CURVES LENGTH-MINIMIZING MODULO ν IN $\mathbf{R}^n$

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Introduction. Several vertical pegs inserted between two horizontal glass plates can bound strips of soap film which have certain length-minimization properties. Such strips often meet in threes at new junctions distinct from the original peg boundaries. The possibility of such junctions makes the problem of finding the film with a given peg boundary less tractable. Such triple boundary points can be studied conveniently using arithmetic modulo 3. In particular, Taylor [6] studied certain soap film singularities by classifying the singularities of area-minimizing 2-dimensional surfaces modulo 3 in $\mathbf{R}^n$.

A general theory of surfaces modulo ν, for any positive integer ν, has developed and been applied in more classical settings ([4], [7]). Yet for general positive integral ν there have been no complete classifications of singularities, even in the 1-dimensional case of curves length-minimizing modulo ν (defined below). (While a necessary condition on the interior has been known for some time [1], no sufficient condition was known.) We characterize both the interior and the boundary singularities (Theorem 4 below).

THEOREM. *A set of unit vectors with tails (resp., heads) at a common point, say the origin, minimizes length modulo ν if and only if the sum of the vectors has length less than or equal to $\nu - N$, where N is the number of rays comprising the cone:*

$$\left| \sum_{i=1}^{N} u_i \right| \leq \nu - N.$$

We begin by showing that the necessary condition on the interior derived from a first variational argument is sufficient as well. Then, by considering points on the boundary as interior points of other curves, we extend our theorem to the exterior.

As immediate corollaries, we have the following.

COROLLARY. *Let $\nu = 3$. If $N = 2$, the cone is length-minimizing if and only if $|u_1 + u_2| \leq 1$. A cone length-minimizing modulo 3 with 3 rays is planar and equiangular.*

Thus, a cone with 2 rays is length-minimizing modulo 3 if and only if the angle between the two rays is greater than or equal to $2\pi/3$. Three rays is the case of singularities arising from soap films between parallel plates.

COROLLARY. *Let $\nu = 4$. The only nontrivial case is $N = 3$, and it follows from the condition $|u_1 + u_2 + u_3| \leq 1$ that a cone in $\mathbf{R}^2$ with 3 rays is length-minimizing modulo 4 if and only if no included angle exceeds π.*

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The necessary condition on the interior is well known in a broader context by the methods of geometric measure theory; the results here, however, rely on much simpler methods. The sufficient condition, as well as the necessary condition on the exterior, are to my knowledge new and have not appeared before.

The first section of this paper contains definitions. Section 2 gives a complete, but surprisingly simple, classification of singularities.

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1. Definitions. A *curve* is a finite collection of nonconstant, continuous maps with positive integral multiplicities of the unit interval into $\mathbf{R}^n$. We call a component map of a curve a *basic curve,* and we call the curve *linear* if each of its basic curves is linear. In addition, we will allow ourselves the slight abuse of notation and refer occasionally to a *segment* of a linear curve as a synonym for a basic curve. (So as not to worry about the parameterization of curves, we will assume that linear curves are parameterized linearly. It will be clear that our results are invariant under certain changes of parameterization.)

Given a curve, we assign to each point in $\mathbf{R}^n$ a (*boundary*) *multiplicity* equal to the number of times it occurs as the image of one minus the number of times it occurs as the image of zero (cf. Figures 1 and 2). Notice that a point can occur with multiplicity zero. (In fact, only finitely many points may have nonzero multiplicities.)

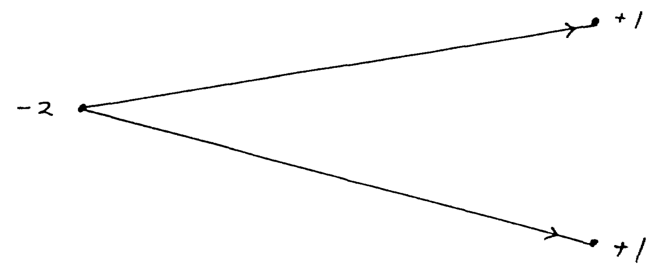

Figure 1 The least length directed curve joining three points.

