in (P, y^2)_t$. Hence $y \in J_v$ for some finitely generated subideal J of (P, y^2). It follows that $y \in J_v D_N \subseteq (J_v D_N)_v = (JD_N)_v = JD_N \subseteq (P, y^2)D_N$, where the v's on the parenthetical expressions denote the v-operation on the valuation domain D_N, the first equality follows from [21, Lemma 4], and the second equality follows from the fact that JD_N is principal. Hence there is an element $s \in D - N$ with $sy \in (P, y^2)$, say $sy = a + xy^2$ with $a \in P$ and $x \in D$. However, this gives $y(s - xy) = a \in P$, so that $s - xy \in P \subseteq N$, a contradiction because $y \in N$ (since $N \in \{B_\alpha\}$).

(3) $\Rightarrow$ (1). Any ring of Krull type D is a PVMD, hence a v-domain. Also $D = \bigcap_\alpha D_{M_\alpha}$, where $\{M_\alpha\}$ is the set of maximal t-ideals, and each D_{M_α} is a valuation domain [9]. Let A be a t-ideal. By [9, Prop. 4], $A = \bigcap AD_{M_\alpha} = \bigcap (AD_{M_\alpha} \cap D)$, so it suffices to show that each $AD_{M_\alpha} \cap D$ is divisorial. Let $M \in \{M_\alpha\}$. Since M is quasi-finite, MD_M is principal, so each nonzero ideal of D_M is divisorial [12, Lemma 5.2]. In particular, we may write $AD_M = \bigcap_\beta a_\beta D_M$, where each $a_\beta \in M$. Thus it suffices to show that $aD_M \cap D$ is divisorial for each $a \in M$. Let $J = aD_M \cap D$. One shows easily that J is a t-ideal. As in the proof of [12, Thm. 5.1: (2) $\Rightarrow$ (1)] we show that $J \not\subseteq N$ for each maximal t-ideal $N \neq M$. Clearly, the radical of J is a prime ideal P contained in M. Since the radical of a t-ideal is a t-ideal, we have by hypothesis that $P \not\subseteq N$, whence $J \not\subseteq N$ for each such N, as claimed. Since a is contained in only finitely many of the N, we may use prime avoidance to choose $b \in J$ so that $(a, b) \not\subseteq N$ for each N. It is then easy to see that $(a, b)_v = (a, b)_v D_M \cap D \supseteq aD_M \cap D = J$. Since J is a t-ideal, it follows that $J = (a, b)_v$, so J is divisorial. □

REMARK 3.2. In [12, Thm. 5.1] Heinzer showed that the class of integrally closed domains all of whose nonzero ideals are divisorial is a certain subclass of the class of Prüfer domains. This might lead one to suspect that an integrally closed TV-domain would necessarily be a PVMD. However, in [1, Examples 3.8] Barucci gives a construction which produces examples of integrally closed Mori (hence TV-) domains which are not Krull domains. (A specific example is $D = k + Yk[X, Y]_{(Y)}$, where k is a field and X and Y are indeterminates.) By [2, Thm. 2.5] localizations at t-invertible maximal t-ideals of such a domain D are rank-one discrete valuation domains. Since D is not a Krull domain, D must contain a non-t-invertible maximal t-ideal, so that D cannot be a PVMD by Theorem 3.1.

REMARK 3.3. The rings of Theorem 3.1 appear to be a natural generalization of Krull domains, and their study in the spirit of [10] may well be fruitful.

4. Comments, examples, and open questions. In this paper we have generalized many of the results in [12]. However, [12, Thm. 2.4] states that a nonzero prime ideal of a domain all of whose nonzero ideals are divisorial is contained in a unique maximal ideal. This does not generalize to TV-domains. Indeed, the structure of Noetherian domains leads one to expect that there would be plenty of Noetherian counterexamples. For the sake of completeness, we give below an explicit example of a Noetherian domain D which has a prime t-ideal contained in two maximal t-ideals.

EXAMPLE 4.1. Let X, Y, Z, V, W be independent indeterminates over the field k; let $R = k[X, Y, Z, V^2, V^3, W^2, W^3, XV, XW, XVW, YV, ZW]$; and let $R' = k[X, Y, Z, V, W]$. Set $P = (X, Y, V)R' \cap R$ and $Q = (X, Z, W)R' \cap R$. The following claims are easily verified.

Claim 1. R is Noetherian with integral closure R'.

Claim 2. The monomial $X^iY^jZ^kV^lW^m$ of R' lies in R if and only if exactly one of the following conditions holds:

(i) $l \neq 1$ and $m \neq 1$;
(ii) $l = 1$, $m \neq 1$, and $i + j \geq 1$;
(iii) $l \neq 1$, $m = 1$, and $i + k \geq 1$;
(iv) $l = m = 1$, $i + j \geq 1$, and $i + k \geq 1$.

Claim 3. Let $f \in R'$ and let $f = f_1 + \cdots + f_n$ denote the canonical representation of f as a sum of monomials. Then $f \in R$ if and only if each $f_r \in R$.

Claim 4. $XR' \subseteq R$, so $XR' = XR' \cap R$ is a (necessarily divisorial) height-one prime of R.

Claim 5.

$$P = XR' + (YR' \cap R) + (VR' \cap R) \quad \text{and} \quad Q = XR' + (ZR' \cap R) + (WR' \cap R).$$

We now assert that $P = R : V$ and $Q = R : W$. Granting this, P and Q are incomparable divisorial primes of R, and each contains the divisorial prime XR'. Since in the Noetherian case localization preserves divisoriality, $D = R_S$, where $S = R - (P \cup Q)$, is the required example.

