GROUP ACTIONS ON TREES AND LENGTH FUNCTIONS

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Introduction. A group of isometries acting on a metric tree gives a length function in the sense of Lyndon [6], associated with each point of the tree. In this paper certain properties of length functions are considered, in particular the existence of extensions, and relationships with corresponding properties of group actions are described.

The quotient of a group action on a metric space is described in Section 1. Group actions on trees and relationships between non-Archimedean elements, bounded actions, and fixed points are considered in Sections 2 and 3. These results are used in Section 4 to relate extensions of length functions to actions of factor groups on quotient trees, in Theorems 4.2 and 4.3. Results on extensions of length functions can therefore be translated to group actions on trees. An example is given in Theorem 4.5, where the action of any hypercentral group is described.

1. Factors of actions on metric spaces. Let a group K act as a group of isometries on a metric space X, equipped with a metric d. Define a relation between the elements of X by $u \sim v$ if $\inf_{x \in K} d(u, xv) = 0$.

LEMMA 1.1. *The relation* $\sim$ *is an equivalence relation on* X.

Proof. The relation is clearly reflexive, and since (for any $x \in K$) $d(u,xv) = d(xv,u) = d(v,x^{-1}u)$, it is symmetric.

For any $x, y \in K$ and $u, v, w \in X$, by the triangle inequality

$$d(u,xyw) \le d(u,xv) + d(xv,xyw) = d(u,xv) + d(v,yw).$$

Thus, if $u \sim v$ and $v \sim w$ then $u \sim w$ and the relation is transitive. □

Let X/K be the set of equivalence classes under $\sim$, and denote the equivalence class of u by $[u]$.

PROPOSITION 1.2. *X/K is a metric space with metric d' defined by*

$$d'([u],[v]) = \inf_{x \in K} d(u,xv).$$

Moreover, if X is complete then so is X/K.

Proof. We first show that d' is well defined. If $u' \sim u$, $v' \sim v$, then for $x, y, z \in K$

$$d(yu', xzv') \le d(yu', u) + d(u, xv) + d(xv, xzv');$$

that is,

$$d(u', y^{-1}xzv') \le d(u, yu') + d(u, xv) + d(v, zv').$$

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Thus $\inf_{x\in K} d(u',xv') \le \inf_{x\in K} d(u,xv)$. By symmetry, $\inf_{x\in K} d(u',xv') = \inf_{x\in K} d(u,xv)$, and d' is well defined.

By definition, $d'([u],[v])=0$ if and only if $[u]=[v]$; and since d is symmetric, so is d'. To prove the triangle inequality let $u,v,w\in K$. Then, for $x,y\in K$,

$$d(u,xv)+d(v,yw)=d(u,xv)+d(xv,xyw)\ge d(u,xyw).$$

Hence, taking infima over K,

$$d'([u],[v])+d'([v],[w])\ge d'([u],[w]).$$

It follows that d' is a metric on X/K.

Suppose now that X is a complete metric space and let $\{[u_n]\}$ be a Cauchy sequence in X/K. Then there exist natural numbers $m_1<m_2<m_3<\cdots$ such that, for $m,n\ge m_r$,

$$\inf_{x\in K} d(u_m,xu_n)=d'([u_m],[u_n])<\frac{1}{2^{r+1}}.$$

Define a sequence $\{v_n\}$ in X as follows. Set $v_1=u_{m_1}$, and since

$$\inf_{x\in K} d(u_{m_1},xu_{m_2})<\frac{1}{2^2}$$

there exists $y\in K$ with $d(u_{m_1},yu_{m_2})<1/2$. Take $v_2=yu_{m_2}$. We proceed so that for $v_1,v_2,\dots,v_r$ we have $d(v_m,v_{m+1})<1/2^m$ for $1\le m<r$. Then $[v_r]=[u_{m_r}]$, and since