Let P be the free $\mathbf{Z}$ module whose basis is the set of points in $\mathbf{R}^n$. The *boundary* of a basic curve f is $f(1)-f(0)$ (as an element of P). The boundary of a curve is the sum of the boundaries of the basic curves counted with multiplicity. Two curves have the *same boundary modulo* ν if the difference of their boundaries is in νP. Note that the definition of boundary is equivalent to stating that the boundary is the set of points with boundary multiplicity not equal to 0 modulo ν. It thus insists that the boundary be finite (and thus bounded as well).

The *length* of a curve is the sum of the lengths of each of its component maps times the multiplicity of the map. We may now define a curve to be *length-minimizing modulo* ν if there is no curve with the same boundary modulo ν having less length. The existence of such curves follows easily from a compactness argument.

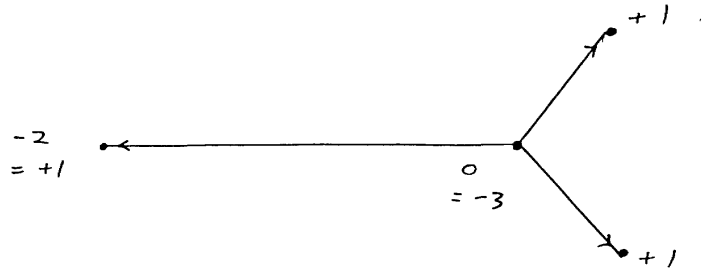

Figure 2 A curve length-minimizing modulo 3 with the same boundary modulo 3 as in Figure 1.

Our interest is to determine the conditions under which a given curve is length-minimizing modulo ν.

We define a *node* to be a point in the image of $\{0,1\}$ with boundary multiplicity an integral multiple of ν (cf. Figure 2). A *cone* is a curve composed entirely of unit rays all beginning or all ending at the origin.

2. Classification. Clearly, if a curve is length-minimizing then every subset of it is as well. Moreover, no curve (of positive length) without a boundary can be length-minimizing. In fact, only linear curves may be length-minimizing. We should note that, according to our definition, a curve with a node at the origin and boundary on the unit circle composed both of segments directed in and of segments directed out does not minimize length modulo ν, except in the trivial case that they all lie in a line. For otherwise one segment directed in and one directed out may be replaced by a single, shorter segment.

An important condition for length minimization is that the tangent cones to the nodes be length-minimizing.

LEMMA 1. *A cone of $k\nu$ $(k>0)$ segments is length-minimizing modulo ν only if $k=1$.*

Proof. If $k\neq 1$, then some subset of ν segments would lie in a halfspace not containing the node, violating our assumption of length minimization. □

The following fundamental lemma leads the way to our main result, Theorem 4.

LEMMA 2. *Consider a cone C centered at the origin with unit rays designated by $u_1, u_2, \ldots, u_N$ (see Figure 3). If C minimizes length modulo ν, then $|\sum_{i=1}^N u_i| \leq \nu - N$.*

Proof. Let $C'(t)$ be the linear curve with the same boundary as C, except with a node at some point $\pi = t\tau$ $(t\neq 0$, τ an arbitrary (unit) vector) and assigning the boundary multiplicity $\nu - N$ to the origin (see Figure 4). Let $l(t)$ be the length of $C'(t)$. Then $l(0)$ denotes the length of C itself, and

$$\begin{aligned} l(t) &= (\nu - N)|t\tau| + \sum_{i=1}^{N} |u_i - t\tau| \\ &= (\nu - N)t + \sum_{i=1}^{N} [u_i^2 - 2t\tau \cdot u_i + t^2\tau^2]^{1/2}. \end{aligned}$$

