HOMOMORPHISMS BETWEEN ALGEBRAS OF DIFFERENTIABLE FUNCTIONS IN INFINITE DIMENSIONS

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Introduction. Let E and F be real Banach spaces. For $n=0,1,\dots,\infty$, let $C^n_{wub}(E;F)$ be the space of n-times continuously differentiable functions f: $E\to F$ such that, for each integer $j\le n$ and each $x\in E$, both the jth derivative mapping $f^j: E\to P(^jE;F)$ and the polynomial $f^j(x)\in P(^jE;F)$ are weakly uniformly continuous on bounded subsets of E. (This space and related notions are reviewed below.) Our primary interest here is in the study of homomorphisms $A: C^n_{wub}(E;R)\to C^m_{wub}(F;R)$. We will show that these homomorphisms are induced by functions $g: F''\to E''$, in a way to be made more precise later. One of the principal purposes of this note is to characterize these functions g in terms of a differentiability property, thereby characterizing the homomorphisms A. An easy consequence will be that every such homomorphism is automatically continuous when the spaces C^n_{wub} are given their natural topology.

By way of defending our interest in $C^n_{wub}(E;F)$, we mention that several quite natural characterizations of this space exist, and are recalled below. In particular, if $E=R^n$ and $F=R^k$ then weak uniform continuity on bounded sets is automatic. In other words, in the case of finite-dimensional spaces E and F, our results reduce to the classical case of homomorphisms $A: C^n(R^q)\to C^m(R^k)$; this is stated as Corollary 3.5 below. (See Glaeser [8] and Bers [4] for discussions of related problems in finite-dimensional real and complex normed spaces, respectively.) Moreover, complex analogs of this space are of independent interest and of some relevance to the Michael problem on automatic continuity of complex-valued homomorphisms on a complex Fréchet algebra. For example, let $E=c_0$ be considered as a complex Banach space, let $F=C$ and $n=\infty$, and call the corresponding space $H_{wub}(c_0)$. Then it is known [5] that if every scalar-valued homomorphism on $H_{wub}(c_0)$ is continuous then every scalar-valued homomorphism on every Fréchet algebra is continuous.

The basic ingredients we will need are few and are all relatively simple. First, under reasonable hypotheses (such as E' having the bounded approximation property), $C^n_{wub}(E;R)$ can be characterized as the completion of the unital algebra generated by E' under the topology of uniform convergence of a function and its first k derivatives on bounded sets, where $k\in N$ and $k\le n$. Therefore, a continuous homomorphism $\Phi: C^n_{wub}(E;R)\to R$ is determined by its action on E'.

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Second, any function in $C^n_{wub}(E;R)$ has a (necessarily unique) extension to an element in a space of differentiable functions on E'', which we will examine later. Moreover, we will see that there is a natural, well-known locally convex topology on E'', called the bw^* topology, such that $C^n_{wub}(E;R)$ is algebraically and topologically isomorphic to $C^n(E''_{bw^*};R)$. We will see that this isomorphism, which essentially "trades off" a subspace of the space of differentiable functions on a Banach space with the space of all continuously differentiable functions on a locally convex space, has considerable value for us. Finally, given any homomorphism $A: C^n_{wub}(E;R)\to C^m_{wub}(F;R)$ and any point $y\in F$, a functional $g(y)\in E''$ can be defined by $g(y)(\phi)=A(\phi)(y)$ for each $\phi\in E'$. In this way we get a function $g: F\to E''$ which we will be able to extend to $g: F''\to E''$. Note that with the topology described above $C^n_{wub}(E;R)$ is a Fréchet algebra over the reals, and so every multiplicative linear functional on $C^n_{wub}(E;R)$ is automatically continuous (cf. [12]). From this we will be able to deduce that A is continuous, and we will also be able to derive the differentiability properties which determine g.

This paper is in four parts. Section 1 reviews the key concepts, in particular the bw^* topology and the spaces $C^n_{wub}(E;F)$ and $C^n(E''_{bw^*};F''_{bw^*})$. In Section 2, the relationship between these two spaces will be studied. Section 3 is the main body of the paper; here the homomorphisms between the algebras $C^n_{wub}(E;R)$ and $C^m_{wub}(F;R)$ $(0\le m,n\le\infty)$ are characterized in terms of mappings $g: F''\to E''$, which are differentiable when E'' and F'' are endowed with the bw^* topologies. It is reasonable to ask what relation exists between differentiability of a function in this sense and the usual Fréchet derivative between Banach spaces F'' and E''. This topic is discussed at the end of Section 3 and examples are given in Section 4. In brief, our treatment shows that bw^* differentiability is natural and, in any case, unavoidable in this context.

In a separate note [1], Aron and Llavona examine questions concerning the ranges of homomorphisms between algebras C^0_{wub} of functions which are weakly uniformly continuous when restricted to bounded sets.

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1. Notation, definitions, and preliminary results. E and F will always denote real Banach spaces, with duals E' and F', second duals E'' and F''. The weak topology on E, $\sigma(E,E')$, will often be denoted by the letter w, and the weak* topology on E'', $\sigma(E'',E')$, will often be denoted w^*. For each $n\in N$, $B_n=\{x\in E: \|x\|\le n\}$ and $B''_n=\{x\in E'': \|x\|\le n\}$. For any Hausdorff locally convex spaces X and Y and $n=1,2,\ldots$, $L(^nX,Y)$ denotes the space of continuous n-linear mappings on $X\times\cdots\times X\to Y$, with the topology of uniform convergence on bounded subsets of $X\times\cdots\times X$. The space of n-homogeneous polynomials from X to Y, $P(^nX;Y)$, consists of all compositions $A\circ\Delta$, where $A\in L(^nX;Y)$ and $\Delta: X\to X\times\cdots\times X$ is the diagonal mapping. $P(^nX;Y)$ is assigned the locally convex topology generated by all seminorms of the form $P\in P(^nX;Y)\to\sup\{\alpha(P(x)): x\in B\}$, where $B \subseteq X$ is bounded and α is a continuous seminorm on Y. We agree that $L(^0X; Y) = P(^0X; Y)$ is associated with the constant mappings from X to Y, and so is associated with Y. $P_f(^nX; Y)$ denotes the vector subspace of $P(^nX; Y)$ generated by all n-homogeneous polynomials of the form $P(x) = \phi(x)^n y$, where $\phi \in X'$ and $y \in Y$. $P(X; Y)$ and $P_f(X; Y)$ denote the (algebraic) direct sums

$$\sum_{n=0}^{\infty} P(^nX; Y) \quad \text{and} \quad \sum_{n=0}^{\infty} P_f(^nX; Y),$$

respectively. We will also use the standard notation $L(X; Y)$ to denote $L(^1X; Y)$. For $p = 0, 1, \ldots, \infty$, $C^p(X; Y)$ is the space of all p-times continuously Fréchet differentiable mappings from X to Y (see Yamamuro [15, §1.8]). Throughout, if the range space is omitted in the notation then it is understood that the range is R. Thus, for example, $C^p(X)$ denotes $C^p(X; R)$. For further information on differentiable approximation theory in infinite dimensions, see [7; 11; 15].

In 1.2–1.4 below, we define the space of differentiable functions considered in this paper, and we give some of its properties. The remainder of the section is devoted to recalling several topologies, first introduced by Day and Wheeler, and to making some preliminary observations concerning differentiable mappings between spaces endowed with these topologies.

