A NEWLANDER-NIRENBERG THEOREM FOR MANIFOLDS WITH BOUNDARY

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The Newlander-Nirenberg theorem [7] states that if M is a manifold with an integrable almost complex structure, then M is actually a complex manifold. Thus, about any $z_0 \in M$ there exist coordinate functions $z_1, \dots, z_n$ such that the almost complex structure defined by $z_1, \dots, z_n$ coincides with the given almost complex structure on M. If $\bar{M}$ is a manifold with smooth boundary such that the almost complex structure extends smoothly to the boundary, then it is natural to ask if the assumption of integrability still implies the conclusions of the Newlander-Nirenberg theorem. Hill [4] has constructed counterexamples showing that such a theorem does not hold in general. In this paper we show that the analog of the Newlander-Nirenberg theorem does hold if the boundary of $\bar{M}$ is pseudoconvex near z_0.

We now state precisely what we mean by an almost complex structure that extends smoothly to the boundary. Suppose that $\bar{M}$ is a manifold of real dimension $2n$ with smooth boundary, and let U be a neighborhood in the relative topology of $\bar{M}$ of a given boundary point z_0. We shall say that an *almost complex structure* is defined in U if there exists a subbundle $\mathcal{L}$ of fiber dimension n of the complexified tangent bundle $\mathbf{C}T(\bar{M})$ such that for each $z \in U$, $\mathcal{L}_z \cap \bar{\mathcal{L}}_z = 0$. The structure is said to be *integrable* if $\mathcal{L}$ is closed under brackets; that is, if L' and L'' are arbitrary sections of $\mathcal{L}$, then $[L', L'']$ is again a section of $\mathcal{L}$.

Observe that if $\bar{N}$ is a smoothly bounded complex manifold, then the bundle $T^{1,0}$ of holomorphic tangent vectors defines an integrable almost complex structure which is called the complex structure of $\bar{N}$. To show that $\bar{M}$ possesses a complex structure, it suffices to construct, in a neighborhood U of each point $z_0 \in \bar{M}$, a set of smooth functions $f_1, \dots, f_n$ with linearly independent differentials such that each function f_j is "holomorphic" with respect to the almost complex structure, that is, such that $\bar{L} f_j \equiv 0$ for every section L of $\mathcal{L}$. In fact, if we view $f = (f_1, \dots, f_n)$ as a coordinate map into $\mathbf{C}^n$, then the bundle $\mathcal{L}$ satisfies $f_* \mathcal{L} = T^{1,0}$, which is the complex structure of $\mathbf{C}^n$. Thus near z_0 we may view $\bar{M}$ as a complex manifold with smooth boundary.

On an integrable almost-complex manifold, the usual $\bar{\partial}$-formalism carries through with no changes. If $\mathcal{L}_z^p$ denotes the p-fold product $\mathcal{L}_z \oplus \cdots \oplus \mathcal{L}_z$, then $\Lambda_z^{p,q}$ is the space of alternating tensors on $\mathcal{L}_z^p \oplus \bar{\mathcal{L}}_z^q$. Because of the integrability assumption, it follows that if w is a section of $\Lambda^{p,q}$ then dw can be written as a sum $\partial w + \bar{\partial} w$, where $\partial w \in \Lambda^{p+1,q}$ and $\bar{\partial} w \in \Lambda^{p,q+1}$. From this one also obtains the familiar identity $\bar{\partial} \circ \bar{\partial} = 0$.

Now suppose that $r(z)$ is a boundary-defining function for $\bar{M}$. This means that $r < 0$ on M, $bM = \{z \in \bar{M};\, r(z) = 0\}$, and $dr(z) \neq 0$ when $z \in bM$. We say that bM

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is pseudoconvex if the form $\partial\bar{\partial}r(L,\bar{L})$ is nonnegative for all $L \in \mathcal{L}_z \cap \mathbf{C}T_z(bM)$, $z \in bM$. In this paper we will prove the following result.

