INVARIANTS OF COMPLEX FOLIATIONS AND THE MONGE–AMPÈRE EQUATION

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0. Introduction. In this paper we study the local and global geometric invariants associated to a foliation of a complex manifold by complex submanifolds. We call such foliations *complex foliations* and do not require them to be transversely holomorphic (i.e., the leaves of the foliation might not fit together in a holomorphic way)—indeed, one of the objects of the present paper is to determine global conditions which force complex foliations to be transversely holomorphic.

Our primary motivation for undertaking such a study comes from the complex Monge–Ampère equation. In [2] it is observed that a smooth, real-valued function u on a complex n-dimensional manifold M, satisfying the equations $(\partial\bar{\partial}u)^{p+1}=0$ and $(\partial\bar{\partial}u)^{p}\neq 0$, gives rise to a foliation of M by complex submanifolds of complex codimension p, a Monge–Ampère foliation, in the following way. The closed $(1,1)$-form $\partial\bar{\partial}u$ defines a distribution $L=\{X\in TM \mid i(X)\partial\bar{\partial}u=0\}$, which is easily shown to be integrable.

The technique of exploiting the geometry of Monge–Ampère foliations to study the Monge–Ampère equation has been used a great deal in recent years ([1], [5], [10], [11], and [13]). We call particular attention to the paper of Lempert, where the Kobayashi metric is used to associate a canonical complex foliation, singular at one point, to a smooth, strongly convex domain in $\mathbb{C}^n$. The results of [3] show that the geometry of the entire domain can be recovered from the foliation germ at the singularity together with metric data on the leaves.

In most of the above work the solution of the Monge–Ampère equation is used heavily in the analysis of the associated foliation, but there are interesting examples of foliations which are not Monge–Ampère (e.g., those arising in twistor theory [6]), and even when a foliation does arise from a solution of the Monge–Ampère equation, the solution may not be known but the geometry of the foliation itself may be of interest. For example, it would be interesting to have a characterization of which germs of singular foliations can arise from the Lempert foliation of a convex domain. A systematic study of complex foliations is therefore of interest, and this paper initiates such a study.

The outline of the paper is as follows. Our notational conventions, as well as some basic definitions and useful local formulas, are presented in Section 1.

In Section 2 the relative de Rham complex $(\Omega^k_{\mathfrak{F}}, d_{\mathfrak{F}})$ of a complex foliation is presented. Here $\Omega^k_{\mathfrak{F}}$ denotes the sheaf of sections of the bundle $\wedge^k L^*$, where L is the tangent bundle of $\mathfrak{F}$ and the operator $d_{\mathfrak{F}}$ is exterior differentiation in the

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directions tangent to the leaves of $\mathfrak{F}$. This complex is a resolution of the sheaf $C_{\mathfrak{F}}^{\infty}$ of germs of functions which are locally constant along the leaves of $\mathfrak{F}$, and it is possible to define *relative Chern classes* of the foliation, denoted by $c_k(\mathfrak{F}) \in H^{2k}(M, C_{\mathfrak{F}}^{\infty})$, which contain information about its nonholomorphic nature.

The *complex Bott partial connection* is introduced in Section 3. (A partial connection is a covariant derivative which is defined only in directions tangent to the leaves of $\mathfrak{F}$.) The torsion tensor of the complex Bott connection is shown to vanish precisely when the foliation is transversely holomorphic (this tensor first appears in the work of Bedford–Burns [1] under the name *antiholomorphic twist*). We compute the Cartan structure equations for the complex Bott connection and use them to produce explicit representatives $C_k(\mathfrak{F}) \in \Gamma(M, \Omega_{\mathfrak{F}}^{2k})$ for the relative Chern classes of $\mathfrak{F}$.

In Section 4 we consider the problem of determining when a foliation is Monge–Ampère. Suppose that the foliation $\mathfrak{F}$ is Monge–Ampère; then the covariant derivative of the nonnegative, real $(1,1)$-form $\sigma = i\partial\bar{\partial}u$ vanishes in the directions tangent to $\mathfrak{F}$. More generally, a *tangentially Monge–Ampère foliation* is a complex foliation together with a real, $d_{\mathfrak{F}}$-closed, $(1,1)$-form, say σ, satisfying the condition

$$L = \{X \in TM \mid i(X)\sigma = 0\}.$$

It is shown, in the special case where the foliation is tangentially Monge–Ampère and the complex Bott connection agrees with the unique Hermitian connection on Q defined by the Hermitian inner product associated to the $(1,1)$-form yielding a generalization of the result of [1], that the Ricci form of a Monge–Ampère foliation vanishes if and only if the foliation is transversely holomorphic.

The main result of this section is a determination of a set of necessary and sufficient conditions for a complex foliation to be tangentially Monge–Ampère. These conditions are a finite set of linear algebraic conditions on the antiholomorphic twist tensor, first appearing in [1] as necessary conditions for $\mathfrak{F}$ to be Monge–Ampère. We conclude the section with some comments on the more difficult problem of determining necessary and sufficient conditions for $\mathfrak{F}$ to be Monge–Ampère.

Section 5 is concerned with an analysis of the first Chern class of Q. There is a natural map

$$\iota^*: H^{\cdot}(M, C) \to H^{\cdot}(M, C_{\mathfrak{F}}^{\infty}),$$

where $C_{\mathfrak{F}}^{\infty}$ is the sheaf of germs of C^{∞} complex valued functions which are locally constant along the leaves of $\mathfrak{F}$, and the relative Chern classes of Q can then be defined as the images under j^* of the Chern classes of Q. Under certain conditions the vanishing of the first relative Chern class implies that the foliation $\mathfrak{F}$ is transversely holomorphic. (It is easy to show that all relative Chern classes of a transversely holomorphic foliation vanish.) Several corollaries follow, of which the following example is typical.