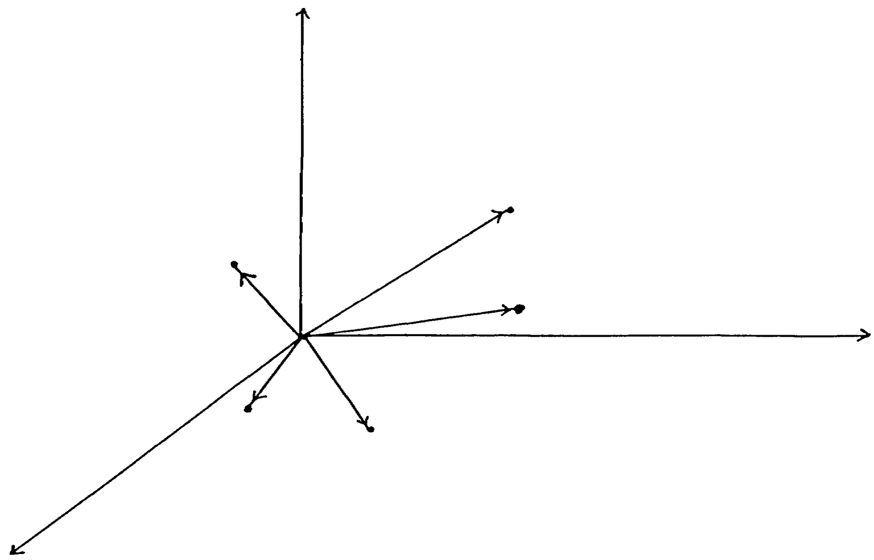

Figure 3 The original cone in Lemma 2.

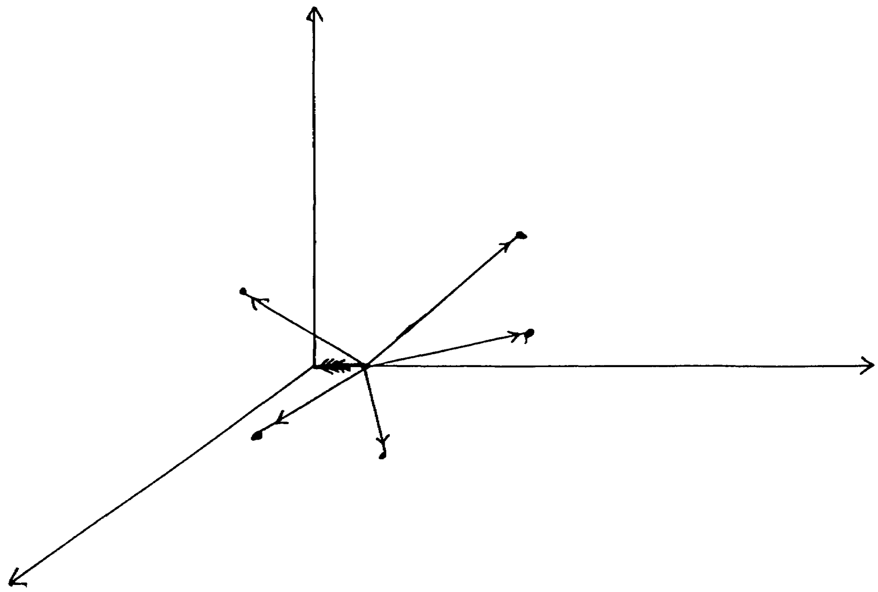

Figure 4 The new structure formed in Lemma 2.

Taking one-sided derivatives,

$$l'(t) = (\nu - N) - \sum_{i=1}^{N} \frac{2\tau \cdot u_i - 2t\tau^2}{2(1 - 2t\tau \cdot u_i + t^2\tau^2)^{1/2}}.$$

Thus,

$$l'(0) = (\nu - N) - \sum_{i=1}^{N} \tau \cdot u_i$$
$$= (\nu - N) - \tau \cdot \sum_{i=1}^{N} u_i.$$

Since τ was arbitrary, let us choose it so that τ is parallel to $\sum u_i$. Then $l'(0) = (\nu - N) - |\sum u_i|$, which is greater than or equal to 0 by hypothesis. □

Where $N = \nu$ (a node), the cone is length-minimizing modulo ν only if $|\sum_{i=1}^{N} u_i| \leq 0$. Thus, in order for a cone to be length-minimizing modulo ν it is necessary that the vector sum of the rays equal 0. In fact, it is sufficient as well.