THEOREM. *Let $\bar{M}$ denote a $2n$-real dimensional manifold with smooth boundary. Let z_0 be a given point in bM and suppose that there is a neighborhood U of z_0 such that $\bar{M} \cap U$ has an integrable almost complex structure that is smooth up to the boundary $bM \cap U$. Suppose further that, with respect to this structure, $bM \cap U$ is pseudoconvex. Then there exist a neighborhood U_1 of z_0 with $U_1 \subset U$ and functions $f_j \in C^\infty(U_1)$, $j=1,\dots,n$, such that $\bar{\partial}f_j = 0$, $j=1,\dots,n$, and such that the differentials of f_j at z_0 are linearly independent, $j=1,\dots,n$. Thus, $\bar{M}$ is a smoothly bounded complex manifold near z_0.*

Proof. The main idea is to adapt the proof of Theorem 5.2.10 of Hörmander [5], which is the result of Grauert that if a complex manifold admits a strongly plurisubharmonic exhaustion function, then it is a Stein manifold; in particular, there are globally defined functions which form a local coordinate system near any given point. In our case the point in question will be in the boundary of an integrable almost complex manifold. Since Hörmander's proof is based on weighted $\bar{\partial}$-estimates, it must be modified so that the solutions of the $\bar{\partial}$-equation are smooth up to the boundary. This means we must combine Hörmander's proof with Kohn's proof of the existence of smooth solutions (up to the boundary) of the $\bar{\partial}$-equation on pseudoconvex domains [6]. Since this was already done by the author in [3], we will not need to prove any new $\bar{\partial}$-estimates.

The first step is to show that we can replace the local assumption that bM is pseudoconvex near z_0 with the stronger assumption that bM is smooth and pseudoconvex everywhere. To do this, we first choose a sufficiently small neighborhood U of z_0, where we can easily construct a smooth strictly plurisubharmonic function λ on U (this means that $\partial\bar{\partial}\lambda$ is a positive definite form on $\mathcal{L}_z$, $z \in U$). We can also assume that a smooth Hermitian metric is defined in $U \cap \bar{M}$.

Let $r(z)$ denote any defining function for bM in U. The argument of Range in [8] applies without change to the case of an integrable almost complex manifold to show that, for suitably chosen positive constants c and η, the function $-e^{-c\lambda}(-r)^\eta$ is a bounded strictly plurisubharmonic exhaustion function in $M \cap U$. By using this function in the same way as in the lemma in §4 of Bell [2], one obtains the following proposition (this result was first obtained by Amar [1]).

PROPOSITION 1. *There exists a neighborhood V of z_0 (in the relative topology of $\bar{M}$) such that $\bar{V} \subset\subset U$, and such that if $N = V \cap M$ then N has smooth pseudoconvex boundary. Thus N is a small open subset of $M \cap U$ such that its boundary is smooth and pseudoconvex and coincides with the boundary of M near z_0.*

By working on the manifold $\bar{N}$, we can apply the machinery of the $\bar{\partial}$-Neumann problem and obtain smooth solutions of the $\bar{\partial}$-equation. For any $f \in L^2(N)$, we define the weighted norm

$$\|f\|_\varphi^2 = \int_N |f|^2 e^{-\varphi}\, dV, \tag{1}$$

where φ is a smooth function on $\bar{N}$ and dV is the volume form of the metric on N. If we assume that the neighborhood was chosen to be sufficiently small, then we can assert that there exist smooth sections $L_1, \ldots, L_n$ of $\mathcal{L}$ on $\bar{N}$ such that at each $z \in \bar{N}$, $L_1, \ldots, L_n$ form an orthonormal basis of $\mathcal{L}_z$. Let $\omega^1, \ldots, \omega^n$ denote the corresponding dual basis of $\Lambda^{1,0}$. For $\alpha = \sum_{k=1}^n \alpha_k \bar{\omega}^k$, we define

$$\|\alpha\|_\varphi^2 = \int_N \sum_{k=1}^n |\alpha_k|^2 e^{-\varphi}\, dV. \tag{2}$$

