SOLVABILITY OF INVARIANT SECOND-ORDER DIFFERENTIAL OPERATORS ON METABELIAN GROUPS

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Introduction. This paper is the third in a series (see [3], [4]) devoted to the idea of applying group representations in order to obtain solvability properties of differential operators on groups and homogeneous spaces. A typical theorem in the subject asserts that if L is a left-invariant differential operator on a connected Lie group G and if its infinitesimal components $\pi(L)$, $\pi \in \hat{G}$, have bounded inverses with suitable norm-growth properties, then L is semiglobally solvable. Theorems 2.1 and 2.4 of [3] are of this variety. In [4] these theorems are improved by allowing for Sobolev-type norms on the representation spaces. The main theorem of [4] is then applied to prove new solvability results for differential operators on the affine motion group of the line (i.e., the $ax+b$ group). In this paper we broaden the applications of [4] considerably.

Until [3], virtually all group representations–related work on solvability concentrated on *nilpotent* groups and transversally elliptic operators. The framework for much of my work in solvability has been *solvable* groups $G=SN$ which are semidirect products of a normal simply connected nilpotent subgroup N and a vector group S. Within this framework I have considered parabolic and hyperbolic operators in addition to transversally elliptic operators. For example, a typical operator of interest is the variable-coefficient heat operator $\mathcal{H}=\sum_i A_i - \sum_j X_j^2$, where $\{A_i\}$ is a basis of the Lie algebra $\mathfrak{s}$ of S, and $\{X_j\}$ is a basis of $\mathfrak{n}$. If we set $A=\sum_i A_i$ and consider its span, we see it is really no loss of generality to assume that $\dim S=1$. In this paper we study parabolic and hyperbolic second-order operators on $G=SN$ where N is *abelian* and $\dim S=1$. (The $ax+b$ group is the simplest noncommutative example of such a group.) So let $A\in\mathfrak{s}$, $A\neq 0$, and let $\{X_j\}_{j=1}^r$ be a basis of $\mathfrak{n}$. Denote the Laplacian on $\mathfrak{n}$ (with respect to this basis) by $\Delta=\sum_{j=1}^r X_j^2$. We shall consider the following three "classical" second-order operators:

(i) Heat Operator $\mathcal{H}=A-\Delta$,

(ii) Schroedinger Operator $\mathcal{S}=iA-\Delta$,

(iii) Wave Operator $\mathcal{W}=A^2-\Delta$.

We saw in [3] and [4] that the eigenvalues of the matrix $\Lambda=\mathrm{Ad}_{\mathfrak{n}} A$ play a critical role in solvability results on metabelian solvable groups. They continue to play a feature role here; in fact, our main result is the following.

THEOREM. *Suppose that all the eigenvalues of* Λ *have positive real part. Then each of the operators* $\mathcal{H}$, $\mathcal{S}$, $\mathcal{W}$ *is globally solvable.*

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We close the introduction by noting that only when every eigenvalue of Λ is semisimple and purely imaginary do the above operators have constant coefficients. In the situation of the Theorem, they have variable coefficients composed of familiar transcendental and/or polynomial functions.

The main lemma. Let $G=SN$ be a semidirect product of a normal (real) vector group N, acted upon by a one-parameter (real) group S. Let $\mathfrak{g}, \mathfrak{n}, \mathfrak{s}$ denote the Lie algebras and $\mathfrak{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Choose $A \in \mathfrak{s}$, $A \neq 0$. Our basic assumption is that *all* the eigenvalues of the matrix $\Lambda = \mathrm{Ad}_{\mathfrak{n}} A$ have positive real parts.

We recall the representation theory of G via the Mackey machine. The irreducible unitary representations of G consist of two families:

(a) the characters $\chi_\lambda: g = e^{tA}n \to e^{it\lambda}$, $n \in N$, $t, \lambda \in \mathbf{R}$;

(b) the infinite-dimensional representations $\pi_\gamma = \mathrm{Ind}_N^G \gamma$, $\gamma \in \hat{N}$, $\gamma \neq 1$.

