THE NUMERICAL RADIUS OF A COMMUTING PRODUCT

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The numerical radius $w(T)=\sup\{|(Tx,x)|,\ x\in H,\ \|x\|=1\}$ is, apart from the norm and the spectral radius, one of the most important constants attached to a bounded operator T in a complex Hilbert space H. There exists an extensive theory concerning the numerical radius and its relations to the norm.

In the present paper we investigate the following question:

(1) Is it true that $w(TS)\le w(T)\|S\|$ for all commuting operators T and S in a Hilbert space?

This question was probably first considered by Holbrook ([5], [6]) and further studied (usually in the more general context of operator radii w_ρ and C_ρ contractions) by a number of authors (see e.g. [1], [2], [3], [8]). For more about the history and motivations of the problem see [7].

Conjecture (1) looks very reasonable as there are positive results which indicate that the inequality might be true. The inequality is true if T and S are doubly commuting; that is, if $TS=ST$ and $TS^*=S^*T$ (Holbrook [5], Sz.-Nagy [8]). Also, if S is an isometry then $w(TS)\le w(T)$ (Bouldin [2]). Finally, if T and S are arbitrary commuting operators then $w(TS)\le 1.169w(T)\|S\|$ (due to Crabb, communicated by Ando and Okubo [1]).

The aim of this paper, however, is to show that the conjecture is false in general. We exhibit an example of two operators T,S in a 12-dimensional Hilbert space H such that $TS=ST$, $\|S\|\le 1$, $w(T)\le 1$, and $w(TS)>1$. Using a result of Holbrook [7] it is even possible to assume that S is a polynomial of T.

The Hilbert spaces considered in this paper are complex, but the constructed example works in real Hilbert space also.

We start with the following well-known lemma:

LEMMA. *Let n be a positive integer, and let a_{ij} $(i,j=1,\dots,n)$ be complex numbers such that the matrix $(a_{i,j})_{i,j=1}^n$ is positive definite. Then there exist a Hilbert space H $(\dim H=n)$ and linearly independent elements $x_1,\dots,x_n\in H$ such that $(x_i,x_j)=a_{j,i}$ $(i,j=1,\dots,n)$.*

Proof. Let H be an n-dimensional linear space with a basis $x_1,\dots,x_n$. Define the scalar product $(\cdot,\cdot)$ on H by

$$\left(\sum_{i=1}^n \alpha_i x_i,\ \sum_{j=1}^n \beta_j x_j\right)=\sum_{i,j=1}^n \alpha_i\bar{\beta}_j a_{j,i}\quad (\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n\in\mathbf{C}).$$

Clearly, H will become a Hilbert space, $\dim H=n$, and $(x_i,x_j)=a_{j,i}$ for $i,j=1,\dots,n$.

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We shall repeatedly use the classical characterization of positive definite matrices (see e.g. [4, Chap. X, §4]). A self-adjoint matrix $A=(a_{i,j})_{i,j=1}^n$ is positive definite if and only if $D_k=\det(a_{i,j})_{i,j=1}^k>0$ $(k=1,\dots,n)$.

THEOREM. *There exist a Hilbert space H* ($\dim H=12$) *and commuting operators T and S on H such that $w(T)\le 1$, $\|S\|\le 1$, and $w(TS)>1$.*

Proof. Let $H_1,\dots,H_6$ be Hilbert spaces, $\dim H_1=\dim H_6=1$, $\dim H_2=\dim H_5=2$, and $\dim H_3=\dim H_4=3$. Let $H=\bigoplus_{i=1}^6 H_i$ be their orthogonal sum, $\dim H=12$. Take $e_1\in H_1$, $e_2,e_3\in H_2$, $e_4,e_5,e_6\in H_3$, $e_7,e_8,e_9\in H_4$, $e_{10},e_{11}\in H_5$, and $e_{12}\in H_6$ such that