We perturb λ somewhat so that it satisfies $d\lambda(z_0) \neq 0$, $\lambda(z_0) = 0$, and so that λ is still smooth and strongly plurisubharmonic on $\bar{N}$. Let $\chi(\tau)$ denote a convex function such that $\chi(\eta) = 0$ for $\eta < 0$ and $\chi(\eta) > 0$ for $\eta > 0$. For arbitrary parameters s, t, and b, define

$$\varphi_{s,t,b}(z) = s\lambda(z) + t\chi(\lambda(z) - b). \tag{3}$$

We may assume that U was chosen to be sufficiently small so that there are smooth coordinates $(x_1, \ldots, x_{2n})$ defined on $\bar{N}$. (Thus $\bar{N}$ is diffeomorphic to a smoothly bounded domain in $\mathbf{R}^{2n}$.) With respect to these coordinates we define weighted Sobolev norms for any nonnegative integer m by

$$\|f\|_{m,s,t,b}^2 = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{\varphi_{s,t,b}}^2,$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_{2n}^{\alpha_{2n}}}.$$

Similarly, if $g = \sum_{k=1}^n g_k \bar{\omega}^k$ then

$$\|g\|_{m,s,t,b}^2 = \sum_{|\alpha| \leq m} \sum_{k=1}^n \|D^\alpha g_k\|_{\varphi_{s,t,b}}^2.$$

Let $H^m(N)$ and $H^m_{(0,1)}(N)$ denote the spaces of functions and $(0,1)$-forms respectively, such that (1) and (2) are finite.

In Proposition 2.2.3 of [3], the author proved the following result.

PROPOSITION 2. *Let $\bar{N}$ be a smoothly bounded manifold that admits an integrable almost complex structure. Suppose that bN is pseudoconvex and that there exists a smooth strictly plurisubharmonic function λ on $\bar{N}$. Suppose that $\varphi_{s,t,b}$ is defined as in* (3) *and that the function χ used in* (3) *is C^{m+2}. Then there exist constants $s(m)$ and C_m such that if $s \geq s(m)$ and $t \geq 0$, and if g is a $\bar{\partial}$-closed form in $H^m_{(0,1)}(N)$, then there exists a solution $u \in H^m(N)$ of $\bar{\partial}u = g$ that satisfies*

$$\|u\|_{m,s,t,b}^2 \leq C_m(1+s+t)^{2m} \|g\|_{m,s,t,b}^2. \tag{4}$$

The solution u, which depends on the choice of s, t, and b, is the $\bar{\partial}$-Neumann solution corresponding to the weight function $\varphi_{s,t,b}$. We wish to point out that Proposition 2.2.3 of [3] was actually only proved when $\bar{N}$ is a smoothly bounded complex manifold and when χ is smooth. However, by inspecting the proof, it can be easily verified that the proof only requires that $\bar{N}$ have an integrable almost complex structure. Similarly, the proof of (4) only involves derivatives of the weight function $\varphi_{s,t,b}$ up to order $m+2$. Thus it suffices to choose $\chi \in C^{m+2}(\mathbf{R})$.

Holomorphic coordinates are used in Hörmander's argument in [5]. Of course, we do not yet know if they exist in a neighborhood of z_0, but we can prove that they exist up to infinite order at z_0.

PROPOSITION 3. *There exist smooth functions $\zeta_k(z)$, $k=1,\dots,n$, defined in a neighborhood z_0, such that the forms $\bar\partial\zeta_k$, $k=1,\dots,n$, vanish to infinite order at z_0, and such that the differentials of $\zeta_1,\dots,\zeta_n$ at z_0 are linearly independent.*

Proof. Suppose by induction that for a given positive integer m we have found coordinate functions $\zeta_1(z),\dots,\zeta_n(z)$, defined for z near z_0 and which map z_0 to the origin in $\mathbf{C}^n$, and vector fields $L_1',\dots,L_n'$, also defined near z_0, such that: (i) for all z, $L_1'(z),\dots,L_n'(z)$ form a basis of $\mathcal{L}_z$; and (ii) in the ζ-coordinates, each vector field L_j' can be written as

$$L_j' = \frac{\partial}{\partial\zeta_j} + \sum_{i=1}^n a_i^j(z)\frac{\partial}{\partial\zeta_i} + \sum_{i=1}^n b_i^j(\zeta)\frac{\partial}{\partial\bar\zeta_i},$$

where the functions a_i^j and b_i^j, $i,j=1,\dots,n$, vanish to order m at the origin in $\mathbf{C}^n$. (This is trivial when $m=1$.) It follows that if L is any section of $\mathcal{L}$, then the coefficient of $\partial/\partial\bar\zeta_l$ in L must vanish to order at least m.