THEOREM. *Let $\mathfrak{F}$ be a tangentially Monge–Ampère foliation of a compact complex manifold M by complex curves. Then $\mathfrak{F}$ is holomorphic if and only if the relative Chern class $c_1(\mathfrak{F}) \in H^2(M, C_{\mathfrak{F}}^{\infty})$ vanishes.*

We show in Section 6 that when the leaves of $\mathfrak{F}$ are complex curves they inherit a metric which in certain cases (i.e., when the codimension of $\mathfrak{F}$ is 1 or when the codimension of $\mathfrak{F}$ is 2 and the torsion tensor has a certain antisymmetry property) has constant Gaussian curvature. To illustrate we present an example of Calabi [6] of a fibration of complex projective 3-space over quaternionic projective 1-space whose fibers are projective lines. In this example the torsion tensor is antisymmetric and the intrinsic metric on the fibers has constant curvature $K=+2$.

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1. Complex foliations. Throughout the paper M denotes a complex n-dimensional manifold and $\mathfrak{F}$ a C^∞-foliation of M by complex submanifolds of dimension p and complex codimension $q=n-p$. Such a foliation is called *complex* to indicate the holomorphic nature of its leaves. (All objects in the paper are assumed to be of class C^∞.)

Our notational conventions are as follows. If E is a real vector bundle its complexification will be denoted by the symbol $E^{\mathbb{C}}$, and the sheaf of smooth sections of E will be denoted by the boldface letter $\mathbf{E}$. Greek letters range between 1 and q and Roman letters between 1 and p and, except where explicitly stated, the Einstein summation conventions will be in force. Matrices will generally be denoted by boldface letters, superscripts denote column numbers and subscripts row numbers. Finally, holomorphic coordinates on $\mathbb{C}^n=\mathbb{C}^p\times\mathbb{C}^q$ will be written in the form $(w,z)\equiv(w^1,w^2,\dots,w^p,z^1,z^2,\dots,z^q)$ and will be chosen so that the leaves of $\mathfrak{F}$ near the origin are transverse to the fibers of the projection map $(w,z)\to w$.

1.1. RELATIVE AND NORMAL FORMS. Denoting the tangent and normal bundles of $\mathfrak{F}$ by L and Q, respectively, there is a short exact sequence of complex (but not necessarily holomorphic) vector bundles

$$(1.2)\qquad 0\longrightarrow L\xrightarrow{\iota} TM\xrightarrow{\pi} Q\longrightarrow 0.$$

Note that the decomposition $TM^{\mathbb{C}}=TM_{(1,0)}\oplus TM_{(0,1)}$ induces decompositions $L^{\mathbb{C}}=L_{(1,0)}\oplus L_{(0,1)}$ and $Q^{\mathbb{C}}=Q_{(1,0)}\oplus Q_{(0,1)}$ and there are similar decompositions of the dual bundles,

$$\mathrm{Hom}_{\mathbb{C}}(L^{\mathbb{C}},\mathbb{C})=L^{(1,0)}\oplus L^{(0,1)}\quad\text{and}\quad \mathrm{Hom}_{\mathbb{C}}(Q^{\mathbb{C}},\mathbb{C})=Q^{(1,0)}\oplus Q^{(0,1)}.$$

Obvious isomorphisms of complex vector bundles will be used without comment (e.g., $Q\cong Q_{(1,0)}$, $\overline{Q^{(1,0)}}\cong Q^{(0,1)}$, etc.). We will employ the following terminology.

1.3. DEFINITION. A *tangential vector field* is a section of L and a *normal vector field* a section of Q. A *normal* 1-*form* is a section of Q^* and a *relative* 1-*form* is a section of L^*. When necessary we will make finer distinctions according to type (e.g., a section of $L^{(1,0)}$ is called a *relative* 1-*form of type* $(1,0)$).

1.4. LOCAL COORDINATES. When local holomorphic coordinates are needed they will be chosen as follows. Given a point $x_0\in M$ there is a holomorphic coordinate chart

$$\phi: U \to \Delta^p \times \Delta^q \subset \mathbb{C}^p \times \mathbb{C}^q, \tag{1.5}$$

where Δ^k denotes the unit polydisc in $\mathbb{C}^k$, centered at x_0 and such that $\phi^{-1}(\Delta^p \times 0)$ is contained in the leaf of $\mathfrak{F}$ containing x_0 and such that all leaves of $\mathfrak{F}$ intersect U transverse to the fibers of the submersion $\pi_1 \circ \phi$, where π_1 denotes projection onto Δ^p.

1.6. LOCAL COORDINATE FRAMES. Given local coordinates as above there is a local framing for the bundle $L_{(1,0)}$ given by the vector fields of the form

$$X_{(j)} \equiv \frac{\partial}{\partial w^j} + \lambda_j^\alpha \frac{\partial}{\partial z^\alpha}, \tag{1.7}$$

where $\lambda_j^\alpha \equiv \lambda_j^\alpha(w, z)$ are uniquely determined by the spanning condition. The normal vector fields defined by the formulas

$$\left[\frac{\partial}{\partial z^\alpha}\right] \equiv \pi_* \frac{\partial}{\partial z^\alpha} \tag{1.8}$$

frame $Q \cong Q_{(1,0)}$ and, by duality, the 1-forms

$$\theta^\alpha \equiv dz^\alpha - \lambda_j^\alpha dw^j \tag{1.9}$$

frame the bundle $Q^* \cong Q^{(1,0)}$ and the relative 1-forms

$$[dw^j] = \iota^* dw^j, \quad j = 1, 2, \ldots, p \tag{1.10}$$

frame $L^* \cong L^{(1,0)}$.