DEFINITION 1.1 ([2]). A function $f: E \to F$ is said to be weakly uniformly continuous on bounded subsets of E if for each bounded subset B of E and each $\epsilon > 0$ there is a finite set $\{\phi_1, \ldots, \phi_k\} \subset E'$ and $\delta > 0$ such that if $x, y \in B$ and $|\phi_i(x-y)| < \delta$ $(i = 1, \ldots, k)$ then $\|f(x) - f(y)\| < \epsilon$. $C_{wub}(E; F)$ denotes the space of such functions, and $P_{wub}(^jE; F) = P(^jE; F) \cap C_{wub}(E; F)$, which is a Banach space with the norm induced by $P(^jE; F)$.

DEFINITION 1.2 ([2]). $C^p_{wub}(E; F)$ is the subspace of $C^p(E; F)$ formed by all functions f which satisfy the following conditions:

(a) $f^j(x) \in P_{wub}(^jE; F)$ for all $x \in E$ and $j \le p$; and
(b) $f^j \in C_{wub}(E; P_{wub}(^jE; F))$ for all $j \le p$.

Moreover, $C^\infty_{wub}(E; F) = \bigcap_{p=0}^{\infty} C^p_{wub}(E; F)$.

DEFINITION 1.3 ([2]). Let $f \in C^p(E; F)$. We say that f is uniformly differentiable of order p if, for each bounded set $B \subset E$ and each $\epsilon > 0$, there is $\delta > 0$ such that if $x \in B$ and $y \in E$ with $\|y\| \le \delta$ then

$$\left\| f(x+y) - f(x) - f'(x)(y) - \cdots - \frac{f^p(x)}{p!}(y) \right\| \le \epsilon \|y\|^p.$$

THEOREM 1.4 ([2]).

(1.4.1) *If* $f \in C_{wub}(E; F)$ *and* $B \subseteq E$ *is bounded, then* $f(B)$ *is precompact in* F.

(1.4.2) *For every* $j \in N$, $P_{wub}(^jE; F)$ *is complete with the norm induced by* $P(^jE; F)$.

(1.4.3) $(C^p_{wub}(E; F), \tau^p_b)$ *is complete for all* $p = 0, 1, \ldots, \infty$, *where* τ^p_b *denotes the locally convex topology generated by all seminorms of the form* $f \in C^p_{wub}(E; F) \to \sup\{\|f^j(x)\| : x \in B\}$, *where* B *is a bounded subset of* E *and* $j \in N$, $j \le p$.

(1.4.4) *$C^p_{wub}(E;F)$ is the set of functions $f \in C^p(E;F)$ which satisfy the following condition:*

(a) $f \in C_{wub}(E;F)$;

(b) $f^j(x) \in P_{wub}({}^jE;F)$ *for all* $x \in E$ *and* $j \leq p$;

(c) *f is uniformly differentiable of order j for every $j \leq p$.*

(1.4.5) *If E' has the bounded approximation property then $P_f(E;F)$ is τ_b^p dense in $C^p_{wub}(E;F)$.*

We recall that a Banach space G is said to have the bounded approximation property if there is a constant $M > 0$ such that for all compact subsets $K \subset G$ and all $\epsilon > 0$ there is a finite rank continuous linear operator $T: G \to G$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$ and $\|T\| \leq M$. L^p spaces, $C(K)$ spaces, and spaces with Schauder basis have the bounded approximation property.

DEFINITION 1.5 ([6]). The bounded weak (bw) topology on E is the finest topology on E which agrees with the weak topology on bounded subsets of E. E_{bw} will denote E with this topology.

DEFINITION 1.6 ([14]). The convex bounded weak (cbw) topology on E is the finest locally convex topology on E which agrees with the weak topology on bounded subsets of E.

DEFINITION 1.7 ([6]). The bounded weak star (bw^*) topology on E'' is the finest topology on E'' which agrees with the weak* topology on bounded sets. The space E'', endowed with the bw^* topology, will be denoted E''_{bw^*}.

It is known that the bw^* topology is a locally convex topology. The following is a useful characterization of the cbw topology.

THEOREM 1.8 ([14]). *The cbw topology on E is the restriction to E of the bw* topology on E''. In particular, the bw topology on E is a locally convex topology if E is reflexive.*

Wheeler [14] shows that the bw topology on c_0 is not locally convex. In [9], Gomez proved that, in fact, the bw topology on a Banach space E is locally convex if and only if E is reflexive.

It is immediate that in E'', the bounded sets for the norm, the weak*, and the bw^* topologies all coincide. An application of the Grothendieck completeness theorem [13] yields that $(E''_{bw^*})' = E'$. Also from the definition of the bw^* topology, it is immediate that E''_{bw^*} is the topological direct limit of the topological spaces (B''_n, w^*), and so for a given function $f: E'' \to X$ into a locally convex space X, f is continuous for the bw^* topology if and only if $f|_B: (B, w^*) \to X$ is continuous for all bounded sets $B \subset E''$. Since closed balls in E'' are compact in E''_{bw^*} the following is immediate.

LEMMA 1.9. *If $g \in C(E''_{bw^*};X)$ and $B \subset E''$ is bounded, then $g(B)$ is precompact. In particular, $g(B)$ is bounded.*

For future use, we record the following, which is a direct application of the above definitions.

REMARK 1.10. For $p \geq 1$, $C^p(E''_{bw^*})$ is the space of all functions $f \in C^p(E'')$ which satisfy the following properties:

(a) for all $x \in E''$, $j \in N$, and $j \leq p$, $f^j(x) \in P({}^jE''_{bw^*})$;

(b) for all $j \in N$ and $j \leq p$, $f^j \in C(E''_{bw^*}; P({}^jE''_{bw^*}))$.

LEMMA 1.11. *If* $g \in C^p(E''_{bw^*}; F''_{bw^*})$ *then for each* $j \in N$ $(1 \leq j \leq p)$ *and each bounded subset* B *in* E'', $\sup\{\|g^j(x)(y)\| : x, y \in B\} < \infty$.

Proof. For each $\phi \in F'$, let $V(\phi, B, 1) = \{P \in P({}^jE''_{bw^*}; F''_{bw^*}) : |P(x)(\phi)| < 1$ for all $x \in B\}$. Since $g^j \in C(E''_{bw^*}; P({}^jE''_{bw^*}; F''_{bw^*}))$, $g^j(B)$ is precompact by Lemma 1.9, and so there are points $y_1, \ldots, y_s \in B$ such that

$$g^j(B) \subset \bigcup_{i=1}^{s} \{g^j(y_i) + V(\phi, B, 1)\}.$$

Since each $g^j(y_i) \in P({}^jE''_{bw^*}; F''_{bw^*})$, there is $M < \infty$ such that

$$\sup\{|g^j(y_i)(y)(\phi)| : y \in B, 1 \leq i \leq s\} < M.$$

Therefore

$$\sup\{|g^j(x)(y)(\phi)| : x, y \in B\} < M + 1$$

and an application of the uniform boundedness principle completes the proof. □

There is no confusion in using the notation τ_b^p to also denote the locally convex topology on $C^p(E''_{bw^*})$ of uniform convergence of order p on bounded subsets of E''. That is, τ_b^p is generated by all seminorms of the form

$$f \in C^p(E''_{bw^*}) \to \sup\{\|f^j(x)\| : x \in B\},$$

where B is allowed to range over all bounded subsets of E'' and where $j \leq p$.