Since $\mathcal{L}$ is integrable, it follows that $[L_i', L_j']$ is a section of $\mathcal{L}$. If we let $\tilde b_i^j$ denote the terms of order m in the Taylor polynomial of b_i^j at the origin, then modulo terms of order m or more, the coefficient of $\partial/\partial\bar\zeta_l$ in $[L_i', L_j']$ equals

$$\frac{\partial}{\partial\zeta_i}\tilde b_l^j - \frac{\partial}{\partial\zeta_j}\tilde b_l^i,$$

which must vanish identically since it is of order $m-1$. It follows that for each $l=1,\dots,n$, $\sum_{i=1}^n \tilde b_l^i d\zeta_i$ is a ∂-closed $(1,0)$-form in $\mathbf{C}^n$, and therefore there exists a smooth function g_l that satisfies

$$\frac{\partial g_l}{\partial\zeta_i} = \tilde b_l^i, \quad i=1,\dots,n.$$

We may assume that the function g_l is homogeneous of degree $m+1$. In fact the homogeneity of $\tilde b_l^i$ implies that the Taylor series of g_l contains no term of the form $\zeta^\alpha\bar\zeta^\beta$ with $|\alpha+\beta|\neq m+1$ and $\alpha\neq 0$, and we may obviously discard from g_l all terms of the form $\bar\zeta^\beta$.

Thus we have shown that, modulo terms of order at least $m+1$,

$$L_i' = \frac{\partial}{\partial\zeta_i} + \sum_{l=1}^n \frac{\partial g_l}{\partial\zeta_i}\frac{\partial}{\partial\bar\zeta_l} + \cdots.$$

Now define coordinates $w_1,\dots,w_n$ by

$$w_j = \zeta_j - \overline{g_j(\zeta)}, \quad j=1,\dots,n.$$

In the w-coordinates, the vector field $\partial/\partial\zeta_i$ can be expressed as

$$\sum_{j=1}^n \frac{\partial w_j}{\partial\zeta_i}(\zeta^{-1}(w))\frac{\partial}{\partial w_j} + \sum_{j=1}^n \frac{\partial\bar w_j}{\partial\zeta_i}(\zeta^{-1}(w))\frac{\partial}{\partial\bar w_j}$$
$$= \frac{\partial}{\partial w_i} - \sum_{j=1}^n \frac{\partial\bar g_j}{\partial\zeta_i}(\zeta^{-1}(w))\frac{\partial}{\partial w_j} - \sum_{j=1}^n \frac{\partial g_j}{\partial\zeta_i}(\zeta^{-1}(w))\frac{\partial}{\partial\bar w_j},$$

which modulo terms of order at least $m+1$ equals

$$\frac{\partial}{\partial w_i}-\sum_{j=1}^{n}\frac{\partial \bar{g}_j}{\partial \zeta_i}(w)\frac{\partial}{\partial w_j}-\sum_{j=1}^{n}\frac{\partial g_j}{\partial \zeta_i}(w)\frac{\partial}{\partial \bar{w}_j},$$

since $\zeta^{-1}(w)=w+$terms of order $m+1$. Similarly, the vector field

$$\frac{\partial g_j}{\partial \zeta_i}(\zeta)\frac{\partial}{\partial \bar{\zeta}_j},$$

modulo terms of order $m+1$, transforms into

$$\frac{\partial g_j}{\partial w_i}(w)\frac{\partial}{\partial \bar{w}_j}.$$

We conclude that, modulo terms of order $m+1$,

$$L_j'=\frac{\partial}{\partial w_j}-\sum_{i=1}^{n}\frac{\partial \bar{g}_j}{\partial w_i}\frac{\partial}{\partial w_i}+\cdots.$$