Recall that $\pi_\gamma \cong \pi_{\gamma'}$ if and only if $s \cdot \gamma = \gamma'$ for some $s \in S$. Note also that we are using the fact that for any character $\gamma \in \hat{N}$, $\gamma \neq 1$, the stability group S_γ is trivial; this is an easy consequence of our assumption about the eigenvalues of Λ. It's also easy to see from the basic assumption that the orbits of S in $\hat{N}$ are locally closed. In particular, the Lie group G is type I. Furthermore, it is a final simple consequence of the basic assumption that every nontrivial S-orbit in $\hat{N}$ tends to both ∞ and 0. In particular, we may find a *relatively compact* set $\mathcal{C} \subset \hat{N}$ (that is also bounded away from 0) which serves as a cross-section for $\hat{N}/S$. Thus a complete list of representatives for the generic classes in $\hat{G}$ is given by $\mathcal{L} = \{\pi_\gamma : \gamma \in \mathcal{C}\}$. (Generic means that $\hat{G} \backslash \mathcal{L}$ has Plancherel measure zero.)

The representations $\pi_\gamma = \mathrm{Ind}_N^G \gamma$ may be realized in $L^2(S)$ via the formula

$$\pi_\gamma(ns)f(\sigma) = \gamma(\sigma n \sigma^{-1}) f(\sigma s), \quad s, \sigma \in S, \ n \in N, \ f \in L^2(S)$$

(see [3, p. 1277]). Furthermore, we identify $L^2(S)$ to $L^2(\mathbf{R})$ via the identification $f(e^{tA}) = f(t)$, $t \in \mathbf{R}$. Now let $L \in \mathfrak{U}(\mathfrak{g})$ be considered as a right-invariant differential operator on G. (Formulation of the main result of [4], Theorem 4.5, requires right invariance for the operator.) We shall write L_γ to denote $\pi_\gamma(L)$: $L_\gamma = \pi_\gamma(L)$. Then L_γ may be considered as a differential operator on $\mathbf{R}$. Here is the main Lemma—it is the analog of [4, Thm. 5.2].

LEMMA. *Fix* $\beta \geq 0$. *Suppose that every* L_γ, $\gamma \in \mathcal{C}$, *has the following property:* $\forall f \in C_c^\infty(\mathbf{R})$, $\exists u \in C^\infty(\mathbf{R})$ *such that* $u \equiv 0$ *for large negative* t, $e^{-\beta t} u \in L^2$ *for large positive* t, *and* $L_\gamma u = f$. *Then* L *is globally solvable.*

Proof. We begin with the assumption that $\{X_j\}_{j=1}^r$ is a modified Jordan basis for Λ in the sense of [3, pp. 1291, 1292]. Then we can explicate the Sobolev norms of [2] for the spaces of the representations π_γ, $\gamma \in \mathcal{C}$. The collection $\{A, X_j : j = 1, \ldots, r\}$ forms a basis of $\mathfrak{g}$. Then a simple computation shows that, when we realize π_γ on $L^2(\mathbf{R})$, we have

$$\pi_\gamma(A) = \frac{d}{dt} \quad \text{and} \tag{1}$$

$$\pi_\gamma(X_j) = p_{\gamma,j}(t), \tag{2}$$

a multiplication operator given by the purely imaginary function

$$p_{\gamma,j}(t)=\frac{d}{dx}\gamma(\exp e^{\operatorname{ad} tA}xX_j)_{x=0}. \tag{3}$$