1.11. THE INTEGRABILITY CONDITION. The Frobenius integrability condition,

$$d\Gamma(M, Q^*) \subseteq \Gamma(M, Q^*) \wedge \Gamma(M, \Omega^\bullet), \tag{1.12}$$

where $\Omega^\bullet$ denotes the sheaf of germs of smooth complex-valued differential forms, yields three important identities. To get them begin by observing that the integrability of the complex structure on M yields the formula

$$\begin{aligned} d\theta^\alpha = {} & \frac{\partial \lambda_j^\alpha}{\partial z^\beta} dw^j \wedge \theta^\beta + \frac{\partial \lambda_j^\alpha}{\partial \bar{z}^\beta} dw^j \wedge \overline{\theta^\beta} \\ & + \left(\overline{\lambda_j^\beta} \frac{\partial \lambda_k^\alpha}{\partial \bar{z}^\beta} + \frac{\partial \lambda_j^\alpha}{\partial \bar{w}^j} \right) dw^k \wedge d\bar{w}^j \\ & + \left(\lambda_j^\beta \frac{\partial \lambda_k^\alpha}{\partial z^\beta} + \frac{\partial \lambda_j^\alpha}{\partial w^j} \right) dw^k \wedge dw^j. \end{aligned}$$

By (1.12) the last two terms on the right-hand side vanish, giving the three identities:

$$d\theta^\alpha = \frac{\partial \lambda_j^\alpha}{\partial z^\beta} dw^j \wedge \theta^\beta + \frac{\partial \lambda_j^\alpha}{\partial \bar{z}^\beta} dw^j \wedge \overline{\theta^\beta}, \tag{1.13}$$

$$\left(\frac{\partial}{\partial w^j} + \overline{\lambda_j^\beta} \frac{\partial}{\partial \bar{z}^\beta} \right) \lambda_k^\alpha = 0, \tag{1.14}$$

and

$$\left(\frac{\partial}{\partial w^j}+\lambda_j^\beta\frac{\partial}{\partial z^\beta}\right)\lambda_k^\alpha=\left(\frac{\partial}{\partial w^k}+\lambda_k^\beta\frac{\partial}{\partial z^\beta}\right)\lambda_j^\alpha. \tag{1.15}$$

Equations (1.14) and (1.15) can be rewritten in the forms:

$$[\overline{X_{(j)}},X_{(k)}]=0 \tag{1.16}$$

and

$$[X_{(j)},X_{(k)}]=0, \tag{1.17}$$

respectively. These identities are basic and will be used to obtain the Cartan structure equations for the complex Bott connection of $\mathfrak{F}$ (see Section 3).

2. The relative de Rham complex. The *relative de Rham complex* of $\mathfrak{F}$ makes precise the notation of exterior differentiation in the directions tangent to leaves, and will be used extensively here. Detailed discussions of this complex can be found in the works of Heitch [8], Kamber–Tondeur [9], and Vaisman [12], and we give here a rather short presentation adapted to the case of complex foliations.

2.1. DEFINITION. A *relative form of degree i* is a smooth section of the sheaf $\Omega^i_{\mathfrak{F}}$ of germs of sections of the bundle $\wedge^i L^*$; a *relative form of type* (j,k) is a smooth section of the sheaf $\Omega^{(j,k)}_{\mathfrak{F}}$ of germs of sections of the bundle $\wedge^j L^{(1,0)}\otimes\wedge^k L^{(0,1)}$; a *normal form of degree* α is a section of the sheaf Ω^α_Q of germs of smooth sections of the bundle $\wedge^\alpha Q^*$; a *normal form of type* (α,β) is a section of the bundle $\wedge^\alpha Q^{(1,0)}\otimes\wedge^\beta Q^{(0,1)}$. More generally, a section of $\Omega^i_{\mathfrak{F}}\otimes_{C^\infty_{\mathfrak{F}}}\Omega^\alpha_Q$ is called a $\wedge^\alpha Q^*$-*valued i-form* while a section of $\Omega^{(j,k)}_{\mathfrak{F}}\otimes_{C^\infty_{\mathfrak{F}}}\Omega^\alpha_Q$ is called a $\wedge^\alpha Q^*$-*valued form of type* (j,k).

The graded sheaf $\Omega^\bullet_{\mathfrak{F}}$ inherits the structure of a bi-graded, exterior differential algebra from the Dolbeault complex $(\Omega^{(\bullet,\bullet)}_M, d=\partial+\bar\partial)$ via the restriction map i^*: $\Omega^{(\bullet,\bullet)}_M\to\Omega^{(\bullet,\bullet)}_{\mathfrak{F}}$ induced by the inclusion $\iota: L\hookrightarrow TM$. The Frobenius integrability condition, the integrability of the complex structure on M, and the fact that the leaves of $\mathfrak{F}$ are holomorphic are used to show that the operators d, ∂, and $\bar\partial$ induce operators

$$d_{\mathfrak{F}}:\Omega^\bullet_{\mathfrak{F}}\to\Omega^{\bullet+1}_{\mathfrak{F}},$$
$$\partial_{\mathfrak{F}}:\Omega^{(\bullet,\bullet)}_{\mathfrak{F}}\to\Omega^{(\bullet+1,\bullet)}_{\mathfrak{F}},$$
$$\bar\partial_{\mathfrak{F}}:\Omega^{(\bullet,\bullet)}_{\mathfrak{F}}\to\Omega^{(\bullet,\bullet+1)}_{\mathfrak{F}},$$

with $d_{\mathfrak{F}}=\partial_{\mathfrak{F}}+\bar\partial_{\mathfrak{F}}$, making ι^* a map of bi-graded differential algebras.