Now set

$$L_j''=L_j'+\sum_{i=1}^{n}\frac{\partial \bar{g}_j}{\partial w_i}L_i'.$$

Then the coordinates $w_1,\dots,w_n$ and the vector fields L_j'', $j=1,\dots,n$, obviously satisfy properties (i) and (ii) with respect to $m+1$. By induction we obtain, for each $m=1,2,\dots$, coordinate functions $\zeta_1^m,\dots,\zeta_n^m$ and vector fields $L_1^m,\dots,L_n^m$, defined near z_0, which satisfy (i) and (ii). Since ζ_k^{m+1} differs from ζ_k^m by terms of order $m+1$, it is clear that the formal power series of ζ_k^m converges to a formal power series ζ_k as m approaches infinity. By the classical result of Borel, there is a smooth function $\zeta_k(z)$ defined near z_0 which has this as its series. Since $\zeta_j-\zeta_j^m$ vanishes to order $m+1$, we see from (ii) that $\bar{L}_k^m\zeta_j$ vanishes to order at least m at z_0 for $j,k=1,\dots,n$. Hence (i) implies that if L is any section of $\mathcal{L}$, then $\bar{L}\zeta_j$ vanishes to order m at z_0. This holds for all m, so we conclude that $\bar{L}\zeta_j$ vanishes to infinite order at z_0. This completes the proof of Proposition 3. □

It is now a matter of adapting Hörmander's argument in [5]. Recall that $d\lambda(z_0)\neq 0$ and $\lambda(z_0)=0$.

If we work in the coordinates $\zeta_1,\dots,\zeta_n$ given by Proposition 3, then λ satisfies

$$\lambda(z(\zeta))=\operatorname{Re}\left(\sum_{j=1}^{n}a_j\zeta_j+\sum_{i,j=1}^{n}a_{ij}\zeta_i\zeta_j\right)+\sum_{i,j=1}^{n}c_{i,j}\zeta_i\bar{\zeta}_j+\mathcal{O}(|\zeta|^3).$$

Without loss of generality we may assume that $a_n\neq 0$. Since λ is strictly plurisubharmonic, it follows that $[c_{i,j}]$, $1\le i,j\le n$, is positive definite. Define new coordinates $w=(w_i,\dots,w_n)$ by $w_i=\zeta_i$, $i=1,\dots,n-1$, and

$$w_n=\sum_{j=1}^{n}a_j\zeta_j+\sum_{i,j=1}^{n}a_{i,j}\zeta_i\zeta_j.$$

It follows that if w is close to the origin, say $|w|<\delta_0$, then

$$\lambda(w)-\operatorname{Re} w_n\ge c|w|^2,\quad |w|<\delta_0, \tag{5}$$

for c a small constant.

Let $\psi(\eta)$, $\eta \in \mathbf{R}$, and denote a smooth real-valued function satisfying $\psi(\eta)=1$, $\eta \le 1$ and $\psi(\eta)=0$, $\eta \ge 2$. Define $\Psi_\tau(w)=\psi(\tau|w|)$. For all $i=1,\dots,n$ and $\tau \ge 2\delta_0^{-1}$, set $g_\tau^i=\bar\partial v_\tau^i$, where

$$v_\tau^i(w)=\Psi_\tau(w)w_i e^{A(\tau^2\log\tau)w_n},$$

and where A is a positive constant still to be chosen.

We want to choose a weight function in the form $\varphi_{s,t,b}$ to obtain a solution of $\bar\partial u = g_\tau^i$. Set $m=n+2$ and define $\chi(\eta)=\eta^{m+3}$, $\eta\ge 0$, and $\chi(\eta)=0$, $\eta<0$. Thus $\chi \in C^{m+2}(\mathbf{R})$. With this choice of χ, we define $\varphi_{s,t,b}$ as in (3). We set $s=s(m)$ (the constant occurring in Proposition 2), $t=t(\tau)=\tau^M$, where $M=2m+9$, and $b=b(\tau)=\frac{c}{4}\tau^{-2}$, where c is the same constant occurring in (5). Now set $\varphi(\tau)=\varphi_{s(m),t(\tau),b(\tau)}$. Let u_τ^i denote the solution of $\bar\partial u = g_\tau^i$ given by Proposition 2 for this choice of the weight function.