We prove now that $P = R : V$; the proof that $Q = R : W$ is similar. To prove that $PV \subseteq R$, it suffices by Claim 5 to show that $XR' \cdot V \subseteq R$, $(YR' \cap R) \cdot V \subseteq R$, and $(VR'\cap R)\cdot V\subseteq R$. The first inclusion is obvious. Let $f\in YR'\cap R$. We may assume that f is a monomial, say $f=aX^iY^jZ^kV^lW^m\cdot Y$, $a\in k$. It is now straightforward to show that $fV\in R$, using Claim 2. The third inclusion is handled similarly.

Finally, suppose that $gV\in R$ and $g\in R$. We may assume that g is a monomial, say $g=aX^iY^jZ^kV^lW^m$. If $l>0$ then $g\in VR'\cap R\subseteq P$. If $l=0$ then, by Claim 2, $gV\in R$ implies that $i+j\geq 1$. Hence $g\in (X,Y)R'\cap R\subseteq P$. This completes the proof. □

REMARK 4.2. Let D be a TV-domain in which every prime t-ideal is contained in a unique maximal t-ideal. Then, with relatively minor modifications in the proof of [12, Thm. 3.6], we can show that each localization of D at a maximal t-ideal is a TV-domain. As Example 4.1 shows, however, this extra assumption is not necessary, since any localization of a Noetherian domain is a TV-domain. We have been unable to prove or disprove the following:

OPEN QUESTION. If M is a maximal t-ideal of the TV-domain D, is D_M necessarily a TV-domain? (By Corollary 1.8 and Proposition 2.4, D_M is an H-domain.)

From the proof of Theorem 3.1 (1) ⇒ (2), one is tempted to conjecture that if A is a nonzero finitely generated ideal in a TV-domain D then A^{-1} is quasi-finite. That this is false is demonstrated by the next example. Before presenting the example, we prove that every pseudo-valuation domain (PVD) is a TV-domain. PVD's were introduced in [11]; a PVD may be defined as a quasi-local domain (D,M) having a valuation overring V whose maximal ideal is also M. If (D,M) is a PVD which is not a valuation domain, then M^{-1} is the valuation overring with maximal ideal M.

PROPOSITION 4.3. *If D is a PVD which is not a valuation domain, then D is a TV-domain.*

Proof. Let M denote the maximal ideal of D, and let $V=M^{-1}$. It suffices to show that every nonprincipal t-ideal A of D is an ideal of V [11, Thm. 2.13]. Accordingly, let $a\in A$ and pick $b\in A-aD$. Then $(a,b)_v\subseteq A$. If $(a,b)=(c)$ is principal in D, write $a=rc$. Since $b\notin (a)$, $c\notin (a)$, whence $r\in M$. Therefore, $aV=rcV\subseteq Mc\subseteq A$, as desired. If (a,b) is not principal in D, then $(a,b)_v=(a,b)V$ [11, Prop. 2.14]. In this case $aV\subseteq (a,b)_v\subseteq A$, and the proof is complete. □

REMARK. Easy examples show that a quasi-local TV-domain need not be a PVD.

EXAMPLE 4.4. Let (D,M) be a PVD, not a valuation domain, such that M is not principal in $V=M^{-1}$. (If V is a nondiscrete valuation ring of the form $K+M$ and F is a proper subfield of the field K, then $D=F+M$ is such an example [11, Example 2.1].) We shall produce a finitely generated fractional ideal of D whose inverse is not quasi-finite. (The existence of such an integral ideal then follows easily.) Let $x\in V-D$. Then $(1,x)$ is the desired ideal. It is easy to see that $M=(1,x)^{-1}$. Suppose that M is quasi-finite, say $M=J_v$, where J is a finitely generated subideal of M. Since $ax \notin aD$ for each $a \in M$, M is not principal in D. Therefore [11, Prop. 2.14], $M = J_v = JV$, contradicting the nonfiniteness of M in V.

We now consider polynomial rings over TV-domains. We begin with a modification of [19, Lemme 2].

LEMMA 4.5. *Let A be a t-ideal of the integrally closed domain D.*

(1) *If $A \cap D \neq 0$ then $A = (A \cap D)D[x]$.*
(2) *If $A \cap D = 0$ then $A = fJD[x]$ for suitably chosen $f \in D[x]$ and fractional t-ideal J of D.*

Proof. (1) Pick $g \in A$ and $a \in A \cap D$, $a \neq 0$. Then $(a, g)_v \cap D \neq 0$, so by [19, Lemme 2(1)] $g \in (a, g)_v = ((a, g)_v \cap D)D[x] \subseteq (A \cap D)D[x]$.

(2) This requires only a minor modification of the proof of [19, Lemme 2(2)]. □

PROPOSITION 4.6. *If D is an integrally closed TV-domain, then so is $D[x]$.*

Proof. This follows easily from Lemma 4.5 and the fact that divisoriality is preserved upon passage from D to $D[x]$. □

We have not been able to remove the hypothesis that D be integrally closed. Therefore, we close with the following:

OPEN QUESTION. If D is a TV-domain, is $D[x]$ necessarily a TV-domain? (Note that by [8, 3.2c], $D[X]$ is an H-domain.)

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