2. Representation of uniformly weakly differentiable functions. We show here that functions in $C^m_{wub}(E; F)$ have extensions to functions defined on E'' having the same degree of differentiability. The importance of this result comes from the fact that we can thus obtain a topological and algebraic isomorphism between $C^p_{wub}(E)$ and $C^p(E''_{bw^*})$, which will be useful in the sequel. Our first result is a simple version of the above remarks, and will be extended in Theorem 2.4.

PROPOSITION 2.1. *Every function* f *in* $C_{wub}(E; F)$ *can be extended in a unique way to a function* $\tilde{f} : E'' \to F$, *where* $\tilde{f} \in C(E''_{bw^*}; F)$. *Moreover, the mapping* $f \to \tilde{f}$ *is a homomorphism and, for all* n, $\sup\{\|f(t)\| : t \in B_n\} = \sup\{\|\tilde{f}(t)\| : t \in B''_n\}$.

Proof. For each n, $f_n = f \mid B_n$ is uniformly continuous on B_n with the induced weak topology. Thus (see e.g. [10, p. 196]), there exists a unique extension $\tilde{f}_n$ to B''_n with the induced weak* topology. It is clear that the functions $\tilde{f}_n$ give rise to a coherently defined function $\tilde{f} : E'' \to F$. That $\tilde{f} \in C(E''_{bw^*}; F)$ follows from the definition of the bw^* topology. The rest of the proof is straightforward, using the weak* density of B_n in B''_n. □

COROLLARY 2.2. *The following pairs of spaces are topologically isomorphic:* $C_{wub}(E; F)$ *and* $C(E''_{bw^*}; F)$; *and* $P_{wub}({}^jE; F)$ *and* $P({}^jE''_{bw^*}; F)$ *for* $j \in N$.

One consequence of 2.1 and 2.2 is that if $f \in C^p_{wub}(E)$ then each $f^j: E \to P_{wub}(^jE)$ can be extended to $\widetilde{f^j} \in C(E''_{bw^*}; P(^jE''_{bw^*}))$.

LEMMA 2.3. *If $f \in C^p_{wub}(E)$, then for each $j \in N$, $j \le p$, and each bounded subset $B \subset E''$, the mapping $\Phi: B \times B \to R$, $\Phi(x,y) = \widetilde{f^j}(x)(y)$, is continuous when B has the induced weak* topology.*

Proof. By Lemma 1.9, $\widetilde{f^j}(B)$ is precompact in $P(^jE''_{bw^*})$. Therefore, given the 0-neighborhood $V = \{P \in P(^jE''_{bw^*}): |P(x)| < \epsilon \text{ for all } x \in B\}$, there exists a finite set $\{b_1, \dots, b_k\} \subset B$ such that

$$\widetilde{f^j}(B) \subset \bigcup_{l=1}^{k} (\widetilde{f^j}(b_l) + V). \tag{2.3.1}$$

Since each $\widetilde{f^j}(b_l) \in P(^jE''_{bw^*})$, we can find a finite set $\{\phi_1, \dots, \phi_s\} \subset E'$ and $\delta_1 > 0$ such that, for all $x, y \in B$ satisfying $|\phi_i(x-y)| < \delta_1$ $(i = 1, \dots, s)$,

$$|\widetilde{f^j}(b_l)(x) - \widetilde{f^j}(b_l)(y)| < \epsilon \quad (l = 1, \dots, k). \tag{2.3.2}$$

On the other hand, since $\widetilde{f^j} \in C(E''_{bw^*}; P(^jE''_{bw^*}))$, we can find a finite set $\{\phi_{s+1}, \dots, \phi_t\} \subset E'$ and $\delta_2 > 0$ such that if $x, y \in B$ satisfy $|\phi_i(x-y)| < \delta_2$ $(i = s+1, \dots, t)$ then, for all $z \in B$,

$$|\widetilde{f^j}(x)(z) - \widetilde{f^j}(y)(z)| < \epsilon. \tag{2.3.3}$$

Let $\delta = \min(\delta_1, \delta_2)$, and choose $x_1, x_2, y_1, y_2 \in B$ so that $|\phi_i(x_1 - x_2)| < \delta$ and $|\phi_i(y_1 - y_2)| < \delta$ $(i = 1, \dots, t)$. Then, by applying (2.3.1)–(2.3.3), we conclude that for some b_i,

$$\begin{aligned}
&|\widetilde{f^j}(x_1)(y_1) - \widetilde{f^j}(x_2)(y_2)| \\
&\quad \le |\widetilde{f^j}(x_1)(y_1) - \widetilde{f^j}(x_2)(y_1)| + |\widetilde{f^j}(x_2)(y_1) - \widetilde{f^j}(x_2)(y_2)| \\
&\quad \le \epsilon + |\widetilde{f^j}(x_2)(y_1) - \widetilde{f^j}(b_i)(y_1)| + |\widetilde{f^j}(b_i)(y_1) - \widetilde{f^j}(b_i)(y_2)| \\
&\qquad + |\widetilde{f^j}(b_i)(y_2) - \widetilde{f^j}(x_2)(y_2)| \\
&\quad \le 4\epsilon.
\end{aligned}$$

□

The following is the promised generalization of Proposition 2.1.

THEOREM 2.4. *Every function $f \in C^p_{wub}(E)$ can be extended in a unique way to a function $\tilde{f}: E'' \to R$, so that $\tilde{f} \in C^p(E''_{bw^*})$. Moreover, the mapping $f \to \tilde{f}$ is a homomorphism and, for all $j, n \in N$ and $j \le p$,*

$$\sup\{\|f^j(x)\|: x \in B_n\} = \sup\{\|\tilde{f}^j(x)\|: x \in B''_n\}.$$

Proof. Proposition 2.1 yields a function $\tilde{f} \in C(E''_{bw^*})$ which extends f and, more generally, functions $\widetilde{f^j} \in C(E''_{bw^*}; P(^jE''_{bw^*}))$ which extend f^j, for $j \le p$. Also, for each such j and $n \in N$, $\sup\{\|f^j(x)\|: x \in B_n\} = \sup\{\|\widetilde{f^j}(x)\|: x \in B''_n\}$. The theorem will be proved once we show that $\tilde{f}^j = \widetilde{f^j}$ for all $j \le p$, that is, once we show that the derivatives of the extension $\tilde{f}$ agree with the extension of the corresponding derivatives of f. We show this by induction. Let $p = 1$ and let $n \in N$ and $\epsilon > 0$ be arbitrary. By (1.4.4) there is a real number $\delta > 0$ such that for all $x \in B_n$ and $y \in E$, $\|y\| \le \delta$,

$$(2.4.1) \qquad |f(x+y)-f(x)-f'(x)(y)| \leq \epsilon\|y\|.$$

Let $s \in B_n''$ and $t \in E''$ with $\|t\| \leq \delta$. Let $(x_\alpha) \in B_n$ be a net converging weak* to s and let (y_α) be a net in E converging weak* to t, with $\|y_\alpha\| \leq \|t\|$. By Lemma 2.3, $\lim|f(x_\alpha+y_\alpha)-f(x_\alpha)-f'(x_\alpha)(y_\alpha)| = |f(s+t)-f(s)-f'(x)(t)|$, and therefore by (2.4.1) $|\tilde{f}(s+t)-\tilde{f}(s)-\widetilde{f'}(s)(t)| \leq \epsilon\|t\|$. This implies that $\widetilde{f'} = \tilde{f}'$.