We first study how rapidly $\|g_\tau^i\|_{\varphi(\tau)}$ approaches zero. Define

$$\Pi_1=\{z\in \bar N;\ w(z) \text{ satisfies } |w|<\tau^{-1} \text{ and } \lambda(w)\le \tfrac{c}{2}\tau^{-2}\}.$$

The inequality (5) implies that if $z\in\Pi_1$ then

$$|e^{A(\tau^2\log\tau)w_n}|\le \tau^{(Ac/2)}.$$

Since $|w|<\tau^{-1}$ implies that $\Psi_\tau(w)\equiv 1$ and since all of the derivatives $D^\alpha\bar\partial w_i$, $i=1,\dots,n$, vanish to infinite order at the origin, it follows that

$$\sup_{\Pi_1}|D^\alpha g_\tau^i|^2\le \tau^{-2l+Ac}, \quad |\alpha|\le m,$$

holds for any positive integer l. Finally, $\varphi(\tau)(z)\ge\varphi(0)(z)$ for all $\tau\ge 0$ and $z\in \bar N$. Thus $\varphi(\tau)$ is bounded below by $-D$ for some constant D. This means that $\sup_N e^{-\varphi(\tau)}\le e^D$ for all τ. We conclude that for any positive integer p and for all α, $|\alpha|\le m$,

(6) $$\int_{\Pi_1}|D^\alpha g_\tau^i|^2e^{-\varphi(\tau)}\,dV\le\tau^{-p}, \quad \tau \text{ large}.$$

Define

$$\Pi_2=\{z\in\bar N;\ \lambda(z)\le\tfrac{c}{2}\tau^{-2},\ \tau^{-1}\le|w(z)|\}.$$

For $z\in\Pi_2$, it follows that

$$\tfrac{c}{2}\tau^{-2}\ge\lambda(z(w))\ge \operatorname{Re} w_n+c|w|^2\ge\operatorname{Re} w_n+c\tau^{-2},$$

so that $\operatorname{Re} w_n\le-\frac{c}{2}\tau^{-2}$. Hence

$$|e^{A(\tau^2\log\tau)w_n}|\le\tau^{-(cA/2)}.$$

The derivatives of the terms in $\bar\partial g_\tau^i$ which involve $\bar\partial w_i$, $i=1,\dots,n$, obviously vanish to infinite order at z_0, and so these derivatives can be estimated exactly as above. The only term left in $\bar\partial g_\tau^i$ comes from the term containing $\bar\partial\Psi_\tau$. Derivatives of order m of this term are of size τ^{m+1}. Thus we conclude that if $|\alpha|\le m$ then

(7) $$\int_{\Pi_2}|D^\alpha g_\tau^i|^2e^{-\varphi(\tau)}\,dV\le C\tau^{2m+2-cA-2n}.$$

Finally, set

$$\Pi_3=\{z\in\bar N;\ \lambda(z)\ge\tfrac{c}{2}\tau^{-2}\}.$$

Since Ψ_τ is supported in $|w| \le 2\tau^{-1}$, it follows that $\operatorname{Re} w_n \le 2\tau^{-1}$ on Π_3. Hence

$$|e^{A(\tau^2 \log \tau) w_n}| \le e^{2A\tau \log \tau}$$

holds on Π_3.

We conclude, as in the case of Π_1 and Π_2, that

$$\sup_{\substack{\Pi_3 \\ |\alpha| \le m}} |D^\alpha g_\tau^i|^2 \le \tau^{2m+2} e^{2A\tau \log \tau} \le e^{\tau^2}, \tag{8}$$

provided that τ is sufficiently large (for any fixed choice of A). On the other hand, the choice of $b(\tau)$ means that

$$\begin{aligned} \inf_{\Pi_3} \varphi(\tau) &\ge c' \left(\frac{c\tau^{-2}}{4} \right)^{m+3} \tau^M \\ &\ge c'' \tau^{M-(2m+6)} = c'' \tau^3. \end{aligned}$$

Hence, if $|\alpha| \le m$, we obtain from (8) that

$$\int_{\Pi_3} |D^\alpha g_\tau^i|^2 e^{-\varphi(\tau)} \, dV \le C e^{\tau^2 - c'' \tau^3}, \tag{9}$$

which goes to zero extremely rapidly as τ approaches infinity. We conclude from (6), (7), and (9) that

$$\sum_{|\alpha| \le m} \|D^\alpha g_\tau^i\|_{\varphi(\tau)} \le C \tau^{2m+2-cA-2n}.$$