2.2. DEFINITION. The complex $(\Omega^\bullet_{\mathfrak{F}}, d_{\mathfrak{F}})$ is called the *relative de Rham complex of* $\mathfrak{F}$ and the bi-complex $(\Omega^{(\bullet,\bullet)}_{\mathfrak{F}}, \partial_{\mathfrak{F}}, \bar\partial_{\mathfrak{F}})$ the *relative Dolbeault complex of* $\mathfrak{F}$.

It is easily checked that the above derivations are $C^\infty_{\mathfrak{F}}$-linear and therefore extend to give the tensor product $\Omega^{(\bullet,\bullet)}_{\mathfrak{F}}\otimes_{C^\infty_{\mathfrak{F}}}\Omega^\alpha_Q$ the structure of a bi-complex. Define a wedge product operation

$$\wedge:\left(\Omega^{t_1}_{\mathfrak{F}}\otimes_{C^\infty_{\mathfrak{F}}}\Omega^{\alpha_1}_Q\right)\otimes\left(\Omega^{t_2}_{\mathfrak{F}}\otimes_{C^\infty_{\mathfrak{F}}}\Omega^{\alpha_2}_Q\right)\to\Omega^{t_1+t_2}_{\mathfrak{F}}\otimes_{C^\infty_{\mathfrak{F}}}\Omega^{\alpha_1+\alpha_2}_Q \tag{2.3}$$

by the formula

(2.4) $$\eta_1 \wedge \eta_2 = (-1)^{\alpha_1 + t_2} \psi_1 \wedge \psi_2 \otimes \phi_1 \wedge \phi_2,$$

where $\eta_k \equiv \psi_k \otimes \phi_k \in \Omega_{\mathfrak{F}}^{t_k} \otimes_{C_{\mathfrak{F}}^\infty} \Omega_Q^{\alpha_k}$ for $k = 1, 2$. Then the identity

(2.5) $$d_{\mathfrak{F}}(\eta_1 \wedge \eta_2) = d_{\mathfrak{F}}(\eta_1) \wedge \eta_2 + (-1)^{\alpha_1 + k_1} \eta_1 \wedge \delta_{\mathfrak{F}}(\eta_2)$$

is easily verified and shows that $d_{\mathfrak{F}}$, $\partial_{\mathfrak{F}}$, and $\bar{\partial}_{\mathfrak{F}}$ are derivations of degree 1 and bidegrees $(1, 0)$ and $(0, 1)$, respectively. Finally, an interior evaluation $i(X)$ operator can be defined in the obvious manner for $X \in \Gamma(M, L)$.

2.6. REMARK. With respect to local coordinates chosen as in paragraph 1.6, the following formulas characterizing the derivation $d_{\mathfrak{F}}$ hold:

(2.7) $$d_{\mathfrak{F}} f = X_{(j)}(f)[dw^j] + \bar{X}_{(j)}(f)[d\bar{w}^j],$$

(2.8) $$d_{\mathfrak{F}}[dw^j] = d_{\mathfrak{F}}[d\bar{w}^j] = 0,$$

(2.9) $$d_{\mathfrak{F}} \theta^\alpha = \frac{\partial \lambda_j^\alpha}{\partial z^\beta}[dw^j] \otimes \theta^\beta + \frac{\partial \lambda_j^\alpha}{\partial \bar{z}^\beta}[dw^j] \otimes \overline{\theta^\beta},$$

where f is any function. Formula (2.7) follows from the easily derived identity

$$df = \frac{\partial f}{\partial z^\alpha}\theta^\alpha + \frac{\partial f}{\partial \bar{z}^\alpha}\overline{\theta^\alpha} + X_{(j)}(f) dw^j + \overline{X_{(j)}}(f) d\bar{w}^j$$

and the equalities $d_{\mathfrak{F}} f = \iota^*(df)$ and $\iota^*(\theta^\alpha) = \iota^*(\overline{\theta^\alpha}) = 0$, while (2.8) is clear and formula (2.9) is immediate from (1.13).

From the decomposition $d_{\mathfrak{F}} = \partial_{\mathfrak{F}} + \bar{\partial}_{\mathfrak{F}}$ one then obtains the local formulas

(2.10) $$\begin{aligned} \partial_{\mathfrak{F}} f &= X_{(j)}(f)[dw^j], \\ \bar{\partial}_{\mathfrak{F}} f &= \overline{X_{(j)}}(f)[d\bar{w}^j], \end{aligned}$$

and

(2.11) $$\begin{aligned} \partial_{\mathfrak{F}} \theta^\alpha &= \frac{\partial \lambda_j^\alpha}{\partial z^\beta}[dw^j] \otimes \theta^\beta + \frac{\partial \lambda_j^\alpha}{\partial \bar{z}^\beta}[dw^j] \otimes \overline{\theta^\beta}, \\ \bar{\partial}_{\mathfrak{F}} \theta^\alpha &= 0. \end{aligned}$$