$$\begin{aligned}
&\|e_1\|^2=0.28,\\
&\|e_2\|^2=1.9,\quad \|e_3\|^2=0.28,\quad (e_2,e_3)=0.24,\\
&\|e_4\|^2=1,\quad \|e_5\|^2=1.9,\quad \|e_6\|^2=0.28,\quad (e_4,e_5)=1.01,\\
&(e_5,e_6)=0.24,\quad (e_4,e_6)=0.13,\\
&\|e_7\|^2=0.9,\quad \|e_8\|^2=1.5,\quad \|e_9\|^2=0.28,\quad (e_7,e_8)=0.81,\\
&(e_8,e_9)=0.24,\quad (e_7,e_9)=0.13,\\
&\|e_{10}\|^2=0.57,\quad \|e_{11}\|^2=0.46,\quad (e_{10},e_{11})=0.28,\\
&\|e_{12}\|^2=0.29.
\end{aligned}\tag{2}$$

It is possible to choose $e_1,\dots,e_{12}$ satisfying (2) since the matrices

$$A_2=\begin{pmatrix}1.9 & 0.24\\ 0.24 & 0.28\end{pmatrix},\qquad A_3=\begin{pmatrix}1 & 1.01 & 0.13\\ 1.01 & 1.9 & 0.24\\ 0.13 & 0.24 & 0.28\end{pmatrix},$$

$$A_4=\begin{pmatrix}0.9 & 0.81 & 0.13\\ 0.81 & 1.5 & 0.24\\ 0.13 & 0.24 & 0.28\end{pmatrix},\quad \text{and}\quad A_5=\begin{pmatrix}0.57 & 0.28\\ 0.28 & 0.46\end{pmatrix}$$

are positive definite; it is easy to check that

$$\det A_2>0,\qquad \det\begin{pmatrix}1 & 1.01\\ 1.01 & 1.9\end{pmatrix}>0,\qquad \det A_3=0.219686>0,$$

$$\det\begin{pmatrix}0.9 & 0.81\\ 0.81 & 1.5\end{pmatrix}>0,\qquad \det A_4=0.167646>0,\qquad \det A_5>0.$$

Clearly, the elements $e_1,\dots,e_{12}$ are linearly independent. Define the linear operators $T,S\colon H\to H$ by

$$\begin{array}{llll}
Te_1=0, & Te_2=e_1, & Te_3=0, & Te_4=e_2,\\
Te_5=e_3, & Te_6=0, & Te_7=e_5, & Te_8=e_6,\\
Te_9=0, & Te_{10}=e_8, & Te_{11}=e_9, & Te_{12}=e_{11},\\
Se_1=e_3, & Se_2=e_5, & Se_3=e_6, & Se_4=e_7,\\
Se_5=e_8, & Se_6=e_9, & Se_7=e_{10}, & Se_8=e_{11},\\
Se_9=0, & Se_{10}=e_{12}, & Se_{11}=0, & Se_{12}=0.
\end{array}$$

It is easy to check that $TS=ST$, as

$$\begin{aligned}
&TSe_1=STe_1=0, && TSe_2=STe_2=e_3,\\
&TSe_3=STe_3=0, && TSe_4=STe_4=e_5,\\
&TSe_5=STe_5=e_6, && TSe_6=STe_6=0,\\
&TSe_7=STe_7=e_8, && TSe_8=STe_8=e_9,\\
&TSe_9=STe_9=0, && TSe_{10}=STe_{10}=e_{11},\\
&TSe_{11}=STe_{11}=0, && TSe_{12}=STe_{12}=0.
\end{aligned}$$

Further, $\|e_4\|=1$ and $(TSe_4,e_4)=(e_5,e_4)=1.01$; hence $w(TS)\geq 1.01$. We prove now $\|S\|\leq 1$. Clearly, $SH_i\subset H_{i+1}$ $(i=1,\dots,5)$ and $SH_6=\{0\}$. Denote by P_i the orthogonal projection onto H_i $(i=1,\dots,6)$.