We also have

$$-\pi_\gamma(\Delta)=\pi_\gamma(-\Delta)=p_\gamma(t), \tag{4}$$

another multiplication operator, this time by the nonnegative function

$$p_\gamma(t)=-\sum_{j=1}^{r}p_{\gamma,j}(t)^2. \tag{5}$$

But because of the assumption that $\{X_j\}_{j=1}^r$ is a modified Jordan basis, we have: either

$$e^{\operatorname{ad} tA}X_j=e^{t\rho_j}\left[X_j+\sum_{k>j}c_{j,k}(t)X_k\right] \tag{6}$$

or

$$e^{\operatorname{ad} tA}X_j=e^{t\rho_j}\left[a_j(t)X_j+b_j(t)X_{j\pm 1}+\sum_{\substack{k>j+1\\ k>j}}c_{j,k}(t)X_k\right], \tag{7}$$

where either ρ_j is a real positive eigenvalue of Λ and the functions $c_{j,k}(t)$ are polynomial in t; or ρ_j is the real part of a complex eigenvalue of Λ, the $c_{j,k}$'s are as above, and the a_j and b_j are orthogonal sinusoidal functions of t with the same frequency.

Next we express $\gamma\in\hat{N}$ by

$$\gamma(e^X)=e^{i\theta(X)},\quad X\in\mathfrak{n},\ \theta\in\mathfrak{n}^*.$$

Taking $\gamma\neq 1$ amounts to assuming $\theta\neq 0$. Set $p_\theta=p_\gamma$. Now suppose $\gamma\in\mathcal{C}$ is given. Let X_j be chosen so that $\theta(X_j)\neq 0$, and choose it so that the real part of the corresponding eigenvalue is as large as possible—say $\rho_\gamma=\rho_\theta$. Then clearly

$$p_\theta(t)\sim e^{2\rho_\theta t}\quad\text{as } t\to\infty.$$

It is also clear that (up to equivalence) we may realize the Sobolev norms of the representations π_γ on $L^2(\mathbf{R})$ by

$$\|h\|_s=\sum_{j+k=s}\left\{\int\left|e^{j\rho_\gamma t}\frac{d^k}{dt^k}h(t)\right|^2dt\right\}^{1/2}$$

(see [2], [4]).

Now we reason as in [4, Thm. 5.2]—that is, we carry out the usual Hormander argument. Consider the linear form

$$(f,h)=\int_{-\infty}^{\infty}f(t)h(t)\,dt,$$

with topology defined as follows. On the first component we place the Sobolev norms of π_γ (i.e., $\|\cdot\|_s$). On the second component we use the norms $\|L_\gamma^t h\|_s$. Of course, continuity of the form in the first component is obvious. The continuity in the second component comes about because of the solvability hypothesis:

$$|(f,h)|=|(L_\gamma u,h)|=|(u,L_\gamma^t h)|=|(e^{-j\rho_\gamma t}u,e^{j\rho_\gamma t}L_\gamma^t h)|\le C\|L_\gamma^t h\|_j,$$

where we chose j so that $j\rho_\gamma>\beta$. As usual, joint continuity follows and therefore

$$c_\gamma\|L_\gamma^t u\|_s\ge\|u\|_{-s}\quad\text{for some } s\ge 0 \text{ and constant } c_\gamma>0.$$

So L_γ^t has a bounded left inverse, and therefore $L_\gamma=\pi_\gamma(L)$ has a bounded right inverse

$$\pi_\gamma(L)^{-1}\colon \mathcal{H}^s_{\pi_\gamma}\to\mathcal{H}^{-s}_{\pi_\gamma}.$$

Moreover,

$$\|\pi_\gamma(L)^{-1}\|\le c_\gamma,\quad \gamma\in\mathcal{C}.$$

Finally, it is clear from the construction that the bound c_γ varies continuously with γ. Hence, by the (relative) compactness of $\mathcal{C}$, we have that

$$\sup_{\gamma\in\mathcal{C}}\|\pi_\gamma(L)^{-1}\|\le c.$$

Then an application of [4, Thm. 4.5] yields semiglobal solvability of L. But since G is exponential solvable, it follows from [1] that L must actually be globally solvable. ☐

Before going on we note that the Lemma applies to *any* $L\in\mathfrak{U}(\mathfrak{g})$ which satisfies its hypotheses—not just the second-order operators (i)–(iii). Also, in the proof of the Lemma we selected a special kind of basis of $\mathfrak{n}$. The operators (i)–(iii) are taken with respect to *any* basis of $\mathfrak{n}$.