Next assume that $\widetilde{f^j} = \tilde{f}^j$ for $j \leq p-1$. Reasoning as above and using the induction hypothesis, we find that for any B_n and $\epsilon > 0$ there is a number $\delta > 0$ such that for all $s \in B_n''$ and $t \in E''$ with $\|t\| \leq \delta$,

$$\left|\tilde{f}(s+t)-\tilde{f}(s)-\widetilde{f'}(s)(t)-\cdots-\frac{\tilde{f}^{p-1}}{(p-1)!}(s)(t)-\frac{\widetilde{f^p}}{p!}(s)(t)\right| \leq \epsilon\|t\|^p.$$

Therefore $\tilde{f}^p = \widetilde{f^p}$, which concludes the proof. □

COROLLARY 2.5. *The mapping $f \to \tilde{f}$ is a topological isomorphism, in the sense of Fréchet algebras, between $(C_{wub}^p(E), \tau_b^p)$ and $(C^p(E_{bw^*}''), \tau_b^p)$ for all $p = 0, 1, \ldots, \infty$.*

We note in passing that the results of 2.3, 2.4, and 2.5 are also valid for vector-valued functions $f: E \to F$, with virtually identical proofs.

3. Homomorphisms between algebras of uniformly weakly differentiable functions. By Corollary 2.5, every homomorphism $A: C_{wub}^p(E) \to C_{wub}^m(F)$ can be associated in a unique way to a homomorphism, still denoted A, between $C^p(E_{bw^*}'')$ and $C^m(F_{bw^*}'')$. Our object here is to characterize these homomorphisms in terms of mappings they induce between F'' and E''. Since the continuous case is discussed elsewhere (cf. [1]), we will always assume that at least one of p or m is bigger than 0. The first result is basic, albeit easily proved (modulu the aforementioned result of Michael).

PROPOSITION 3.1. *Let $\theta: C^p(E_{bw^*}'') \to R$ be a homomorphism. Then, if E' has the bounded approximation property, there exists a unique point $x \in E''$ such that $\theta(f) = f(x)$ for all $f \in C^p(E_{bw^*}'')$.*

Proof. Since $C^p(E_{bw^*}'')$ is a real Fréchet algebra, θ is continuous. Thus there is some point $x \in E''$ such that $\theta(\phi) = \phi(x)$ for all $\phi \in E'$. As a result, $\phi(P) = P(x)$ for all $P \in P_f(E)$, and the result follows by (1.4.5). □

COROLLARY 3.2. *Let $A: C^p(E_{bw^*}'') \to C^m(F_{bw^*}'')$ be a homomorphism. Then if E' has the bounded approximation property, A is induced by a function g: $F'' \to E''$. That is, $A(f) = f \circ g$ for every $f \in C^p(E_{bw^*}'')$.*

Proof. For each $y \in F''$, $\delta_y \circ A: C^p(E_{bw^*}'') \to R$ is a homomorphism, where δ_y denotes evaluation at y. Hence, there corresponds a unique point $x \in E''$ such that $\delta_y \circ A(f) = f(x)$ for all $f \in C^p(E_{bw^*}'')$. The required function g is given by $g(y) = x$. □

Having established the existence of some function g inducing every homomorphism, we now study differential properties of g. Our principal result here is the following theorem.

THEOREM 3.3. *Let* $A: C^p(E''_{bw^*}) \to C^m(F''_{bw^*})$ *be a homomorphism. Then if* E' *has the bounded approximation property,* A *is induced by a function* $g \in C^m(F''_{bw^*}; E''_{bw^*})$. *That is,* $A(f) = f \circ g$ *for every* $f \in C^p(E''_{bw^*})$.

Proof. The case $m=0$ is treated in [1] and so we will suppose that $m \geq 1$. Let g be the function defined in Corollary 3.2. Define $g_j: F'' \to P({}^jF''; E'')$ by

$$g_j(x)(y)(\phi) = [A(\phi)]^j(x)(y),$$

where x and $y \in F''$, $j \in N$, $j \leq m$, and $\phi \in E'$. Since the jth derivative of $A(\phi)$ evaluated at x, $[A(\phi)]^j(x)$, is in $P({}^jF''_{bw^*})$ for each $x \in F''$, it follows that for every $\epsilon > 0$, bounded set $B \subset F''$, and $\phi \in E'$, there exist $\{\psi_1, \ldots, \psi_k\} \subseteq F'$ and $\delta > 0$ such that if $y, z \in B$ and $|\psi_i(y-z)| < \delta$ $(i = 1, \ldots, k)$ then

$$|[g_j(x)(z) - g_j(x)(y)](\phi)| \leq \epsilon. \tag{3.3.1}$$

Consequently, each $g_j(x) \in P({}^jF''_{bw^*}; E''_{w^*}) = P({}^jF''_{bw^*}; E''_{bw^*})$. Next, the definition of $[A(\phi)]'(y)$ implies that, for all $y \in F''$ and all bounded sets $B \subset F''$, there is $\delta > 0$ such that the set

$$\{\epsilon^{-1}[g(y+\epsilon x) - g(y) - g_1(y)(\epsilon x)] : |\epsilon| \leq \delta, x \in B\} \text{ is bounded} \tag{3.3.2}$$

and that

$$\lim_{\epsilon \to 0} \epsilon^{-1}[g(y+\epsilon x) - g(y) - g_1(y)(\epsilon x)](\phi) = 0 \tag{3.3.3}$$

uniformly for $x \in B$. Combining (3.3.2) and (3.3.3) we conclude that, for all $y \in F''$ and all bounded sets $B \subseteq F''$,

$$\lim_{\epsilon \to 0} \epsilon^{-1}[g(y+\epsilon x) - g(y) - g_1(y)(\epsilon x)] = 0 \tag{3.3.4}$$

in E''_{bw^*} uniformly for all $x \in B$.

Assume now that $g_1, \ldots, g_{j-1}$ are the first $j-1$ derivatives of g, where $j < m$, and let us show that g_j is the jth derivative of g. Fix $y \in F''$ and denote by

$$C(y): F'' \times \cdots \times F'' \to E''$$

the unique symmetric j-linear mapping associated to $g_j(y)$. Let

$$u: F''_{bw^*} \to P({}^{j-1}F''_{bw^*}; E''_{bw^*})$$

be the linear mapping given by $u(x)(z) = C(y)(x, z, \ldots, z)$. We must show that, given any bounded set $B \subset F''$,

$$\lim_{\epsilon \to 0} \epsilon^{-1}[g_{j-1}(y+\epsilon x) - g_{j-1}(y) - u(\epsilon x)] = 0 \tag{3.3.5}$$

in E''_{bw^*} uniformly for $x \in B$. As in (3.3.2), we first note that, from the definition of $[A(\phi)]^j(y)$, there is a real number $\delta > 0$ such that the set

$$\{\epsilon^{-1}[g_{j-1}(y+\epsilon x)(z) - g_{j-1}(y)(z) - u(\epsilon x)(z)] : 0 < |\epsilon| \leq \delta,\ x, z \in B\}$$

is bounded. Thus it will be sufficient to show that, for all $\phi \in E'$, (3.3.6) $$\lim_{\epsilon\to 0}(\epsilon^{-1}[g_{j-1}(y+\epsilon x)(z)-g_{j-1}(y)(z)-u(\epsilon x)(z)](\phi))=0$$

uniformly for $x, z\in B$. If $D(y)$ is the unique symmetric j-linear mapping associated to $[A(\phi)]^j(y)$ and v is defined in analogy with u above, using D instead of C, then

$$\lim_{\epsilon\to 0}\epsilon^{-1}([A(\phi)]^{j-1}(y+\epsilon x)-[A(\phi)]^{j-1}(y)-v(\epsilon x))=0$$

uniformly for $x\in B$. In other words,

(3.3.7) $$\lim_{\epsilon\to 0}\epsilon^{-1}([A(\phi)]^{j-1}(y+\epsilon x)(z)-[A(\phi)]^{j-1}(y)(z)-v(\epsilon x)(z))=0$$

uniformly for $x, z\in B$. Now $D(y)(\epsilon x, z, \ldots, z)=C(y)(\epsilon x, z, \ldots, z)(\phi)$, so that $v(\epsilon x)(z)=u(\epsilon x)(z)(\phi)$. Therefore, (3.3.7) implies (3.3.6), and so g_j is the jth derivative of g.