$$\inf_{x\in K} d(v_r,xu_{m_{r+1}})=d'([u_{m_r}],[u_{m_{r+1}}])<\frac{1}{2^{r+1}}$$

there exists $z\in K$ with $d(v_r,zu_{m_{r+1}})<1/2^r$. We take $v_{r+1}=zu_{m_{r+1}}$. Now, for $n>m$,

$$\begin{aligned} d(v_m,v_n)&\le d(v_m,v_{m+1})+d(v_{m+1},v_{m+2})+\cdots+d(v_{n-1},v_n)\\ &\le \frac{1}{2^m}+\frac{1}{2^{m+1}}+\cdots+\frac{1}{2^{n-1}}<\frac{1}{2^{m-1}}.\end{aligned}$$

The sequence $\{v_n\}$ is thus a Cauchy sequence in X, and $\lim v_n=v$ therefore exists. Hence

$$[v]=\lim[v_n]=\lim[u_{m_n}]=\lim[u_n],$$

showing that X/K is complete. □

We now consider the situation where K is a normal subgroup of a larger group of isometries G, as will be the case for applications in Section 4.

PROPOSITION 1.3. *Let G act as a group of isometries on a metric space X, with K a normal subgroup of G. Then G/K acts as a group of isometries on X/K, where for $x\in G$, $xK([u])=[xu]$.*

Proof. Let $u\sim v$ in X. Then for $a\in K$ and $x\in G$,

$$\inf_{a\in K} d(xu,axv)=\inf_{a\in K} d(u,x^{-1}axv)=\inf_{a\in K} d(u,av)=0.$$

Thus $xu \sim xv$ and the action of G/K is well defined. By the definition of $\sim$, K acts trivially on X/K.

G/K then acts as a group of isometries on X/K, since for any $u, v \in X$

$$d'(xK([u]), xK([v])) = d'([xu], [xv]) = \inf_{a\in K} d(xu, axv)$$
$$= \inf_{a\in K} d(u, x^{-1}axv) = \inf_{a\in K} d(u, av) = d'([u], [v]). \quad \square$$

2. Actions on trees and length functions. A (metric) *tree* T is a complete metric space which satisfies the following two axioms:

T0 T has no subspace homeomorphic to a circle.

T1 For any two points $u, v \in T$ there is an isometry $\alpha: [0, r] \to T$ with $\alpha(0) = u$, $\alpha(r) = v$, where $r = d(u, v)$.

The isometry α is unique, and the image $\alpha([0, r])$ is denoted by $[u, v]$ and is called a *segment* of T.

This definition is due to Tits [8]. Imrich and Schwarz ([4] and [5]) have shown that a complete metric space T is a tree if and only if it satisfies T1 and

T2 *The four-point condition:* For any four points $u, v, s, t \in T$, two of the sums

$$d(u,v)+d(s,t), \qquad d(u,s)+d(v,t), \qquad d(u,t)+d(v,s)$$

are equal, with the third no larger.

Morgan and Shalen [7] and Alperin and Bass [1] have studied Λ-trees, where Λ is an ordered abelian group. It follows from Proposition II.1.13 of [7] (or equally from Proposition 2.15 of [1]) that **R**-trees are trees, as above. Conversely, it follows from (1.1)–(1.3) of [8] that trees are **R**-trees.

A *rooted tree* T, u_0 consists of a tree T with a distinguished point $u_0 \in T$. The following is a restatement of Theorem 3.17 of [1] in the case where $\Lambda = \mathbf{R}$.

PROPOSITION 2.1. *Let T be a complete metric space with $u_0 \in T$. Then T, u_0 is a rooted tree if and only if the following conditions hold.*

T1′ *For each $v \in T$ there is an isometry $\alpha: [0, r] \to T$ with $\alpha(0) = u_0$, $\alpha(r) = v$, where $r = d(u_0, v)$.*

T2′ *For any three points $v, s, t \in T$, two of the sums*

$$d(u_0,v)+d(s,t), \qquad d(u_0,s)+d(v,t), \qquad d(u_0,t)+d(v,s)$$

are equal with the third no larger.