Formula (2.11) can be interpreted as follows. If $\iota : N \hookrightarrow M$ is a leaf of $\mathfrak{F}$ then the pull-back bundle $\iota^* Q^{(1,0)}$ is a holomorphic bundle over N, and it is not difficult to show the commutativity of the diagram

(2.12) $$\begin{array}{ccc} Q^{(1,0)} & \xrightarrow{\bar{\partial}_{\mathfrak{F}}} & \Omega_{\mathfrak{F}}^{(0,1)} \otimes_{C_{\mathfrak{F}}^\infty} \Omega_Q^1 \\ \downarrow & & \downarrow \\ Q_N^{(1,0)} & \xrightarrow{\bar{\partial}_N} & \Omega_N^{(0,1)} \otimes_{\mathcal{O}_N} Q_{|N}^{(1,0)}. \end{array}$$

Therefore, the restriction to N of the framing θ yields a holomorphic framing of its conormal bundle.

2.13. DEFINITION. Let θ be a normal framing of type $(1, 0)$. Then θ is said to be *relatively holomorphic* if the equation $\bar{\partial}_{\mathfrak{F}} \theta = 0$ holds.

Formula (2.11) shows that relatively holomorphic framings of $Q^{(1,0)}$ exist.

2.13. RELATIVE CHERN CLASSES. The relative de Rham complex $(\Omega^\bullet_{\mathfrak{F}}, d_{\mathfrak{F}})$ is a fine, free resolution of the sheaf $C^\infty_{\mathfrak{F}}$ of germs of complex-valued functions which are locally constant along the leaves of $\mathfrak{F}$; therefore, there is an isomorphism of cohomology groups,

$$H^k(M, C^\infty_{\mathfrak{F}}) \cong H^k(\Gamma(M, \Omega^\bullet_{\mathfrak{F}}), d_{\mathfrak{F}}),$$

and $H^k(M, C^\infty)$ is called the kth *relative de Rham cohomology group of* $\mathfrak{F}$. The map of complexes, $\iota^*: (\Omega^\bullet_M, d) \to (\Omega^\bullet_{\mathfrak{F}}, d_{\mathfrak{F}})$ induces a map from de Rham cohomology with coefficients in $\mathbb{C}$ into relative de Rham cohomology,

$$\iota^\#: H^\bullet_{DR}(M, \mathbb{C}) \to H^\bullet(M, C^\infty_{\mathfrak{F}}), \tag{2.14}$$

and if $j: N \hookrightarrow M$ is an immersion into a leaf of $\mathfrak{F}$ then there are maps of complexes $j^*: \Omega^\bullet_M \to \Omega^\bullet_N$ and $\hat{j}: \Omega^\bullet_{\mathfrak{F}} \to \Omega^\bullet_N$ induced by the restriction map on forms, and the identity $j^* = \hat{j} \circ \iota^*$ holds. Consequently, induced maps in cohomology factor similarly:

$$j^\# = \hat{j}^\# \circ \iota^\#: H^*(M, \mathbb{C}) \xrightarrow{\iota^\#} H^*(M, C^\infty_{\mathfrak{F}}) \xrightarrow{\hat{j}^\#} H(N, \mathbb{C}). \tag{2.15}$$

2.16. DEFINITION. If $c_k(Q) \in H^{2k}(M, \mathbb{C})$ denotes the kth Chern class of the complex bundle Q then its image $c_k(\mathfrak{F}) \equiv \iota^\#(c_k(Q))$ is called the kth *relative Chern class* of $\mathfrak{F}$.

2.17. REMARK. Note that, by virtue of (2.15), if $c_k(Q_{|N})$ denotes the kth Chern class of the pull-back of the bundle Q to N then

$$c_k(Q_{|N}) = \hat{j}^\# c_k(\mathfrak{F}).$$

In Section 5 the complex Bott connection will be used to exhibit explicit representatives of the relative Chern classes of $\mathfrak{F}$.

3. The Bott connection. In this section we wish to define a *partial connection* on the normal bundle Q. Because it is closely related to the connection given by Bott in [4], we call it the *complex Bott connection*. Unlike Bott's connection, which is partially flat, the complex Bott connection has nonvanishing curvature precisely when the foliation $\mathfrak{F}$ is not holomorphic. By virtue of the results at the end of the previous section, the complex Bott connection can be used to construct relative forms representing the relative Chern classes of the complex vector bundle Q, and these forms vanish identically when $\mathfrak{F}$ is holomorphic. For background material on partial connections we refer the reader to [4] and [9].

For the convenience of the reader we begin with a presentation of the elementary properties of partial connections. If we regard the foliated manifold $(M, \mathfrak{F})$ as a smooth family of manifolds (the leaves of the foliation) parameterized locally by normal coordinates to the leaves, then a vector bundle over M can be viewed as a family of bundles over the leaves of $\mathfrak{F}$ and a partial connection is nothing but a family of connections—hence covariant differentiation is defined only in the directions tangent to the leaves. From this point of view, a relative form is nothing but a family of forms on the leaves of $\mathfrak{F}$ and the relative differential $d_{\mathfrak{F}}$ nothing but the ordinary exterior differential on the leaves. All of the familiar constructs and identities for connections (e.g., the definitions of curvature and torsion and the Cartan structure equations) hold if the connection 1-forms are replaced by relative 1-forms and the exterior derivative operator d is replaced by the relative de Rham differential $d_{\mathfrak{F}}$.

In the notation of the previous section, then, we have the following definition.