Let $z\in H$ and $z_i=P_iz$ $(i=1,\dots,6)$. Because

$$\|z\|^2=\sum_{i=1}^{6}\|z_i\|^2 \quad\text{and}\quad \|Sz\|^2=\sum_{i=1}^{6}\|Sz_i\|^2,$$

it is sufficient to prove $\|Sz_i\|^2\leq\|z_i\|^2$ $(i=1,\dots,6)$. This is clear for $i=6$ as $Sz_6=0$. Further, S/H_1 and S/H_2 are isometries; that is, $\|Sz_1\|^2=\|z_1\|^2$ and $\|Sz_2\|^2=\|z_2\|^2$.

Let $z_3=\alpha e_4+\beta e_5+\gamma e_6$ $(\alpha,\beta,\gamma\in\mathbf{C})$. Then

$$\|z_3\|^2=|\alpha|^2+1.9|\beta|^2+0.28|\gamma|^2+1.01(\alpha\bar\beta+\beta\bar\alpha)+0.24(\beta\bar\gamma+\gamma\bar\beta)+0.13(\alpha\bar\gamma+\gamma\bar\alpha)$$

and

$$\begin{aligned}\|Sz_3\|^2=\|\alpha e_7+\beta e_8+\gamma e_9\|^2&=0.9|\alpha|^2+1.5|\beta|^2+0.28|\gamma|^2\\
&\quad+0.81(\alpha\bar\beta+\beta\bar\alpha)+0.24(\beta\bar\gamma+\gamma\bar\beta)+0.13(\alpha\bar\gamma+\gamma\bar\alpha).\end{aligned}$$

Therefore $\|z_3\|^2-\|Sz_3\|^2=0.1|\alpha|^2+0.4|\beta|^2+0.2(\alpha\bar\beta+\beta\bar\alpha)=0.1|\alpha+2\beta|^2\geq 0$ for every $\alpha,\beta,\gamma\in\mathbf{C}$. Similarly, let $z_4=\alpha e_7+\beta e_8+\gamma e_9$ $(\alpha,\beta,\gamma\in\mathbf{C})$. Then

$$\begin{aligned}\|z_4\|^2-\|Sz_4\|^2&=\|\alpha e_7+\beta e_8+\gamma e_9\|^2-\|\alpha e_{10}+\beta e_{11}\|^2\\
&=0.33|\alpha|^2+1.04|\beta|^2+0.28|\gamma|^2\\
&\quad+0.53(\alpha\bar\beta+\beta\bar\alpha)+0.24(\beta\bar\gamma+\gamma\bar\beta)+0.13(\alpha\bar\gamma+\gamma\bar\alpha)\\
&=0.13|\alpha+\beta+\gamma|^2+0.11|\beta+\gamma|^2+0.2|\alpha+2\beta|^2+0.04|\gamma|^2\geq 0\end{aligned}$$

for every $\alpha,\beta,\gamma\in\mathbf{C}$.

Finally, let $z_5=\alpha e_{10}+\beta e_{11}$, $\alpha,\beta\in\mathbf{C}$. Then

$$\begin{aligned}\|z_5\|^2-\|Sz_5\|^2=\|\alpha e_{10}+\beta e_{11}\|^2-\|\alpha e_{12}\|^2&=0.28|\alpha|^2+0.46|\beta|^2+0.28(\alpha\bar\beta+\beta\bar\alpha)\\
&=0.28|\alpha+\beta|^2+0.18|\beta|^2\geq 0\end{aligned}$$

for every $\alpha,\beta\in\mathbf{C}$. Thus $\|S\|\leq 1$.

It remains to prove $w(T)\leq 1$. Let $\epsilon\in\mathbf{C}$, $|\epsilon|=1$. Define the operator $U_\epsilon: H\to H$ by $U_\epsilon z=\sum_{i=1}^{6}\epsilon^iP_iz$ $(z\in H)$. Obviously, U_ϵ is a unitary operator, and $U_\epsilon^{-1}z=\sum_{i=1}^{6}\bar\epsilon^iP_iz$ $(z\in H)$. Further, $U_\epsilon^{-1}TU_\epsilon z=\sum_{i=1}^{6}\epsilon^iU_\epsilon^{-1}TP_iz=\sum_{i,j=1}^{6}\epsilon^i\bar\epsilon^jP_jTP_iz=\sum_{i=2}^{6}\epsilon P_{i-1}TP_iz=\epsilon T(\sum_{i=2}^{6}P_iz)=(\epsilon T)z$ (we used the inclusions $TH_i\subset H_{i-1}$ $(i=2,\dots,6)$ and $TH_1=\{0\}$, which follow from the definition of T).