Proof of the Theorem. We continue with the same assumptions. Let $A\in\mathfrak{s}$, $A\neq 0$, and assume that every eigenvalue of $\Lambda=\mathrm{Ad}_\mathfrak{n} A$ has positive real part. Let $\{X_j\}_{j=1}^r$ be any basis of $\mathfrak{n}$. Consider the heat, Schroedinger, and wave operators (i)–(iii). We seek to prove that each of these operators is globally solvable. To achieve this we need only verify the hypotheses of the Lemma. We treat the three operators separately.

(i) *Heat operator* $\mathcal{H}=A-\sum_{j=1}^r X_j^2$. First assume that $\{X_j\}$ is a modified Jordan basis. Then, from equations (1)–(4), we have

$$L_\gamma=\pi_\gamma(\mathcal{H})=\frac{d}{dt}+p_\gamma(t),$$

with p_γ as defined in (5). Let us set

$$P_\gamma(t)=\int_0^t p_\gamma(s)\,ds.$$

Then the equation $L_\gamma u=f\in C_c^\infty(\mathbf{R})$ has a solution given by

$$u(t)=e^{-P_\gamma(t)}\int_{-\infty}^t e^{P_\gamma(s)}f(s)\,ds.$$

It is clear from formulas (6) and (7) that

$$p_\gamma(t)\sim e^{2\rho_\gamma t}\quad\text{as } t\to+\infty.$$

Therefore the smooth function $u(t)$ is bounded for large positive t, and hence

$$e^{-\beta t} u \in L^2, \quad t \gg 0$$

for any $\beta > 0$. (For example, select β to be the largest of the real parts of the eigenvalues of Λ.) The requirement that $u \equiv 0$ for large negative t is clearly satisfied since f has compact support. It follows from the Lemma that $\mathcal{H}$ is globally solvable.

If $\{X_j\}$ is not a modified Jordan basis, we still have

$$L_\gamma = \pi_\gamma(\mathcal{H}) = \frac{d}{dt} + q_\gamma(t),$$

where $q_\gamma(t)$ is the nonnegative function

$$q_\gamma(t) = -\sum_j q_{\gamma,j}(t)^2,$$

$$q_{\gamma,j}(t) = \frac{d}{dx}\gamma(\exp e^{\operatorname{ad} tA} xX_j)_{x=0}.$$

Then we reason as in [3, pp. 1297ff] to conclude that there is a positive scalar $\delta > 0$ such that

$$q_\gamma(t) \geq \delta^2 p_\gamma(t).$$

It follows that the growth properties of q_γ as $t \to \infty$ are as strong as that of p_γ. Therefore we can apply the same reasoning to

$$L_\gamma = \frac{d}{dt} + q_\gamma(t)$$

to obtain a solution u with the properties of the Lemma. Once again the conclusion is global solvability of $\mathcal{H}$.

(ii) *Schroedinger operator* $\mathcal{S} = iA - \sum_{j=1}^r X_j^2$. The situation is quite similar to that of the heat operator. We have

$$L_\gamma = \pi_\gamma(\mathcal{S}) = i\frac{d}{dt} + q_\gamma(t).$$

Setting

$$Q_\gamma(t) = \int_0^t q_\gamma(s)\, ds,$$

we see that we may solve the equation $L_\gamma u = f$ by the formula

$$u(t) = -ie^{iQ_\gamma(t)} \int_{-\infty}^t e^{-Q_\gamma(s)} f(s)\, ds.$$

The function $Q_\gamma(t)$ is real-valued, and so it is clear that u is bounded. Also, u vanishes for $t \ll 0$. Hence, it satisfies the conditions of the Lemma, and we obtain global solvability for $\mathcal{S}$.

