SOME REMARKS ON POSITIVELY CURVED 4-MANIFOLDS

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1. The aim of this note is to prove the following.

THEOREM. *Let M^4 be a compact, connected, oriented, positively curved Riemannian 4-manifold without boundary. Then M admits at most one harmonic 2-form of constant length (up to constant multiples). If M admits such a 2-form, then M is definite.*

Background and motivation for the above theorem can be summarized as follows: the Hopf conjecture asserts that $\mathbf{S}^2\times\mathbf{S}^2$ admits no metric of strictly positive sectional curvature. We have attempted to gain insight into this conjecture by starting with a positively curved compact, oriented 4-manifold M (with the metric normalized so that the sectional curvature K satisfies $1\geq K\geq\delta>0$). From Synge's theorem, M is simply connected. Since M is 4-dimensional, it follows that $H_1(M,\mathbf{Z})=H_3(M;\mathbf{Z})=0$ and $H_2(M;\mathbf{Z})\cong H^2(M;\mathbf{Z})$ is torsion-free. It follows from [2] that we know M topologically once we know its intersection form.

From a (Riemannian) geometric point of view we know

$$H^2(M;\mathbf{R})=H^2(M;\mathbf{Z})\otimes\mathbf{R}$$

as the DeRham cohomology and as the space of harmonic 2-forms (relative to the subsumed metric). It seems natural to ask, then, if there are topological restrictions to the "types" of harmonic 2-forms such a manifold can admit? In [3] we showed that if our M admits a *parallel* 2-form, then $\dim H^2(M;\mathbf{R})=1$ and it follows that M is $\mathbf{CP}^2$ (topologically and even biholomorphically).

In the theorem above we relax the assumption of the existence of a parallel 2-form to the existence of a harmonic 2-form of constant length.

While this assumption is clearly quite strong, it is strictly weaker than parallel, and we can at least still conclude that M is definite, so that we conclude that a smooth topological indefinite four manifold can never support such a metric. As $\mathbf{S}^2\times\mathbf{S}^2$, and in fact

$$\underbrace{\mathbf{S}^2\times\mathbf{S}^2\#\cdots\#\mathbf{S}^2\times\mathbf{S}^2}_{n}\#\underbrace{\mathbf{CP}^2\#\cdots\#\mathbf{CP}^2}_{m}\#\underbrace{\overline{\mathbf{CP}^2}\#\cdots\#\overline{\mathbf{CP}^2}}_{k}$$

where either $n>0$, or $n=0$ and $m\cdot k\neq 0$, are all indefinite, our theorem rules out all such manifolds. In fact, it follows from [1] and [2] that a definite, smooth simply connected compact 4-manifold must be topologically

$$\underbrace{\mathbf{CP}^2\#\cdots\#\mathbf{CP}^2}_{b_2}.$$

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Thus, we can summarize as follows: M^4 compact, positively curved admitting a parallel 2-form $\Rightarrow M^4 \approx \mathbf{CP}^2$ (topologically); and M^4 positively curved admitting a harmonic 2-form of constant length $\Rightarrow M^4 \approx \mathbf{CP}^2 \# \cdots \# \mathbf{CP}^2$.

Ultimately, one would like to make conclusions about M without any special assumptions of the types of harmonic 2-forms it admits, and this is the focus of our ongoing work.

Finally, we comment on the significance of the "at most one" portion of our theorem. In [3] we show that on a purely vector space level, if $1 \geq K \geq \delta > 0$ at a point then the Weitzenböck operator R_2 has kernel dimension at most one at that point, and is positive definite on the orthogonal complement to the kernel. Solutions to $R_2 X = 0$ are the "infinitesimal" version of parallel 2-forms, and our results in [3] follow. Here we show more generally (again, at a purely vector space level), that if R_2 has a nonpositive eigenvalue, then the corresponding eigenspace is one-dimensional and R_2 is positive definite on the orthogonal complement to this eigenspace (this is Proposition 3). Solutions to $\langle R_2 X, X \rangle \leq 0$ are an "infinitesimal" version of harmonic forms of constant length, and the "at most one" portion of our current theorem shows that, under this assumption *globally,* uniqueness still follows for a compact 4-manifold.