The operators T and ϵT are unitarily equivalent and their numerical ranges coincide. The set $W(T)=\{(Tx,x),\ x\in H,\ \|x\|=1\}$ is circularly symmetric and the condition $w(T)\leq 1$ is equivalent to the condition $\mathrm{Re}(Tx,x)\leq\|x\|^2$ $(x\in H)$.

For $x\in H$, let $x=\sum_{i=1}^{12}\alpha_i e_i$ $(\alpha_i\in\mathbf{C},\ i=1,\dots,12)$. Then

$$\begin{aligned}\|x\|^2-\mathrm{Re}(Tx,x)&=\sum_{i,j=1}^{12}[\alpha_i\bar{\alpha}_j(e_i,e_j)-\mathrm{Re}\,\alpha_i\bar{\alpha}_j(Te_i,e_j)]\\&=\sum_{i,j=1}^{12}\left[\alpha_i\bar{\alpha}_j(e_i,e_j)-\frac{\alpha_i\bar{\alpha}_j+\alpha_j\bar{\alpha}_i}{2}(Te_i,e_j)\right].\end{aligned}$$

The last expression is nonnegative for all $\alpha_1,\dots,\alpha_{12}\in\mathbf{C}$ if and only if the matrix

$$\left(\begin{array}{ccc|ccc|ccc|ccc}
.28 & -.14 & & & & & & & & & & \\
-.14 & 1.9 & .24 & -.95 & -.12 & & & & & & & \\
 & .24 & .28 & -.12 & -.14 & & & & & & & \\
\hline
 & -.95 & -.12 & 1 & 1.01 & .13 & -.505 & -.065 & & & & \\
 & -.12 & -.14 & 1.01 & 1.9 & .24 & -.95 & -.12 & & & & \\
 & & & .13 & .24 & .28 & -.12 & -.14 & & & & \\
\hline
 & & & -.505 & -.95 & -.12 & .9 & .81 & .13 & -.405 & -.065 & \\
 & & & -.065 & -.12 & -.14 & .81 & 1.5 & .24 & -.75 & -.12 & \\
 & & & & & & .13 & .24 & .28 & -.12 & -.14 & \\
\hline
 & & & & & & -.405 & -.75 & -.12 & .57 & .28 & -.14 \\
 & & & & & & -.065 & -.12 & -.14 & .28 & .46 & -.23 \\
 & & & & & & & & & -.14 & -.23 & .29
\end{array}\right)$$

is positive semidefinite (the remaining terms of the matrix are equal to 0).

It is sufficient to show that the principal minors in the upper left-hand corner $D_1,\dots,D_{12}$ are positive. By calculation we get

$$\begin{aligned}
D_1&=0.28, & D_7&\doteq 3.0880\cdot 10^{-4},\\
D_2&=0.5124, & D_8&\doteq 3.1358\cdot 10^{-6},\\
D_3&\doteq 0.1273, & D_9&\doteq 7.3275\cdot 10^{-7},\\
D_4&\doteq 6.4531\cdot 10^{-2}, & D_{10}&\doteq 9.2473\cdot 10^{-8},\\
D_5&\doteq 3.1638\cdot 10^{-3}, & D_{11}&\doteq 5.9969\cdot 10^{-10},\\
D_6&\doteq 7.7947\cdot 10^{-4}, & D_{12}&\doteq 6.2506\cdot 10^{-11}.
\end{aligned}$$

Therefore the matrix is positive definite and $w(T)\leq 1$ (in fact, even $w(T)<1$). □

REMARK 1. The example was obtained with the essential help of a computer. Because it is not easy to estimate the cumulative rounding error when computing a determinant of higher order and because the determinants above are very close to zero, it is possible to doubt whether the determinants are actually positive. Fortunately, it is possible (after some effort) to obtain the exact values of the determinants by direct calculation and to check that they are indeed positive. For example, the exact value of D_{12} is

$$D_{12}=6.250\,617\,295\,574\,4\cdot 10^{-11}.$$

REMARK 2. It is a natural question to ask what is the best (= the smallest) constant C such that $w(TS) \leq Cw(T)\|S\|$ for all commuting operators T and S. The present example and the result of Crabb give $1.01 < C < 1.169$.