3.1. DEFINITION. Let $E \to M$ be a vector bundle on the foliated manifold M. A *partial connection* is an $\mathbb{R}$-linear map

$$\nabla : \Gamma(M, E) \to \Gamma(M, E \otimes L^*) \cong \operatorname{Hom}(L, E) \tag{3.2}$$

which satisfies the condition

$$\nabla(fe) = f\nabla e + e \otimes d_{\mathfrak{F}} f \tag{3.3}$$

for $e \in \Gamma(M, E)$ and f a smooth function. As usual, we write $\nabla_X e$ for the value of $\nabla e : L \to E$ at $X \in L$. The *curvature 2-form*, $R_\nabla \in \Gamma(M, \wedge^2 L^* \otimes \operatorname{Hom}(E, E))$, is defined by the formula

$$R_\nabla(X, Y)e = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})e \tag{3.4}$$

for $X, Y \in \Gamma(M, L)$ and $e \in \Gamma(M, E)$, and the partial connection ∇ is called *partially flat* if its curvature form vanishes.

3.5. MOVING FRAMES. Consider a rank-r vector bundle E with connection ∇ and suppose that $\mathbf{e} = (e_1, e_2, \ldots, e_r)$ is a local frame. Then we write the matrix equation

$$\nabla \mathbf{e} = \mathbf{e} \otimes \omega, \tag{3.6}$$

where $\omega = (\omega_s^t)$ is the matrix of relative 1-forms defined by the condition $\nabla e_s = e_t \otimes \omega_s^t$. The curvature matrix Ω is the matrix of relative 2-forms defined by the equation $R_\nabla = \Omega_s^t e_s^* \otimes e_t$, and one easily checks that the following identities hold:

$$\Omega = d_{\mathfrak{F}} \omega + \omega \wedge \omega, \quad \text{(curvature formula)}; \tag{3.7}$$

$$d_{\mathfrak{F}} \Omega = \Omega \wedge \omega - \omega \wedge \Omega, \quad \text{(Bianchi identity)}. \tag{3.8}$$

3.9. REMARKS. A partial connection on a *complex vector bundle* is a $\mathbb{C}$-linear map satisfying condition (3.2). Note that a partial connection on a real vector bundle naturally extends to a complex partial connection on the complexified bundle. Note also that if E is a complex vector bundle and ∇ is a complex partial connection then it induces, in the standard way, partial connections on the bundles $\bar{E}$, E^*, and $\wedge^k E$ defined by the following local formulas:

$$\nabla \bar{e}_s = \bar{e}_t \otimes \bar{\omega}_s^t, \tag{3.10}$$

$$\nabla e_s^* = -e_t^* \otimes \omega_t^s, \tag{3.11}$$

$$\nabla(e_{s_1} \wedge e_{s_2} \wedge \cdots \wedge e_{s_k}) = \sum_{j=1}^{k} e_{s_1} \wedge \cdots \wedge e_t \wedge \cdots \wedge \mathbf{e}_{s_k} \otimes \omega^{t_j}. \tag{3.12}$$

Finally, the following change of frame formulas hold:

$$\omega' = \mathbf{g}^{-1} d_{\mathfrak{F}} \mathbf{g} + \mathbf{g}^{-1} \omega \mathbf{g}, \tag{3.13}$$

(3.14) $$\Omega' = \mathbf{g}^{-1}\Omega\mathbf{g},$$

where $\mathbf{g}$ is a $GL(r,\mathbb{C})$-valued function and ω' and Ω' are the connection and curvature forms of ∇ with respect to the local frame $\mathbf{e}' = \mathbf{e}\mathbf{g}$.

There are two natural partial connections on the normal bundle Q, the *smooth Bott connection* $\tilde{\nabla}$, which is partially flat but respects the complex structure only when $\mathfrak{F}$ is holomorphic, and the *complex Bott connection* ∇, which respects the complex structure but is partially flat only when $\mathfrak{F}$ is holomorphic.

3.14. DEFINITION. The *smooth Bott connection* is the partial connection defined by the formula

(3.15) $$\tilde{\nabla}_X Y = \pi[X, \tilde{Y}],$$

for $X \in \Gamma(M, L)$, $Y \in \Gamma(M, Q)$, and $\tilde{Y} \in \Gamma(M, TM)$ such that $\pi(\tilde{Y}) = Y$.

To define the *complex Bott connection,* first recall that there the complex vector bundles Q and $Q_{(1,0)}$ are naturally isomorphic; then observe that the composition

(3.16) $$\nabla : \Gamma(M, Q_{(1,0)}) \hookrightarrow \Gamma(M, Q^{\mathbb{C}}) \xrightarrow{\tilde{\nabla}} \Gamma(M, Q^{\mathbb{C}} \otimes L^*) \xrightarrow{\pi} \Gamma(M, Q_{(1,0)} \otimes L^*)$$

defines a connection on $Q_{(1,0)}$.

3.17. REMARK. There is a useful formula for the smooth Bott connection on the conormal bundle Q^*:

(3.18) $$\tilde{\nabla}_X \eta = i(X) d_{\mathfrak{F}} \eta = L_X \eta.$$

The partial flatness of $\tilde{\nabla}$ then follows from the identity for the Lie derivative,

$$L_X L_Y - L_Y L_X - L_{[X,Y]} \equiv 0.$$

Formula (3.18) holds on all of the exterior product bundles $\wedge^s Q^*$.