A *length function* on a group G is a function $\ell: G \to \mathbf{R}$ such that, if $c(x,y) = \frac{1}{2}(\ell(x)+\ell(y)-\ell(xy^{-1}))$, then the following axioms hold for all $x, y, z \in G$:

A1′ $\ell(1) = 0$;

A2 $\ell(x) = \ell(x^{-1})$;

A4 $c(x,y) < c(x,z)$ implies $c(x,y) = c(y,z)$.

The definition is due to Lyndon [6], and an equivalent statement of A4 is that of the three numbers $c(x,y)$, $c(x,z)$, $c(y,z)$ two are equal with the third no smaller.

An element $x \in G$ is *Archimedean* if $\ell(x^2) > \ell(x)$ and is *non-Archimedean* otherwise. The set of non-Archimedean elements of G is denoted by N, and a length function is said to be *Archimedean* if $N=\{1\}$ and *non-Archimedean* if $N=G$. It follows from Propositions 2.1 and 3.4 of [10] that N is a normal subset of G, and $x\in N$ if and only if the set $\{\ell(x^n);\ n$ an integer$\}$ is bounded.

Let a group G act as a group of isometries on a tree T and let $u\in T$. Then a length function $\ell_u\colon G\to\mathbf{R}$ is defined by $\ell_u(x)=d(u,xu)$. Axioms A1′ and A2 follow immediately, and A4 is an easy consequence of T2, the four-point condition. In the other direction, Chiswell [3] and Alperin and Moss [2] have shown that given a length function $\ell\colon G\to\mathbf{R}$ there exists an essentially minimal rooted tree T, u_0, such that G acts as a group of isometries on T with $\ell_{u_0}=\ell$.

Let G act on T with $u, v\in T$. Then, for $x\in G$,

$$\begin{aligned}\ell_v(x^n)=d(v,x^nv)&\le d(v,u)+d(u,x^nu)+d(x^nu,x^nv)\\&=d(u,x^nu)+2d(u,v)\\&=\ell_u(x^n)+2d(u,v).\end{aligned}$$

The set $\{\ell_v(x^n);\ n\in Z\}$ is therefore bounded if and only if $\{\ell_u(x^n);\ n\in Z\}$ is bounded. Thus x is non-Archimedean with respect to ℓ_v if and only if it is non-Archimedean with respect to ℓ_u. An element x is therefore said to be *non-Archimedean* if it is non-Archimedean with respect to ℓ_u for some (and hence all) $u\in T$. N denotes the set of non-Archimedean elements of G.

Let $T^x=\{u\in T; xu=u\}$, the set of points fixed by x. If T^x is nonempty it is a subtree of T. If T^x is empty then the tree T contains an *axis* for x, that is, an isometric image of $\mathbf{R}$ on which x acts as a translation; see Theorem II.2.3 of [7]. G is said to *act without fixed points* if $T^x=\phi$ for each $1\ne x\in G$.

The results in the following proposition have also been considered elsewhere; see Lemmas 1 and 3 of [5], Theorem 3 of [3], and Proposition (6.3) and Corollary (6.13) of [1].

PROPOSITION 2.2. *If G acts on a tree T then $x\in N$ if and only if $T^x\ne\phi$. If $x\in N$ and $u\in T$ then the mid-point of $[u,xu]$ is fixed by x.*

Proof. Let $u\in T^x\ne\phi$; then $x^2u=xu=u$. Hence $\ell_u(x^2)=\ell_u(x)=0$ and $x\in N$.