REMARK 3. As pointed out by the referee, the above example does not enable one to see how it was obtained; we shall indicate that briefly in this remark.

We are looking for commuting operators T and S in a Hilbert space H and an element $x \in H$, $\|x\| = 1$, such that $\|S\| < 1$, $w(T) \leq 1$, and $(TSx, x) > 1$. Clearly, we may assume that $H = \bigvee_{i,j=0}^{\infty} T^i S^j x$; that is, we must determine the scalar products $(T^i S^j x, T^k S^l x)$ for all i, j, k, l. Using an "orthogonalization technique" analogous to that developed by Crabb (see the above-mentioned inequality in [1]), one can assume $(T^i S^j x, T^k S^l x) = 0$ for $i - j \neq k - l$. (On the other hand, we cannot require this for $i - j = k - l$ as (TSx, x) should be nonzero.)

Now we should like to make the example as simple as possible. The assumption $T^2 x = 0$ is too strong as then $T^2 = 0$, $T^{*2} = 0$, and Crabb's estimates give $w(TS) \leq 1$. If $T^3 x = 0$ then Crabb's proof gives a better estimate than the general $w(TS) \leq 1.169$, but it does not give $w(TS) \leq 1$. So it is reasonable to put $T^3 x = 0$ (then $w(TS)$ will necessarily be very close to 1). Again Crabb's proof gives $T^2 S^2 x \neq 0$ (otherwise $w(TS) \leq 1$); therefore $T^i S^j x \neq 0$ for $0 \leq i, j \leq 2$ and we are looking for an example of at least 9 dimensions. Actually the example was found on the 12-dimensional Hilbert space $H = \bigvee_{i=0}^{2} \bigvee_{j=0}^{3} T^i S^j x$ (these vectors $T^i S^j x$ are denoted by $e_1, \ldots, e_{12}$ in the construction). It remains to find the scalar products $(T^i S^j x, T^k S^j x)$, $0 \leq i, k \leq 2$, $0 \leq j, l \leq 3$, $i - j = k - l$ so that these vectors would form a Hilbert space $(TSx, x) > 1$, $\|S\| \leq 1$, and $w(T) \leq 1$. All these conditions but the last one are rather easy to check. The condition $w(T) \leq 1$ reduces to the positivity of some determinants, which was checked using a computer.

An interesting question is to ask what is the minimal dimension of a Hilbert space H on which a similar example can be constructed. The previous reasoning gives no estimate from below, as the dimension might have increased in the process of orthogonalization. The only known result in this direction is that such an example cannot be found in a 2-dimensional Hilbert space (T. Ando, personal communication).

In [7], the following result was proved: If T and S are commuting operators in a finite-dimensional Hilbert space H and if $\epsilon > 0$, then there exist an operator T' in H and a polynomial p such that $\|T - T'\| < \epsilon$ and $S = p(T')$. Using this result we obtain

COROLLARY. *There exist an operator A in the* 12-*dimensional Hilbert space H and a polynomial p such that*

$$w(A) \leq 1, \quad \|p(A)\| \leq 1, \quad \textit{and} \quad w(Ap(A)) > 1.$$

Similarly, there exist an operator B and a polynomial q such that $\|B| \leq 1$, $w(q(B)) \leq 1$, and $w(Bq(B)) > 1$.

Proof. Let H, T, and S be as before. Put $\epsilon = 1 - w(T) > 0$. Then the first part of the corollary follows immediately from the above-mentioned result. The second part can be proved analogously by considering the operators $S \cdot w(T)$ and $T/w(T)$ instead of T and S. □

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