3.19. LOCAL COORDINATES. To get some feeling for both the smooth and complex Bott connections on the complexified bundle $Q^{\mathbb{C}}$, it is useful to compute their connection 1-forms relative to the framing of Q defined in paragraph 1.6. Equations (1.14)–(1.17) can be used to derive the following identities:

(3.20) $$\tilde{\nabla}\left[\frac{\partial}{\partial z^\alpha}\right] = \left[\frac{\partial}{\partial z^\beta}\right] \otimes \omega_\alpha^\beta + \left[\frac{\partial}{\partial \bar{z}^\beta}\right] \otimes \overline{\Lambda_{\bar{\alpha}}^\beta},$$

(3.21) $$\nabla\left[\frac{\partial}{\partial z^\alpha}\right] = \left[\frac{\partial}{\partial z^\beta}\right] \otimes \omega_\alpha^\beta,$$

where

(3.22) $$\omega_\alpha^\beta = -\frac{\partial \lambda_j^\beta}{\partial z^\alpha}[dw^j] \quad \text{and}$$

(3.23) $$\Lambda_{\bar{\alpha}}^\beta = -\frac{\partial \lambda_j^\beta}{\partial \bar{z}^\alpha}[dw^j].$$

In this notation, formula (2.9) assumes the form

(3.24) $$d_{\mathfrak{F}} \theta^\alpha = -\omega_\beta^\alpha \wedge \theta^\beta - \Lambda_{\bar{\beta}}^\alpha \wedge \overline{\theta^\beta}.$$

The following proposition summarizes the important properties of the complex Bott connection.

3.25. PROPOSITION. *Let* $\mathbf{e} = (e_1, e_2, \ldots, e_q)$ *be a local framing for* $Q_{(1,0)}$, *let* $\boldsymbol{\theta} = (\theta^1, \theta^2, \ldots, \theta^q)$ *be its dual coframe, and let* $\boldsymbol{\omega} = (\omega^\alpha_\beta)$ *and* $\boldsymbol{\Lambda} = \Lambda^\alpha_{\bar\beta}$ *be the* $q \times q$ *matrices of relative connection* 1*-forms defined by the equations* $\nabla\mathbf{e} = \mathbf{e} \otimes \boldsymbol{\omega}$ *and* $\tilde\nabla\mathbf{e} = \mathbf{e} \otimes \boldsymbol{\omega} + \bar{\mathbf{e}} \otimes \bar{\boldsymbol{\Lambda}}$. *Let* $\boldsymbol{\Omega} = (\Omega^\alpha_\beta)$ *denote the curvature matrix of* ∇ *on the bundle* $Q_{(1,0)}$. *Then*

(a) *the following structure equations are satisfied:*

$$d_{\mathfrak{F}}\boldsymbol{\theta} = -\boldsymbol{\omega} \wedge \boldsymbol{\theta} - \boldsymbol{\Lambda} \wedge \bar{\boldsymbol{\theta}}, \tag{3.26}$$

$$\boldsymbol{\Omega} = -\boldsymbol{\Lambda} \wedge \bar{\boldsymbol{\Lambda}}, \tag{3.27}$$

$$d_{\mathfrak{F}}\boldsymbol{\Lambda} = -\boldsymbol{\omega} \wedge \boldsymbol{\Lambda} - \boldsymbol{\Lambda} \wedge \bar{\boldsymbol{\omega}}; \quad \textit{and} \tag{3.28}$$

(b) *if the frame* $\mathbf{e}$ *is relatively holomorphic then the connection* 1*-forms* ω^α_β *are of type* $(1,0)$.

Proof. (a) Formula (3.26) follows from (3.24). To obtain (3.27) and (3.28) take the exterior derivative of (3.26) and substitute equation (3.26) and its complex conjugate into the resulting expression to arrive at the equation

$$\begin{aligned} 0 = &-(d_{\mathfrak{F}}\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega} + \boldsymbol{\Lambda} \wedge \bar{\boldsymbol{\Lambda}}) \wedge \boldsymbol{\theta} \\ &-(d_{\mathfrak{F}}\boldsymbol{\Lambda} + \boldsymbol{\omega} \wedge \boldsymbol{\Lambda} + \boldsymbol{\Lambda} \wedge \bar{\boldsymbol{\omega}}) \wedge \bar{\boldsymbol{\theta}}, \end{aligned}$$

from which the results follow.

(b) Choose local coordinates (w, z) as in paragraph 1.4 and write $\mathbf{e} = \mathbf{e}_0\mathbf{g}$, where $\mathbf{e}_0$ is the framing

$$e^\alpha = \left[\frac{\partial}{\partial z^\alpha}\right].$$

Let $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_0$ be the dual coframes. By assumption, $\bar\partial_{\mathfrak{F}}\boldsymbol{\theta} = 0$ and $\bar\partial_{\mathfrak{F}}\boldsymbol{\theta}_0 = 0$ by (2.17). It follows that $\bar\partial_{\mathfrak{F}}\mathbf{g} = 0$. But then

$$\boldsymbol{\omega} = \mathbf{g}^{-1}d_{\mathfrak{F}}\mathbf{g} + \mathbf{g}^{-1}\boldsymbol{\omega}_0\mathbf{g} = \mathbf{g}^{-1}\bar\partial_{\mathfrak{F}} + \mathbf{g}^{-1}\boldsymbol{\omega}_0\mathbf{g},$$

where $\boldsymbol{\omega}_0$ is the matrix of connection forms relative to the framing $\mathbf{e}_0$ and hence $\boldsymbol{\omega}$ is of type $(1,0)$. □