Finally, it is easy to see that for each j $(1\le j\le m)$ and for all bounded sets $B\subset F''$, $\sup\{\|(g_j(x)-g_j(y))(z)\|: x, y, z\in B\}<\infty$. Therefore, to prove that $g_j\in C(F''_{bw^*}; P(^jF''_{bw^*}; E''_{bw^*}))$ we need only show that for all bounded sets $B\subset F''$, all $\phi\in E'$, and all $\epsilon>0$, there are $\psi_1, \ldots, \psi_k\in F'$ and $\delta>0$ such that if $x, y\in B$ and $|\psi_i(x-y)|<\delta$ $(i=1, \ldots, k)$ then $|(g_j(x)(z)-g_j(y)(z))(\phi)|<\epsilon$ for all $x\in B$. However, this is immediate from the fact that $[A(\phi)]^j\in C(F''_{bw^*}; P(^jF''_{bw^*}))$. □

It is of interest to note that the above proof shows that in fact $C^m(F''_{bw^*}; E''_{bw^*})$ consists of precisely those functions $g: F''\to E''$ such that for all $\phi\in E'$, $\phi\circ g\,|_F\in C^m_{wub}(F)$. To see this, let $g\in C^m(F''_{bw^*}; E''_{bw^*})$ and let $\phi\in E'$ be arbitrary. Then $\phi\circ g\,|_F\in C^m_{wub}(F)$ by Corollary 2.5. On the other hand, if $g: F\to E''$ is such that $\phi\circ g\in C^m_{wub}(F)$ for all $\phi\in E'$, define $\tilde{g}: F''\to E''$ by $\tilde{g}(y)(\phi)=(\widetilde{\phi\circ g})(y)$, as in Theorem 2.4. It follows that $\phi\circ\tilde{g}\in C^m(F''_{bw^*})$. Summarizing, $C^m(F''_{bw^*}; E''_{bw^*})=\{g: F''\to E'': \text{for all } \phi\in E', \ \phi\circ g\in C^m(F''_{bw^*})\}\cong\{g: F\to E'': \text{for all } \phi\in E', \ \phi\circ g\in C^m_{wub}(F)\}$.

An immediate consequence of Theorem 3.3, Lemma 1.9, and the definition of the τ_b^p topology is the following.

COROLLARY 3.4. *If E' has the bounded approximation property, then every homomorphism $A: C^p(E''_{bw^*})\to C^m(F''_{bw^*})$ is continuous.*

The finite-dimensional interpretation of Theorem 3.3 is given next.

COROLLARY 3.5. *Let $A: C^p(R^n)\to C^m(R^k)$ be a homomorphism. Then A is automatically continuous and is induced by a function $g\in C^m(R^k; R^n)$, via the mapping $A(f)=f\circ g$.*

Proof. Since all Hausdorff locally convex topologies on a finite-dimensional space coincide and since such spaces are automatically reflexive and have the bounded approximation property, it is immediate that Theorem 3.3 implies the existence of such a mapping g. □

Our next result will yield as a corollary a complete characterization of homomorphisms between two spaces of the form $C^p_{wub}(E)$.

THEOREM 3.6. *Suppose that E' has the bounded approximation property. Let $g \in C^m(F''_{bw^*}; E''_{bw^*})$ and let $p \in N$, $m \leq p$. Then for every $f \in C^p(E''_{bw^*})$, $f \circ g \in C^m(F''_{bw^*})$.*

Proof. By [15, 1.8.3], $f \circ g$ is m-times differentiable and the chain rule holds. By [15, 1.7.2], it suffices to prove that $(f \circ g)^m \in C(F''_{bw^*}; P(^m F''_{bw^*}))$.

The proof for general m is too complicated to be very enlightening. Therefore, we only prove the result for the case $m=2$, since this case contains all the ideas needed for the general case. So, let $R>0$ and $\epsilon>0$ be arbitrary. By Lemma 1.9, there exists $M \geq R$ such that

$$\sup\|g^j(z)(\Phi)\| : j \leq 2,\ z, \Phi \in F'',\ \|z\| \leq R,\ \|\Phi\| \leq 1\} \leq M. \tag{3.6.1}$$

By Corollary 2.5 and (1.4.5), there is a finite type polynomial $P \in P_f(E)$ such that

$$\sup\{\|f^j(x) - \tilde{P}^j(x)\| : j \leq 2,\ \|x\| \leq M,\ x \in E''\} < \epsilon, \tag{3.6.2}$$

where $\tilde{P}$ is the extension of P to E''.

Since $f \in C^p(E''_{bw^*})$, there are $\psi_1, \ldots, \psi_s \in E'$ and $\delta_1 > 0$ such that, if $z_1, z_2 \in E''$, $\|z_1\| \leq M$, $\|z_2\| \leq M$, and $|\psi_i(z_1 - z_2)| < \delta_1$ $(i=1,\ldots,s)$, then

$$\|f^j(z_1) - f^j(z_2)\| \leq \epsilon \quad (j \leq 2). \tag{3.6.3}$$

For each j $(1 \leq j \leq s)$, $\psi_j \circ g \in C(F''_{bw^*})$ and so there exist $\delta_2 > 0$ and $\phi_1, \ldots, \phi_k \in F'$ such that, if $u_1, u_2 \in F''$, $\|u_1\| \leq R$, $\|u_2\| \leq R$, and $|\phi_i(u_1 - u_2)| < \delta_2$ $(i=1,\ldots,k)$, then

$$|\psi_j(g(u_1) - g(u_2))| < \delta_1 \quad (j=1,\ldots,s). \tag{3.6.4}$$

Next, for each $j \leq 2$, $g^j \in C(F''_{bw^*}; P(^j F''_{bw^*}; E''_{bw^*}))$. Hence there exist $\phi_{k+1}, \ldots, \phi_l \in F'$ and $\delta_3 > 0$ such that if $u_1, u_2 \in F''$, $\|u_1\| \leq R$, $\|u_2\| \leq R$, and $|\phi_i(u_1 - u_2)| < \delta_3$ $(i=k+1,\ldots,l)$, then

$$\begin{cases} |\tilde{P}''(g(x))(g'(u_1)(\Phi)) - \tilde{P}''(g(x))(g'(u_2)(\Phi))| \leq \epsilon \\ |\tilde{P}'(g(x))(g''(u_1)(\Phi)) - \tilde{P}'(g(x))(g''(u_2)(\Phi))| \leq \epsilon \end{cases} \tag{3.6.5}$$

whenever $\Phi \in F''$, $\|\Phi\| \leq 1$, $x \in F''$, and $\|x\| \leq R$.