Conversely, suppose that $x\in N$. Then for any $u\in T$, $\ell_u(x^2)\le\ell_u(x)$. Consider points $u, xu, x^{-1}u\in T$ and let $[u,xu]\cap[u,x^{-1}u]=[u,v]$, with

$$s=\ell_u(x)=d(u,xu)=d(u,x^{-1}u)\quad\text{and}\quad t=d(u,v).$$

Then

$$2s-2t=d(x^{-1}u,xu)=d(u,x^2u)=\ell_u(x^2)\le\ell_u(x)=s,$$

and so $s/2\le t$ (see Figure 1).

Let w be the mid-point of $[u,xu]$; then $d(u,w)=s/2$ and w is also the mid-point of $[u,x^{-1}u]$. The isometry x sends $[x^{-1}u,u]$ to $[u,xu]$, and since w is the unique point a distance $s/2$ from $x^{-1}u$ and from u it follows that $xw=w$ and $T^x\ne\phi$. □

3. Bounded actions. Let a group K act as a group of isometries on a tree T. K is said to have *bounded action* on T if for some $u\in T$ the set of lengths $\{\ell_u(x); x\in K\}$ is bounded. By the triangle inequality,

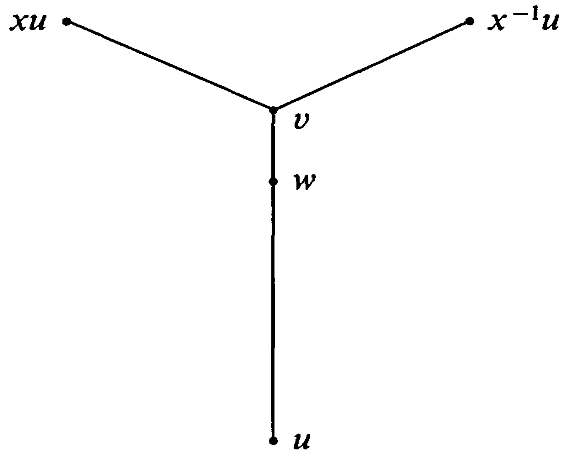

Figure 1

$$\begin{aligned}\ell_v(x) = d(v, xv) &\le d(v, u) + d(u, xu) + d(xu, xv)\\ &= d(u, xu) + 2d(u, v) = \ell_u(x) + 2d(u, v)\end{aligned}$$

for $v \in T$. Thus K has bounded action if the set $\{\ell_v(x);\, x \in K\}$ is bounded for each $v \in T$. Since the lengths with respect to a given base are bounded, K will consist entirely of non-Archimedean elements.

LEMMA 3.1. *Let G act on a tree T with $u \in T$. If $x, y, xy \in N$ then x and y have a common fixed point in T, namely the mid-point of $[u, zu]$ where z is the element of greater length from x, y.*

Proof. Write $\ell = \ell_u$; then, by Proposition 3.3 of [10], two of $\ell(x), \ell(y), \ell(xy)$ are equal with the third no larger. We first consider the case where $\ell(x) = \ell(y) \ge \ell(xy)$ (see Figure 2).

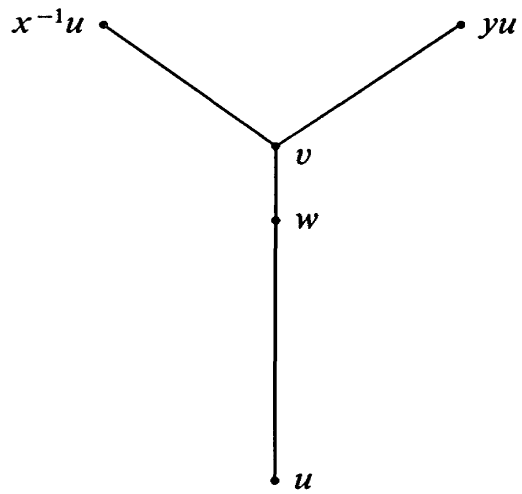

Figure 2

Let $[u, x^{-1}u] \cap [u, yu] = [u, v]$, with

$$s = \ell(x) = \ell(y) = d(u, x^{-1}u) = d(u, yu) \quad \text{and} \quad t = d(u, v).$$

Then

$$2s - 2t = d(x^{-1}u, yu) = d(u, xyu) = \ell(xy) \leq \ell(x) = \ell(y) = s,$$

and so $s/2 \leq t$. Hence w, the mid-point of $[u, x^{-1}u]$, is also the mid-point of $[u, yu]$. As was seen in the proof of Proposition 2.2, this is also the mid-point of $[u, xu]$. By Proposition 2.2, w is fixed by both x and y, and hence also by xy.