Our notation follows that of [3]. In particular, R_2 is the Weitzenbock operator, and harmonic forms X satisfy $0 = \frac{1}{2}\Delta|X|^2 + |\nabla X|^2 + \langle R_2 X, X \rangle$, and the sectional curvature of M, K, satisfies $1 \geq K \geq \delta > 0$. Also, $e_{ij} = e_i \wedge e_j$ where e_i, e_j are orthonormal vectors, and $\langle R_2 e_{ij}, e_{ij} \rangle = K^{ij} = \sum_{k \neq i} K_{jk} + \sum_{k \neq j} K_{ik} \geq 4\delta$, where K_{ij} = sectional curvature of $\{e_i, e_j\}$.

Finally, since our initial concerns are linear algebraic in R_2 at a point, we identify 2-forms with 2-vectors as in [3].

2. The linear algebra of R_2. From the definition of R_2 (cf. (3) in [3]), we know that R_2 is a symmetric operator on $\Lambda^2(T_p M)$ for each p. As M is oriented, $\Lambda^2(M) = \Lambda^2_+(M) \oplus \Lambda^2_-(M)$, the eigenspace splitting for the Hodge $*$-operator.

PROPOSITION 1. *$R_2 \colon \Lambda_+ \to \Lambda_+$ and $R_2 \colon \Lambda_- \to \Lambda_-$.*

Proof. It suffices to show $R_2 \colon \Lambda_+ \to \Lambda_+$ since R_2 is symmetric. To show this we need only show

$$\langle R_2 X_+, X_- \rangle = 0 \quad \forall X_+ \in \Lambda^2_+(T_p M), \; X_- \in \Lambda^2_-(T_p M). \tag{1}$$

There is an oriented orthonormal basis $e_1, \ldots, e_4$ of $T_p M$ such that

$$X_\pm = \frac{\sqrt{2}}{2} |X_\pm| (e_{12} \pm e_{34}). \tag{2}$$

Thus the left-hand side of (1) is

$$\langle R_2 X_+, X_- \rangle = \tfrac{1}{2} |X_+| |X_-| \langle R_2(e_{12} + e_{34}), e_{12} - e_{34} \rangle. \tag{3}$$

This last term is equal to 0 using the definition of R_2 ((3) in [3]). ❑

PROPOSITION 2. *Fix* $p \in M$ *and let* $r_1 \le r_2 \le \cdots \le r_6$ *be the eigenvalues of* R_2 *on* $\Lambda^2(T_pM)$. *Assume* $r_1 \le 0$. *Then* $r_i + r_1 \ge 8\delta$ $\forall i = 2, \dots, 6$.

Proof. Assume $R_2X = r_iX$. Then splitting $X = X_+ + X_-$, we get from Proposition 1 that $R_2X_\pm = r_iX_\pm$. Now take *any* $X = X_+ + X_-$. Then from (2),

$$|X_-|X_+ + |X_+|X_- = \sqrt{2}|X_-||X_+|e_{12}.$$

Again from Proposition 1 we get

$$\begin{aligned}(4)\qquad &\langle R_2(|X_-|X_+ + |X_+|X_-), |X_-|X_+ + |X_+|X_-\rangle \\ &= |X_-|^2\langle RX_+X_+\rangle + |X_+|^2\langle RX_-X_-\rangle.\end{aligned}$$

From (4) we have

$$(5)\qquad |X_-|^2\langle RX_+X_+\rangle + |X_+|^2\langle RX_-X_-\rangle = 2|X_-|^2|X_+|^2K^{12} \ge 8\delta|X_-|^2|X_+|^2.$$

Now assume $R_2X = r_1X$ ($r_1 \le 0$). Then (5) shows that either $X = X_+$ or $X = X_-$. We may assume (by using $-*$ in place of $*$) that $X = X_+$. Then (5) again shows that, if $RX_- = r_iX_-$,

$$r_1 + r_i \ge 8\delta.$$

Suppose now that $R_2X_i = r_iX_i$, $i \ge 2$. The above remarks show that, in order to conclude $r_1 + r_i \ge 8\delta$, we need only consider $X_i = X_+ \in \Lambda_+$. Also assume $RX_1 = r_1X_1$ ($X_1 \in \Lambda_+$). Taking X_1, X_i orthogonal unit vectors, there is an orthonormal basis $F_1, \dots, F_4$ of T_pM such that

$$(6)\qquad X_1 = \frac{1}{\sqrt{2}}[-F_{12} + F_{34}], \qquad X_i = \frac{1}{\sqrt{2}}[F_{13} + F_{24}]$$

as in (9), (10) of [1]. As in that paper, we conclude:

$$(7)\qquad r_1 = K^{12} + 2R_{1243}, \qquad r_i = K^{13} - 2R_{4213}$$

and adding these, we obtain

$$(8)\qquad r_1 + r_i = \tfrac{1}{2}\rho + K_{14} + K_{23} + 2R_{1423},$$

where ρ = scalar curvature.