THEOREM 3. *If* $N = \nu$ *then the cone is length-minimizing modulo* ν *if and only if* $\sum u_i = 0$.

Proof. By Lemma 2, a zero vector sum is a necessary condition for minimization modulo ν.

To show sufficiency, consider a length-minimizing comparison curve C. We may assume C is linear. We shall begin by showing that C has at most one node, which then must be at the center. For the sake of the following discussion, consider a boundary point of multiplicity d as $|d|$ separate boundary points.

Clearly, we may assume that C contains no circuits. If C has m nodes ($m \geq 2$), then at least two of them must have only one segment (of arbitrary multiplicity) not extending to the boundary, for otherwise C would contain some circuit, contrary to our hypothesis. Furthermore, since no segment may have multiplicity greater than or equal to $\nu/2$, each of these nodes must have segments extending to more than $\nu/2$ boundary points. Together, then, they must extend to more than ν boundary points. C, however, has only ν boundary points, so it may not have even two nodes. Therefore, C has exactly one node.

To show that the one node must be at the center, let us consider creating the comparison curve C by moving the node from the center along the x-axis (as in Lemma 2). Then the vector sum of the rays will have a negative x component that can only equal 0 when the node is at the origin. Since the initial orientation of the cone with respect to the x-axis was arbitrary, C is the original cone. □

We thus have a necessary and sufficient condition on the interior points for any curve to be length-minimizing modulo ν. Moreover, it follows immediately that any boundary point which is the junction of $N > \nu$ segments cannot minimize length modulo ν.

We generalize our theorem to all exterior points as well with our main result.

THEOREM 4. *Given a cone of* N *segments (counting multiplicities), it is length-minimizing modulo* ν *if and only if* $|\sum_{i=1}^{N} u_i| \leq \nu - N$.

Proof. By Lemma 2, if the cone is length-minimizing then $|\sum_{i=1}^{N} u_i| \leq \nu - N$.

To prove the converse, suppose $|\sum_{i=1}^{N} u_i| \leq \nu - N$. If $N \neq \nu - 1$, then some set of $\nu - N$ vectors will satisfy

$$\sum_{i=1}^{N} u_i = -\sum_{i=N+1}^{\nu} u_i.$$

Then the ν vectors form a cone. Furthermore, $\sum_{i=1}^{\nu} u_i = 0$, and so the cone is length-minimizing. Moreover, any subset of it is length-minimizing as well, and in particular our original cone.

If $N = \nu - 1$, on the other hand, then consider the cone created by doubling the multiplicity of the original cone. Clearly

$$\left|\sum_{i=1}^{2N} u_i\right| = 2\left|\sum_{i=1}^{N} u_i\right| \le 2(\nu - N) = 2\nu - 2N.$$

Thus, it is length-minimizing modulo 2ν. By the following lemma (Lemma 5), then, so is the original cone modulo ν. □

LEMMA 5. *A cone is length-minimizing modulo ν if the cone with all multiplicities doubled is length-minimizing modulo 2ν.*

Theorem 4 provides us with a local characterization of all curves which minimize length modulo ν.

Our results for curves defined as finite collections of images of the unit interval actually give a complete description of the interior and boundary singularities in the general class of length-minimizing flat chains modulo ν with finite boundaries in $\mathbf{R}^n$. (For definitions and examples see [2, §4.2.26] and [3].) At an interior singularity, a length-minimizing flat chain modulo ν can be viewed as a stationary varifold with integer multiplicities. Work of Allard and Almgren [1, §5] shows that such singularities are isolated points where a finite number of line segments meet. Hence, our results apply. At a boundary singularity, say the origin, adding to the varifold its image under the map $x \to -x$ yields a stationary varifold. Again, by [1, §5], the stationary varifold—and hence also the original one—consist locally of a finite number of line segments, and our results apply.

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