The cases where $\ell(xy) > \ell(x)$ or $\ell(y)$ follow by considering either the triple xy, y^{-1}, x or the triple x^{-1}, xy, y in place of x, y, xy. □

THEOREM 3.2. *Let K act as a group of isometries on a tree T. Then K has bounded action if and only if it fixes some point of T.*

Proof. If K has a fixed point u then $\ell_u(x) = 0$ for each $x \in K$, and K has bounded action.

Conversely, suppose K has bounded action, in which case K consists entirely of non-Archimedean elements. Let $u \in T$, write $\ell = \ell_u$, and let $r = \sup\{\ell(x);\ x \in K\}$. If there exists $y \in K$ with $\ell(y) = r$ then (by Lemma 3.1) w, the mid-point of $[u, yu]$, is a fixed point for each $x \in K$.

If there is no $y \in K$ with $\ell(y) = r$, then for each natural number n choose $x_n \in K$ with $d(u, x_n u) = \ell(x_n) > r - 1/n$ and $\ell(x_m) > \ell(x_n)$ for $m > n$. Let w_n be the mid-point of $[u, x_n u]$. If $m > n$ then, by Proposition 3.3 of [10], $\ell(x_m^{-1}x_n) = \ell(x_m) > \ell(x_n)$. Thus

$$d(u, x_m u) = \ell(x_m) = \ell(x_m^{-1}x_n) = d(u, x_m^{-1}x_n u) = d(x_m u, x_n u)$$

and so $[u, x_m u] \cap [u, x_n u] = [u, w_n]$ (see Figure 3). Hence, on $[u, x_m u]$,

$$d(w_m, w_n) = \frac{1}{2}(\ell(x_m) - \ell(x_n)) \leq \frac{1}{2}\left(r - \left(r - \frac{1}{n}\right)\right) = \frac{1}{2n}.$$

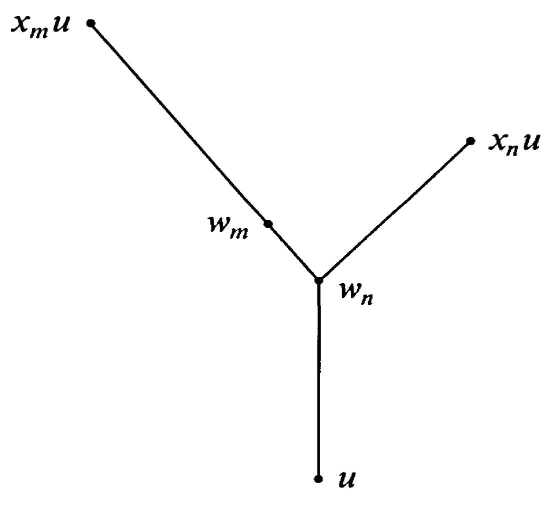

Figure 3

Thus $\{w_n\}$ is a Cauchy sequence, and since T is complete $\lim w_n = w \in T$ exists. Suppose there exists $x \in K$ with $xw \neq w$. Let $d(w, xw) = \epsilon > 0$, and choose x_n with $\ell(x_n) > \ell(x)$ and with $d(w, w_n) < \epsilon/2$. By Lemma 3.1, w_n is fixed by x and so

$$d(w_n, xw) = d(xw_n, xw) = d(w_n, w) < \epsilon/2.$$

Hence, $d(w, xw) \leq d(w, w_n) + d(w_n, xw) < \epsilon$, giving a contradiction. It follows that w is fixed by K. □

An example of a bounded action is given by the action of K on any tree T, where K is an (ℓ, m, n)-group generated by elements x, y, where $x^\ell = y^m = (xy)^n = 1$. Since x, y, and xy have finite order they are in N. By Lemma 3.1, x and y have a common fixed point, which is a fixed point for K.