The argument in [3] ((14) through (17)) now shows that $r_1 + r_i \ge 8\delta$. □

PROPOSITION 3. *Let* X, Y *be nonzero vectors in* Λ^2T_pM *such that* $\langle R_2X, X\rangle \le 0$ *and* $\langle X, Y\rangle = 0$. *Then* $\langle R_2Y, Y\rangle \ge 8\delta|Y|^2$.

Proof. Let e_1 be a unit eigenvector for the minimum R_2 eigenvalue r_1 (≤ 0). Proposition 2 shows that $r_i \ge 8\delta + |r_1|$ for $i = 2, \dots, 6$. Decompose orthogonally $X = x_1e_1 + X'$, $Y = y_1e_1 + Y'$. Then $\langle R_2Y', Y'\rangle \ge (8\delta + |r_1|)|Y'|^2$ and $\langle R_2X', X'\rangle \ge (8\delta + |r_1|)|X'|^2$.

Now $0 \ge \langle R_2X, X\rangle \ge x_1^2r_1 + (8\delta + |r_1|)|X'|^2$, so we have

$$(9)\qquad \frac{|r_1|}{8\delta + |r_1|} \ge \frac{|X'|^2}{x_1^2}.$$

Also, $0=x_1y_1+\langle X',Y'\rangle$ so $y_1^2=\langle X',Y'\rangle^2/x_1^2$. Hence

$$\langle R_2Y,Y\rangle=y_1^2r_1+\langle RY',Y'\rangle\geq\frac{\langle X',Y'\rangle^2}{x_1^2}r_1+(8\delta+|r_1|)|Y'|^2$$
$$\geq|Y'|^2\frac{|X'|^2}{x_1^2}r_1+(8\delta+|r_1|)|Y'|^2.$$

Using (9), we now obtain:

$$\begin{aligned}\langle R_2Y,Y\rangle&\geq|Y'|^2\left(\frac{-r_1^2}{8\delta+|r_1|}+8\delta+|r_1|\right)\\ &=|Y'|^2\left(\frac{8\delta|r_1|}{|r_1|+8\delta}\right)+8\delta|Y'|^2.\end{aligned}\tag{10}$$

Again, we have

$$y_1^2=\frac{\langle X',Y'\rangle^2}{x_1^2}\leq|Y'|^2\frac{|X'|^2}{x_1^2}\leq\frac{|Y'|^2|r_1|}{8\delta+|r_1|}$$

(by (9)). Therefore, $8\delta y_1^2\leq|Y'|^2\cdot 8\delta|r_1|/|r_1|+8\delta$, and so (10) yields $\langle R_2Y,Y\rangle=8\delta y_1^2+8\delta|Y'|^2=8\delta|Y|^2$. □

3. Proof of the theorem in §1. If X is a harmonic form, then as noted,

$$0=\tfrac{1}{2}\Delta|X|^2+|\nabla X|^2+\langle R_2,X,X\rangle,\tag{11}$$

where $|\nabla X|_p^2=\sum_i|\nabla_{e_i}X|_p^2$, e_i an orthonormal basis for T_pM. In particular, for any harmonic X, integrating (13) over M yields

$$\int_M\langle R_2XX\rangle\leq 0\tag{12}$$

with equality if and only if X is parallel. Polarizing (11) we obtain, for any harmonic X and Y,

$$0=\tfrac{1}{2}\Delta\langle XY\rangle+\langle\nabla X,\nabla Y\rangle+\langle R_2X,Y\rangle\tag{13}$$

where $\langle\nabla X,\nabla Y\rangle_p=\sum_i\langle\nabla_{e_i}X,\nabla_{e_i}Y\rangle$.