Finally, let $u_1, u_2 \in F''$ such that $\|u_1\| \leq R$, $\|u_2\| \leq R$, and

$$|\phi_i(u_1 - u_2)| < \min(\delta_2, \delta_3) \quad \text{for } i=1,\ldots,l.$$

For every $\Phi \in F''$ $(\|\Phi\| \leq 1)$, $(f \circ g)''(u_1)(\Phi) - (f \circ g)''(u_2)(\Phi)$ can be written as a sum of terms of the following type, using the chain rule:

$$[f''(g(u_1)) - f''(g(u_2))](g'(u_1)(\Phi)), \tag{3.6.6}$$

$$[f''(g(u_2)) - \tilde{P}''(g(u_2))](g'(u_1)(\Phi)), \tag{3.6.7}$$

$$\tilde{P}''(g(u_2))(g'(u_1)(\Phi)) - \tilde{P}''(g(u_2))(g'(u_2)(\Phi)), \tag{3.6.8}$$

$$[f'(g(u_1)) - f'(g(u_2))](g''(u_1)(\Phi)), \tag{3.6.9}$$

$$[f'(g(u_2)) - \tilde{P}'(g(u_2))](g''(u_1)(\Phi)), \tag{3.6.10}$$

$$\tilde{P}'(g(u_2))(g''(u_1)(\Phi)) - \tilde{P}'(g(u_2))(g''(u_2)(\Phi)). \tag{3.6.11}$$

(3.6.1), (3.6.3), and (3.6.4) imply that the norms of each of the terms in (3.6.6) and (3.6.9) are bounded by ϵM. Using (3.6.1) and (3.6.2) we see that (3.6.7) and (3.6.10) are each bounded in norm by ϵM. Finally, by (3.6.5) the norms of the terms of the form (3.6.8) and (3.6.11) are bounded by ϵ. Therefore,

$$\|(f\circ g)''(u_1)-(f\circ g)''(u_2)\| \leq CM\epsilon$$

for some absolute constant C, and the proof is complete. □

COROLLARY 3.7. *If E' has the bounded approximation property, then the space of homomorphisms $A: C^p_{wub}(E) \to C^m_{wub}(F)$, where $m \leq p$, can be identified with the space of all functions $g: F \to E''$ such that, for all $\phi \in E'$, $\phi \circ g \in C^m_{wub}(F)$ via the formula $A(f) = \tilde{f}\circ g$ for $f \in C^p_{wub}(E)$.*

It is natural to consider the relation between the two spaces $C^m(F''_{bw^*}; E''_{bw^*})$ and $C^m(F''; E'')$, the latter space consisting of all Fréchet differentiable mappings between the Banach spaces F'' and E''. In fact, we prove below the somewhat surprising (albeit easily proved) result that $C^m(F''_{bw^*}; E''_{bw^*}) \subset C^{m-1}(F''; E'')$ for every $m = 1, \ldots, \infty$. In particular, a C^∞ mapping for the bw^* topologies is also C^∞ for the norm topologies. In Section 4, examples are given which show that $C^{m-1}(F''; E'')$ cannot be replaced by $C^m(F''; E'')$.

PROPOSITION 3.8. *Let $g \in C^m(F''_{bw^*}; E''_{bw^*})$. Then $g \in C^{m-1}(F''; E'')$.*

Proof. The proof consists of a straightforward induction argument, combined with several applications of the mean value theorem. For $m=1$, $\|g(x)-g(y)\| = \sup\{|\phi(g(x)-g(y))| : \phi \in E', \|\phi\| \leq 1\} \leq \sup\{\|\phi\circ g'(z)\| \|x-y\|\}$, where the supremum is taken over all $z \in F''$ which lie on the segment $[x,y]$ and over all ϕ of norm ≤ 1. Since g' is locally bounded, it follows that g is continuous.

To prove the result for general m, it will be convenient to regard the kth derivative of $g: F'' \to E''$ at a point $x \in F''$ as a k-linear mapping of $F'' \times \cdots \times F'' \to E''$, $d^k g(x) \in L(^kF''; E'')$, rather than as a k-homogeneous polynomial $g^k(x) \in P(^kF''; E'')$. Thus, since $d^k g: F'' \to L(^kF''; E'')$, $d(d^k g)(x) \in L(F''; L(^kF''; E''))$ and so $d(d^k g)(x)(h) \in L(^kF''; E'')$ for all $h \in F''$. Now, assuming the result for $j = 1, \ldots, m$, let $g \in C^{m+1}(F''_{bw^*}; E''_{bw^*})$. For $x, h \in F''$ and $\phi \in E'$, consider

$$\frac{\|\phi\circ[d^{m-1}g(x+h) - d^{m-1}g(x) - d(d^{m-1}g)(x)(h)]\|}{\|h\|}.$$

Let $f(u) = \phi\circ[d^{m-1}g(x+u) - d(d^{m-1}g)(x)(u)]$, so that the numerator in the above expression is $\|f(h) - f(0)\|$.

By the mean value theorem, this quantity is dominated by $\sup\{\|f'(v)\| \|h\|\}$, where the supremum is taken over all points v lying on the segment $[0,h]$. However, $f'(v) = \phi\circ[d(d^{m-1}g)(x+v) - d(d^{m-1}g)(x)]$, which by another application of the mean value theorem is bounded above by

$$\sup\{\|\phi\circ d(d(d^{m-1}g))(x+y)\| \|v\|\},$$

where the supremum is taken over $y \in [x, x+u]$. Since $d(d(d^{m-1}g))(x+y)$ can be naturally associated to $d^{m+1}g(x+y)$, which is locally bounded, it follows by taking the supremum over all $\phi \in E'$ ($\|\phi\| = 1$) that $g \in C^m(F''; E'')$. □

Thus far, we have excluded the case $p < m$. The reason for this is apparent from the following.

PROPOSITION 3.9. *Let $A: C^p(E''_{bw^*}) \to C^m(F''_{bw^*})$ be a homomorphism, where $p < m$. If E' has the bounded approximation property, then A is induced by a constant function $g: F'' \to E''$.*

The proof depends on the following elementary lemma.