Any bounded action of a group K will essentially be made up of trees and actions given by Chiswell's construction in [3], following from non-Archimedean length functions given by chains of subgroups of K, as described in [9].

4. Quotient trees and extensions of lengths.

LEMMA 4.1. *If K acts on a tree T as a group of isometries with bounded action then T/K is a tree.*

Proof. By Proposition 1.2, T/K is complete. By Theorem 3.2, K fixes some point of T, say the point u_0. To show that T/K is a tree we prove that conditions T1′ and T2′ of Proposition 2.1 are satisfied.

The point u_0 is fixed by K, so in T/K, for any $v \in T$ and any $x \in K$, $d(u_0, v) = d(u_0, xv)$ so that $[u_0] = \{u_0\}$ and, writing $[u_0] = u_0$, $d'(u_0, [v]) = d(u_0, xv)$. If $d(u_0, v) = r$ then since T is a tree there exists an isometry $\alpha: [0, r] \to T$ with $\alpha(0) = u_0$, $\alpha(r) = v$. Let $\beta = p\alpha: [0, r] \to T/K$, where $p: T \to T/K$ is the projection. Now if $w_1, w_2 \in [u_0, v] = \alpha([0, r])$ with $w_1 = \alpha(r_1)$, $w_2 = \alpha(r_2)$ then $d(w_1, w_2) = |r_1 - r_2|$. For $[w_1], [w_2] \in \beta([0, r])$,

$$d'(u_0, [w_1]) = d(u_0, w_1) = r_1 \quad \text{and} \quad d'(u_0, [w_2]) = d(u_0, w_2) = r_2,$$

and so by the triangle inequality $d'([w_1], [w_2]) = |r_1 - r_2|$. The function β is therefore an isometry, and condition T1′ is satisfied by T/K.

To establish T2′ we need to show that for $v, s, t \in T$ the four-point condition is satisfied for u_0, $[v]$, $[s]$, $[t]$ in T/K. In fact, we show that the following equivalent condition is satisfied. If for some m, two of

$$d'(u_0, [v]) + d'([s], [t]), \qquad d'(u_0, [s]) + d'([v], [t]), \qquad d'(u_0, [t]) + d'([v], [s])$$

are $\leq m$, then the third must be $\leq m$.

Suppose $d'(u_0, [v]) + d'([s], [t])$ and $d'(u_0, [s]) + d'([v], [t])$ are $\leq m$. Then, since

$$d'([s], [t]) = \inf_{x \in K} d(s, xt), \qquad d'([v], [t]) = \inf_{x \in K} d(v, xt),$$

given $\epsilon > 0$ there exist $y, z \in K$ such that

$$d(u_0, v) + d(ys, t) \leq m + \epsilon, \qquad d(u_0, s) + d(zv, t) \leq m + \epsilon.$$

This can be written

$$d(u_0, zv)+d(ys, t) \leq m+\epsilon, \qquad d(u_0, ys)+d(zv, t) \leq m+\epsilon.$$

T is a tree and therefore satisfies the four-point condition for u_0, zv, ys, and t, and hence

$$d(u_0, t)+d(v, z^{-1}ys) = d(u_0, t)+d(zv, ys) \leq m+\epsilon.$$

Thus

$$d'(u_0, [t])+d'([v], [s]) = d(u_0, t)+\inf_{x \in K} d(v, xs) \leq m+\epsilon.$$

This holds for any $\epsilon > 0$ and so $d'(u_0, [t])+d'([v], [s]) \leq m$, showing that T2′ is satisfied, and completing the proof. □