(iii) *Wave operator* $\mathcal{W} = A^2 - \sum_{j=1}^r X_j^2$. This time the infinitesimal components $L_\gamma = \pi_\gamma(\mathcal{W})$ of our differential operator $\mathcal{W}$ become

$$L_\gamma = \frac{d^2}{dt^2} + q_\gamma(t).$$

Now we reason as in case (iii) of [4, Cor. 5.3]. Suppose that u_1, u_2 are two linearly independent solutions of the homogeneous equation $L_\gamma u = 0$. Let $f \in C_c^\infty(\mathbf{R})$ and suppose that $L_\gamma u = f$ for some $u \in C^\infty(\mathbf{R})$. (The existence of u is guaranteed by standard ODE—we must adjust u so that it satisfies the remaining two conditions of the Lemma.) In any event, for large negative t, u is a solution of the homogeneous equation. So $u = c_1u_1 + c_2u_2$, $t \ll 0$, for some scalars c_1, c_2. Replace u by $v = u - c_1u_1 - c_2u_2$. Then $L_\gamma v = f$ and $v = 0$ for large negative t. Finally, for large positive t we again have that v solves the homogeneous equation. Hence, the growth of v as $t \to \infty$ is completely determined by that of u_1 and u_2. Now we apply [5] to conclude that the growth of u_1 and u_2 is exactly the same as the growth of the solutions to the (asymptotically similar) homogeneous equation

$$\frac{d^2}{dt^2} + e^{2\rho t} = 0, \quad \rho = \rho_\gamma. \tag{8}$$

(Note that [5] applies since $q_\gamma(t)$ obeys the hypothesis on [5, p. 193].) But in fact, if J_0, Y_0 denote the Bessel functions of order zero, then two linearly independent solutions of (8) are

$$J_0\left(\frac{1}{\rho}e^{\rho t}\right) \quad \text{and} \quad Y_0\left(\frac{1}{\rho}e^{\rho t}\right).$$

These are bounded functions as $t \to \infty$. Hence the solution v for $L_\gamma u = f$ constructed above obeys the conditions of the Lemma. As before, the conclusion is global solvability of $\mathcal{W}$. □

Concluding remarks. 1. A familiar category of groups covered by the methods of this paper are (many) exponential solvable groups AN, where MAN is a maximal parabolic subgroup of a connected semisimple Lie group. Of course the $ax+b$ group is the simplest of these. It was precisely such groups that were not treatable by the balanced eigenvalue technique of [3]—see [3, Remark 4.4(iii)].

2. In spite of the preceding remark, we observe that Conjecture 5.1 of [4] remains open. Namely, it is still not known if *every* right invariant differential operator on the $ax+b$ group is (even) locally solvable.

3. We remark that under the situation of our main Theorem, we may replace $A \in \mathfrak{s}$ by λA, $\lambda > 0$. However, the results do not apply if we change A to $-A$. For example, if G is the $ax+b$ group with Lie algebra generators A, X satisfying the bracket relation $[A, X] = X$, then the operator $A + X^2$ does not seem to be amenable to the methods of this paper.

4. The author believes it is likely that the conclusion of the main Theorem is true provided only *one* of the eigenvalues of Λ has positive real part. That is good enough to carry forward (generically) the growth estimates in the proof. However, in that case we cannot assert relative compactness for the cross-section $\mathcal{C}$ of $\hat{N}/A$, and the final reasoning in the proof of the Lemma breaks down.

5. The author contemplates future work with second-order operators on exponential solvable groups $G = SN$, with dim $S = 1$ but with N only simply connected nilpotent. It is interesting to consider the operators (i)–(iii) in that circumstance, where Δ may be replaced by a sub-Laplacian in generators of the Lie algebra $\mathfrak{n}$.

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