We shall now prove that if X and Y are harmonic 2-forms with constant length then X is a constant multiple of Y. Assume $|Y|^2\equiv 1$. Now (11) yields that $\langle R_2X,X\rangle=-|\nabla X|^2\leq 0$, and $\langle R_2Y,Y\rangle=-|\nabla Y|^2\leq 0$ everywhere on M. Also, $X-\langle XY\rangle Y$ is everywhere orthogonal to Y, so Proposition 3 of §2 yields

$$\langle R_2(X-\langle XY\rangle Y),X-\langle XY\rangle Y\rangle\geq 8\delta|X-\langle XY\rangle|^2,\tag{14}$$

which reduces to

$$\langle R_2XX\rangle-2\langle XY\rangle\langle R_2X,Y\rangle+\langle XY\rangle^2\langle R_2Y,Y\rangle\geq 8\delta|X\wedge Y|^2.\tag{15}$$

Using (13) and the above remarks, (15) yields

$$-|\nabla X|^2+2\langle XY\rangle\langle\nabla X,\nabla Y\rangle-\langle XY\rangle^2|\nabla Y|^2+\langle XY\rangle\Delta\langle X,Y\rangle\geq 8\delta|X\wedge Y|^2.\tag{16}$$

Now Proposition 3 implies that $\langle XY\rangle$ is never 0, since both $\langle R_2X, X\rangle \le 0$ and $\langle R_2Y, Y\rangle \le 0$, and X, Y are never 0. We may therefore assume that $\langle XY\rangle > 0$ everywhere on M. Also, using the Cauchy-Schwartz inequality twice, we have $\langle \nabla X, \nabla Y\rangle \le |\nabla X||\nabla Y|$. Using this in (16), we get

$$-(|\nabla X| - \langle XY\rangle|\nabla Y|)^2 + \langle XY\rangle\Delta\langle XY\rangle \ge 8\delta|X\wedge Y|^2; \tag{17}$$

hence $\langle XY\rangle\Delta\langle XY\rangle \ge 8\delta|X\wedge Y|^2$ and since $\langle XY\rangle > 0$, we get $\Delta\langle XY\rangle \ge 0$ everywhere, so $\langle X, Y\rangle$ is constant and $|X\wedge Y| \equiv 0$. Therefore $X = fY$ for some function f. But then constant $= |X|^2 = \langle X, fY\rangle = f\cdot$constant, so f is constant. Thus $X = \text{constant}\cdot Y$.

We now will show that if M has a 2-form of constant length then M is definite.

Let $X = X_+ + X_-$. Now $0 \ge \langle R_2XX\rangle = \langle R_2X_+, X_+\rangle + \langle R_2X_-, X_-\rangle$ so (5) implies that at no point can we have $|X_+|^2 = |X_-|^2$. We can assume $|X_+|^2 > |X_-|^2$ everywhere. We have $|X_-|^2\langle R_2XX\rangle \le 0$ so

$$|X_-|^2\langle R_2X_+, X_+\rangle + |X_-|^2\langle R_2X_-, X_-\rangle \le 0,$$

and from (5) we get

$$\begin{aligned}|X_-|^2\langle R_2X_+, X_+\rangle &+ |X_-|^2\langle R_2X_-, X_-\rangle \\ &\le 0 \le 8\delta|X_+|^2|X_-|^2 \\ &\le |X_+|^2\langle R_2X_-, X_-\rangle + |X_-|^2\langle R_2X_+, X_+\rangle.\end{aligned}$$

Hence we obtain:

$$0 \le (|X_+|^2 - |X_-|^2)\langle R_2X_-, X_-\rangle \quad \text{everywhere on } M, \tag{18}$$

which implies $\langle RX_-, X_-\rangle \ge 0$ everywhere on M. If $X_- \ne 0$ then (12) now implies X_- is parallel and, from [3], $b_2 = 1$ which contradicts the existence of X_+. Thus $X_- \equiv 0$. We now know $X = X_+$ and hence $\langle RX_+, X_+\rangle \le 0$. One more application of (5) now shows that, for *any* X_-, $\langle RX_-, X_-\rangle \ge 8\delta|X_-|^2$ which via (12) means there are no harmonic forms of the type X_-, so M is (positive) definite. □

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