LEMMA 3.10. *Let $g \in C^m(R)$, where $m \geq 1$. If $p < m$ and $f \circ g \in C^m(R)$ for all $f \in C^p(R)$, then g is constant.*

Proof. For simplicity, suppose that $g(0) = 0$ and that $g'(0) \neq 0$. Let $f(x) = |x|^{p+1/2}$. Then $f \circ g \notin C^{p+1}(R)$ since $(f \circ g)^{p+1}$ is unbounded near 0, which is a contradiction. The general case follows easily. □

Proof of Proposition 3.9. By Theorem 3.3, there is a function $g \in C^m(F''_{bw^*}; E''_{bw^*})$ such that $A(f) = f \circ g$ for every $f \in C^p(E''_{bw^*})$. Let us suppose that $g(0) = 0$ and that, for some $v \in F''$, $g(v) \neq 0$. Let F_0 be the span of $\{v\}$ in F'' and E_0 the span of $\{g(v)\}$ in E''. Let $\pi: E'' \to E_0$ be the projection $\pi(\Phi) = \Phi(\phi)g(v)$ for $\Phi \in E''$, where $\phi \in E'$ is chosen to satisfy $g(v)(\phi) = 1$. It is immediate that π is a linear mapping which lies in $C^\infty(E''_{bw^*}; E_0)$. Thus, by Theorem 3.6, $h \circ \pi \in C^p(E''_{bw^*})$ for every $h \in C^p(E_0)$. Therefore,

$$A(h \circ \pi) = h \circ \pi \circ g \in C^m(F''_{bw^*}).$$

In particular, $(h \circ \pi \circ g)|_{F_0} \in C^m(F_0)$. However, Lemma 3.9 tells us that $(\pi \circ g)|_{F_0}$ is constant, although $\pi \circ g(0) = 0$ and $\pi \circ g(v) = g(v) \neq 0$. Thus we have a contradiction, and the proof is complete. □

4. Examples. We give three examples in this section which illustrate the extent to which the conclusions of the preceding section are best possible. Example 4.1 gives a situation in which the homomorphism $A: C^1_{wub}(E) \to C^1_{wub}(F)$ induces a mapping $g: F'' \to E''$ which fails to be Fréchet differentiable when E'' and F'' have their norm topologies. The next example shows that the inducing function g can be differentiable without being continuously differentiable. Finally, we adopt an example of [3] to show that not every homomorphism from $C^1(R)$ into a Fréchet algebra need be continuous.

EXAMPLE 4.1. Let F be the Banach space c_0 of null sequences of real numbers and let E be the Banach space of null sequences of complex numbers, considered as a real vector space. It is known that E' can be associated with the space of continuous linear functions $\phi: E \to R$ such that, for some complex sequence $(\alpha_n) \in l_1$, $\phi(y) = \mathrm{Re}[\sum \alpha_n \overline{y_n}]$. For each $x = (x_n)$ in F, let $y = (y_n)$ in E be defined as $y_n = e^{inx_n}/n$. Define $g: F \to E$ by $g(x) = y$. By [7], g is Hadamard differentiable [15] with derivative $g'(x)(y) = z$, where $z = (z_n) \in E$ satisfies $z_n = ie^{inx_n}y_n$. However, g is not Fréchet differentiable. Indeed, if g were Fréchet differentiable then its Fréchet derivative would have to coincide with its Hadamard derivative. Therefore, for each $n \in N$ we would have

$$\left|\frac{1}{t}\right| \|g(te_n)-g(0)-g'(0)(te_n)\| = \left|\left(\frac{1}{nt}\right)\right| |e^{int}-1-int|$$
$$\geq 1-\left|\frac{1-e^{int}}{nt}\right| \geq 1-\frac{2}{nt},$$

which does not tend to 0 as $t\to 0$ uniformly in n.

This lack of differentiability notwithstanding, we now show that g is induced by a homomorphism $A: C^1_{wub}(E)\to C^1_{wub}(F)$. We first show that for all C^1 functions f on F, $f\circ g\in C^1(F)$. It suffices to show that the mapping $(f\circ g)': F\to F'$ is continuous at a fixed point $x\in F$.

To see this, note first that $f\circ g$ is Hadamard differentiable for every $f\in C^1(E)$, and therefore $f\circ g$ has a Hadamard derivative $(f\circ g)'(x)\in F'$ for every $x\in F$. Now $|f\circ g(x+h)-f\circ g(x)-(f\circ g)'(x)(h)|=|[(f\circ g)'(x+\theta h)-(f\circ g)'(x)](h)|$ for some $\theta\in[0,1]$ depending on h, $\leq \|(f\circ g)'(x+\theta h)-(f\circ g)'(x)\|\,\|h\|$, and the Fréchet differentiability follows. So, let $(\alpha_n(y))\in E'$ be the vector associated to $f'(g(x+y))$, and let $(\alpha_n)\in E'$ be associated to $f'(g(x))$. If $z\in F$ and $\|z\|\leq 1$ then

$$[(f\circ g)'(x+y)-(f\circ g)'(x)](z)=f'(g(x+y))(g'(x+y)(z)-f'(g(x))g'(x)(z)$$

$$(4.1.1)\qquad =\mathrm{Re}\left[-i\sum_{n=1}^{\infty}\alpha_n(y)e^{-inx_n}e^{-iny_n}z_n-(-i)\sum_{n=1}^{\infty}\alpha_n e^{-inx_n}z_n\right]$$
$$=\mathrm{Re}\left[(-i)\sum_{n=1}^{\infty}e^{-inx_n}z_n(\alpha_n(y)e^{-iny_n}-\alpha_n)\right].$$

On the other hand, $g(F)\subset K$, where $K=\{(\beta_n)\in E: |\beta_n|\leq 1/n,\ n\in N\}$. Since $f\in C^1(E)$ and K is compact, $f'(K)$ is a compact subset of E'; in particular, every element of $f'(K)$ is bounded in norm by M, say. Moreover, the sequence $(d_n)\in c_0$, where $d_n=\sup\{\sum_{j=n}^{\infty}|x_j|: x=(x_j)\in f'(K)\}$, and so given $\epsilon>0$ we can find $n_0\in N$ such that $|d_n|<\epsilon/6$ whenever $n\geq n_0$. Choose $\delta>0$ so small that $|u|<\delta$ implies

$$(4.1.2)\qquad |e^{iu}-1|<\epsilon/(6n_0M).$$

Let $0<\delta_1<\delta$ such that if $y\in E$ and $\|y\|<\delta_1$ then

$$(4.1.3)\qquad \|f'(g(x+y))-f'(g(x))\|<\epsilon/6.$$

Combining (4.1.1)-(4.1.3) we conclude that if $\|y\|<\min(\delta_1,\delta/(n_0+1))$ is satisfied by $y\in F$ then

$$\|(f\circ g)'(x+y)-(f\circ g)'(x)\|\leq \sum_{n=1}^{\infty}|\alpha_n(y)e^{-iny_n}-\alpha_n|=\sum_{n=1}^{\infty}|\alpha_n(y)-e^{iny_n}\alpha_n$$
$$\leq \sum_{n=1}^{\infty}|\alpha_n(y)-\alpha_n|+\sum_{n=1}^{n_0}|\alpha_n|\,|1-e^{iny_n}|$$
$$+\sum_{n=n_0+1}^{\infty}|\alpha_n|\,|1-e^{iny_n}|$$
$$<\frac{\epsilon}{6}+\frac{n_0M\epsilon}{6n_0M}+\frac{2\epsilon}{6}=\epsilon.$$

This shows that $(f\circ g)'$ is continuous.