Proposition 2 implies that u_τ^i satisfies

$$\sum_{|\alpha| \le m} \|D^\alpha u_\tau^i\|^2_{\varphi(\tau)} \le C \tau^{2mM + 2(m-n) - cA + 2}. \tag{10}$$

In order to estimate $\|u_\tau^i e^{-\frac{1}{2}\varphi(\tau)}\|_m$, it suffices to estimate $(D^\alpha u_\tau^i)(D^\beta e^{-\frac{1}{2}\varphi(\tau)})$ for all α, β with $|\alpha| + |\beta| \le m$. From the definition of $\varphi(\tau)$ it follows that

$$|D^\alpha e^{-\frac{1}{2}\varphi(\tau)}| \le C \tau^{M|\beta|} e^{-\frac{1}{2}\varphi(\tau)} \le C \tau^{Mm} e^{-\frac{1}{2}\varphi(\tau)}$$

if τ is large. Hence

$$\int_N |D^\alpha u_\tau^i|^2 |D^\beta e^{-\frac{1}{2}\varphi(\tau)}|^2 \, dV \le C \tau^{2Mm} \int_N |D^\alpha u_\tau^i|^2 e^{-\varphi(\tau)} \, dV. \tag{11}$$

We conclude from (10) and (11) that

$$\|u_\tau^i e^{-\frac{1}{2}\varphi(\tau)}\|_m^2 \le C \tau^{4mM + 2(m-n) - cA + 2}.$$

If we choose A so that $A > c^{-1}(4mM + 2(m-n+1))$, then it follows that

$$\lim_{\tau \to \infty} \|u_\tau^i e^{-\frac{1}{2}\varphi(\tau)}\|_m^2 = 0.$$

Recall that $m = n+2 > \frac{1}{2}(2n) + 1$. Thus the Sobolev lemma implies that the value of $u_\tau^i e^{-\frac{1}{2}\varphi(\tau)}$ at z_0, as well as its first derivatives at z_0, tend to zero as τ approaches infinity. Since $\varphi(\tau)(z_0) = 0$ and since all the derivatives of $\varphi(\tau)$ at z_0 are bounded independently of τ, we conclude that

$$\lim_{\tau \to \infty} |D^\alpha u_\tau^i(z_0)| = 0, \quad |\alpha| \le 1.$$

If τ is sufficiently large, say $\tau=\tau_0$, it follows that the functions $f_i=v^i_{\tau_0}-u^i_{\tau_0}$, $i=1,\dots,n$, satisfy $\bar\partial f_i=0$, are in $H_m(N)$, and have linearly independent differentials at z_0.

All that remains is to show that any function $f\in H_m(N)$ satisfying $\bar\partial f=0$ can be approximated in the H_m-norm by a function $h\in C^\infty(\bar N)$ satisfying $\bar\partial h=0$. Theorem 3.1.4 of [3] gives exactly this result. However it is stated and proved under the assumption that $\bar N$ is a smoothly bounded complex manifold. The proof can be easily modified to work when N is a smoothly bounded manifold with an integrable almost complex structure. In fact, it is easy to construct a smooth 1-parameter family of smooth maps $P_\delta:\bar N\to\bar N$ such that P_0 is the identity map and the image of $\bar N$ under P_δ is contained in N for all $\delta>0$. Now set $f_\delta(z)=f(P_\delta(z))$. Since the Newlander–Nirenberg theorem holds in the interior of N, it follows that f is smooth in the interior and therefore that $f_\delta\in C^\infty(\bar N)$ for all $\delta>0$. Since P_0 is the identity and $f\in H_m(N)$, it follows that $\lim_{\delta\to0}\|\bar\partial f_\delta\|_m=0$.

After using this new definition of f_δ, the proof of Theorem 3.1.4 of [3] goes through with no other changes.

Thus each function f_i can be closely approximated in the H_m-norm by $h_i\in C^\infty(\bar N)$. The Sobolev lemma implies that the differentials h_i at z_0, $i=1,\dots,n$, will still be linearly independent. This completes the proof. □

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