If $\ell: G \to \mathbf{R}$ is a length function and K is a proper normal subgroup of G, with $f: G \to H = G/K$ the projection homomorphism, then ℓ is an *extension* of a length function ℓ_1 on K by a length function ℓ_2 on H if

$$\ell(x) = \begin{cases} \ell_1(x) & \text{if } x \in K, \\ \ell_2(f(x)) & \text{if } x \notin K. \end{cases}$$

By Theorem 4.2 of [10] this occurs precisely when $\ell(ax) = \ell(x)$ for each $a \in K$, $x \in G \backslash K$. The length function ℓ_1, which is the restriction of ℓ to K, is non-Archimedean.

THEOREM 4.2. *Let G act as a group if isometries on a tree T, with K a proper normal subgroup of G. If K has bounded action then G/K acts on T/K, a tree. Moreover, if for $u \in T$ the length function ℓ_u is an extension of ℓ_1 on K by ℓ_2 on G/K, then $\ell_2 = \ell_{[u]}$.*

Proof. By Proposition 1.3, G/K acts on T/K, which is a tree by Lemma 4.1. If ℓ_u is an extension of ℓ_1 on K by ℓ_2 on G/K, then (for each $a \in K$, $x \in G \backslash K$) $\ell_u(ax) = \ell_u(x)$; that is, $d(u, axu) = d(u, xu)$. Thus

$$\begin{aligned} \ell_{[u]}(xK) &= d'([u], xK([u])) = d'([u], [xu]) \\ &= \inf_{a \in K} d(u, axu) = d(u, xu) = \ell_u(x) = \ell_2(xK), \end{aligned}$$

completing the proof. □

If G acts on a tree T, then a proper subgroup K of G *acts with minimal length* if, for each $u \in T$, $a \in K$, $x \in G \backslash K$,

$$\ell_u(a) = d(u, au) \leq d(u, xu) = \ell_u(x).$$

THEOREM 4.3. *Let G act as a group of isometries on a tree T, with K a proper normal subgroup of G. If K acts with minimal length then G/K acts on the tree T/K such that for each $u \in T$, ℓ_u is an extension of ℓ_1 on K by $\ell_2 = \ell_{[u]}$ on G/K.*

Proof. If K acts with minimal length then it has bounded action. For $u \in T$ the length function ℓ_u is minimal on K, which by Theorem 4.2 of [10] is equivalent to

ℓ_u being an extension of ℓ_1 on K by ℓ_2 on G/K. The result follows by Theorem 4.2 above. □

The definition of a bounded action is extended from a group or subgroup to a subset in the obvious way. That is, a subset S of G has bounded action on T if for some (and hence each) $u \in T$, the set $\{\ell_u(x);\ x \in S\}$ is bounded.

Theorem 4.2 can be applied to an action of G on T, where N has bounded action. By Theorem 5.3 of [10], applied for each $u \in T$, N is a normal subgroup of G and ℓ_u is an extension of ℓ_1 on N by an Archimedean ℓ_2 on G/N. By Theorem 4.2, since $\ell_{[u]} = \ell_2$, G/N acts on T/N without fixed points. The following is therefore true.

THEOREM 4.4. *Let G act as a group of isometries on a tree T. If N has bounded action, then it is a normal subgroup of G and G/N acts on the tree T/N without fixed points. Moreover, for each $u \in T$, ℓ_u is an extension of ℓ_1 on N by $\ell_{[u]}$ on G/N.*

Length functions on hypercentral groups have been studied in [11]. Translating the theorem there to actions on trees, as in Theorem 4.4, the result below follows.

THEOREM 4.5. *Let a hypercentral group G act on a tree T. Then $N = G$, or N is a proper subgroup of G with bounded action such that G/N, which is isomorphic to a subgroup of the additive reals, acts on the tree T/N without fixed points.*

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