In addition, $f\circ g\in C^1_{wub}(F)$ for every $f\in C^1(E)$. In fact, we show more; namely that $f\circ g$ and $(f\circ g)'$ are weakly uniformly continuous on F and not just on each bounded subset of F. To see this, let $\epsilon>0$ be arbitrary. Using the relative compactness of $g(F)$, we can find $\delta>0$ such that if $z_1,z_2\in K$ and $\|z_1-z_2\|<\delta$ then $|f(z_1)-f(z_2)|<\epsilon$. Let $\delta_1>0$ be chosen such that if $s,t\in R$ and $|s-t|<\delta_1$ then $|e^{is}-e^{it}|<\delta$. Choose n_0 so that $2/n_0<\delta$, and let V be the weak 0-neighborhood in F defined as $V=\{x\in F: |x_j|<\delta_1/n_0 \text{ for } j=1,\dots,n_0\}$. If $x,y\in F$ satisfy $x-y\in V$ then an easy calculation shows that $\|g(x)-g(y)\|<\delta$. Hence $|f\circ g(x)-f\circ g(y)|<\epsilon$, and so $f\circ g$ is weakly uniformly continuous on F. Next we show that $(f\circ g)'$ is weakly uniformly continuous on F. Let $\epsilon>0$ be arbitrary, and choose $\delta>0$ so small that if $z_1,z_2\in K$ satisfy $\|z_1-z_2\|<\delta$ then

$$\|f'(z_1)-f'(z_2)\|<\epsilon. \tag{4.1.4}$$

Let $M\geq 1$ be such that $\|f'(z)\|\leq M$ for all $z\in K$, and let $n_0\in N$ satisfy

$$\frac{2}{n_0}<\delta \quad\text{and}\quad \sum_{n=n_0}^{\infty}|\alpha_n(y)|<\frac{\epsilon}{4}, \tag{4.1.5}$$

where $(\alpha_n(y))=f'(g(y))$. Let $\delta_1>0$ be such that if $s,t\in R$ and $|s-t|<\delta_1$ then

$$|e^{is}-e^{it}|<\min\left(\frac{\epsilon}{2Mn_0},\delta\right). \tag{4.1.6}$$

Let $V=\{x\in F: |x_j|<\delta_1/n_0 \text{ for } j=1,\dots,n_0\}$. Then for all $x,y\in F$ with $x-y\in V$,

$$\begin{aligned}
&\|(f\circ g)'(x)-(f\circ g)'(y)\|\\
&\quad=\|f'(g(x))g'(x)-f'(g(y))g'(y)\|\\
&\quad\leq\|f'(g(x))g'(x)-f'(g(y))g'(x)\|+\|f'(g(y))g'(x)-f'(g(y))g'(y)\|\\
&\quad\leq\|f'(g(x))-f'(g(y))\|\,\|g'(x)\|+\sup_{\|z\|\leq 1}\operatorname{Re}\left[-i\sum_{n=1}^{\infty}\alpha_n(y)(e^{-inx_n}-e^{-iny_n})z_n\right].
\end{aligned}$$

Since $\|g'(x)\|\leq 1$ for all x, the first term on the right-hand side is bounded above by ϵ, using (4.1.4). The second term is bounded by

$$\sum_{n=1}^{n_0}|\alpha_n(y)|\,|e^{-inx_n}-e^{-iny_n}|+\sum_{n=n_0+1}^{\infty}|\alpha_n(y)|\,|e^{-inx_n}-e^{-iny_n}|\leq\frac{Mn_0\epsilon}{2Mn_0}+2\frac{\epsilon}{4}=\epsilon,$$

using (4.1.5) and (4.1.6). Therefore, $f\circ g$ is a member of $C^1_{wub}(F)$ as required. To summarize, this example shows the existence of a homomorphism $A: C^1_{wub}(E)\to C^1_{wub}(F)$ given by $A(f)=f\circ g$, where g is not a Fréchet differentiable mapping between the Banach spaces F and E. Note that A is automatically continuous, as can be seen by applying Corollary 3.4 or else by a direct computation.

EXAMPLE 4.2. For each $n\in N$, let $\chi_n: R\to[0,1]$ be a C^∞ function with support contained in $[1/(n+1),1/n]$ and such that $\chi_n(t_n)=1$, where

$$t_n=\frac{1/n+1/(n+1)}{2}.$$

Let $g_n: R\to R$ be defined by $g_n(t)=\int_{-\infty}^{t}\chi_n(s)\,ds$. Note that for all $t\in R$ and all $n\in N$, $|g_n(t)|\leq 1/n(n+1)<1/n^2$ and so the function $g: R\to c_0$, $g(t)=(g_n(t))$, is well defined. Moreover, g is a differentiable mapping. In fact, it is obvious that g is differentiable at any $t \neq 0$. In addition, for any $t \in [1/(n+1), 1/n]$,

$$\frac{\|g(t)\|}{|t|} \leq \frac{n+1}{n^2} \to 0 \quad \text{as } n \to \infty.$$

This shows that $g'(0) = 0$. A routine calculation shows that, for each $k \in N$, $g'(t_k) = e_k$, the usual kth unit basis vector of c_0, and so g' is not continuous at 0.

Note that $\phi \circ g \in C^1(R)$ for each $\phi \in l_1 = c_0'$. Indeed, let $u_j \to u_0$ in R. Then, if $\phi = (\phi_j) \in l_1$, $(\phi \circ g)'(u_j) = \sum_{n=1}^{\infty} \phi_n \chi_n(u_j)$. Therefore, if $u_0 \neq 0$, it is clear that $(\phi \circ g)'(u_j) \to \sum_{n=1}^{\infty} \phi_n \chi_n(u_0)$. If $u_0 = 0$, then (given $\epsilon > 0$) choose n_0 such that $\sum_{n=n_0}^{\infty} |\phi_n| < \epsilon$. Therefore, $\sum_{n=1}^{\infty} |\phi_n \chi_n(u_j)| \leq \sum_{n=1}^{n_0-1} |\phi_n \chi_n(u_j)| + \epsilon = \epsilon$ if j is sufficiently large. Define a homomorphism $A: P_f(c_0) \to C^1(R)$ by $A(P) = P \circ g$. Note that the above calculations show that A is well defined. Furthermore, A is $\tau_b^1 - \tau_b^1$ continuous because, for each interval $I_n = [-n, n]$,

$$\sup_{t \in I_n} |A(P)(t)| \leq \sup_{\|x\| \leq 1} |P(x)| \quad \text{and} \quad \sup_{t \in I_n} |(A(P))'(t)| \leq \sup_{\|x\| \leq 1} \|P'(x)\|.$$

Since $C^1(R)$ is complete for the τ_b^1 topology, an appeal to (1.4.5) yields an extension $A: C_{wub}^1(c_0) \to C^1(R)$ as a continuous homomorphism. It is straightforward to prove that if a sequence (P_n) in $P_f(c_0)$ converges to $f \in C_{wub}^1(c_0)$ for the τ_b^1 topology, then $(P_n \circ g) \to f \circ g$ in $C^1(R)$. Therefore $A(f) = f \circ g$ for every $f \in C_{wub}^1(c_0)$, and so we have an example of a homomorphism $A: C_{wub}^1(c_0) \to C^1(R)$ induced by a differentiable function $g: R \to c_0$ which is not C^1.

EXAMPLE 4.3. Let $X = C^1[0,1]$ be the Banach algebra of continuously differentiable functions $x = [0,1] \to R$ with the usual norm, $\|x\| = \sup_{t \in [0,1]} |x(t)| + \sup_{t \in [0,1]} |x'(t)|$. We recall the following theorem of Bade and Curtis, who also showed that $X = B$ satisfies all the conditions in the theorem.

THEOREM 4.4 ([3]). *Let B be a commutative algebra with identity. Suppose that B contains a maximal ideal M such that M^2 is not closed in B. Then there exist a Banach algebra Y and a discontinuous homomorphism $A: B \to Y$.*

Therefore, if we take $T: C^1(R) \to X$ to be the restriction map then we conclude that $A \circ T: C^1(R) \to Y$ is a discontinuous homomorphism. Thus, continuity of homomorphisms from the Fréchet algebra $C^1(R)$ is not automatic.

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