3.29. THE ANTIHOLOMORPHIC TORSION TENSOR. Observe that the connection matrix of $\tilde\nabla$ relative to the framing $(\mathbf{e}, \bar{\mathbf{e}})$ of the bundle $Q^{\mathbb{C}}$ is

$$\tilde{\boldsymbol{\omega}} \equiv \begin{pmatrix} \boldsymbol{\omega} & \boldsymbol{\Lambda} \\ \bar{\boldsymbol{\Lambda}} & \bar{\boldsymbol{\omega}} \end{pmatrix} \tag{3.30}$$

and that the connection matrix relative to a new framing $(\mathbf{e}', \bar{\mathbf{e}}') = (\mathbf{eg}, \overline{\mathbf{eg}})$ is given by the formula

$$\tilde{\boldsymbol{\omega}}' = \begin{pmatrix} \mathbf{g}^{-1}d_{\mathfrak{F}}\mathbf{g} & 0 \\ 0 & \bar{\mathbf{g}}^{-1}d_{\mathfrak{F}}\bar{\mathbf{g}} \end{pmatrix} + \begin{pmatrix} \mathbf{g}^{-1}\boldsymbol{\omega}\mathbf{g} & \mathbf{g}^{-1}\boldsymbol{\Lambda}\bar{\mathbf{g}} \\ \bar{\mathbf{g}}^{-1}\bar{\boldsymbol{\Lambda}}\mathbf{g} & \bar{\mathbf{g}}^{-1}\overline{\boldsymbol{\omega}\mathbf{g}} \end{pmatrix}.$$

Hence the matrix $\boldsymbol{\Lambda}$ transforms according to the formula

(3.31)
$$\Lambda' = \mathbf{g}^{-1}\Lambda\bar{\mathbf{g}}$$

and therefore defines a tensor as follows.

3.32. DEFINITION. The tensor $\tau \in \Gamma(M, L^{(1,0)} \otimes Q^{(0,1)} \otimes Q_{(1,0)})$, defined by the local formula

$$\tau = \Lambda^{\alpha}_{\bar{\beta}} \otimes \overline{\theta^{\beta}} \otimes e_{\alpha},$$

is called the *antiholomorphic torsion tensor* of $\mathfrak{F}$. The foliation $\mathfrak{F}$ is said to be holomorphic at a point $p \in M$ if and only if $\tau_p = 0$.

3.33. REMARKS. The tensor τ is the fundamental tensor associated to a complex foliation. It was first introduced by Bedford and Burns [1] and given the name *antiholomorphic twist* in [5]. However, by virtue of formula (3.26) it is the torsion tensor of the complex Bott connection.

The importance of the antiholomorphic torsion tensor lies in the next lemma, which is contained in [1].

3.34. LEMMA. *The foliation $\mathfrak{F}$ is holomorphic if and only if the torsion tensor τ vanishes.*

Proof. Recall that the foliation $\mathfrak{F}$ is holomorphic if and only if the bundle L is a holomorphic subbundle of the tangent bundle TM, and that this is the case precisely when the spanning vector fields $X_{(j)}$, defined by formula (1.7), are all holomorphic. Now by virture of the easily checked identity $\bar{\partial}X_{(j)} = i(X_{(j)})\tau$, it follows that L is holomorphic if and only if τ vanishes. □

We are now in a position to give representatives for the relative Chern classes $c_k(\mathfrak{F}) \in H^{2k}(M, C^{\infty}_{\mathfrak{F}})$.

3.35. PROPOSITION. *The relative Chern forms of the complex foliation $\mathfrak{F}$, defined as the coefficients $C_k(\mathfrak{F}) \in \Gamma(M, \Omega^{(k,k)}_{\mathfrak{F}})$ of t^k in the polynomial*

$$\det\left(I - \frac{t}{2\pi i}\Omega\right) \equiv \det\left(I - \frac{t}{2\pi i}\Lambda \wedge \bar{\Lambda}\right) = 1 + C_1(\mathfrak{F})t + C_2(\mathfrak{F})t^2 + \cdots + C_q(\mathfrak{F})t^q,$$

are representatives of the relative Chern classes $c_k(\mathfrak{F}) \in H^{2k}(M, C^{\infty}_{\mathfrak{F}})$. In particular, the first relative Chern class is represented by the relative $(1,1)$*-form*

$$C_1(\mathfrak{F}) = -\frac{i}{2\pi} \sum_{\alpha,\beta} \Lambda^{\alpha}_{\bar{\beta}} \wedge \overline{\Lambda^{\beta}_{\bar{\alpha}}}.$$

Proof. It is shown in [9, p. 25] that a partial connection ∇ on a vector bundle $E \to M$ can be extended to an ordinary connection ∇' on E as follows: Let ∇'' be any connection on E and choose a splitting $TM \cong L \otimes Q$; then the connection ∇' is defined by the formula

$$\nabla'_X e \equiv \nabla_{X_L} e + \nabla''_{X_Q} e, \quad e \in \Gamma(M, E),$$

where $X = X_L \oplus X_Q \in TM$. Denoting the curvature matrix of ∇' by Ω', one easily checks the identity $\iota^*\Omega' = \Omega$.

Applying this construction to the complex Bott connection on the normal bundle Q and denoting the Chern forms of Q with respect to the connection ∇' by $C_k(Q)$ then yields the formula $\iota^*(C_k(Q)) = C_k(\mathfrak{F})$, and the proposition follows from Remark 2.17. □

The next proposition follows from Lemma 3.34 and Proposition 3.35, and can be thought of as a generalization of the well-known fact that the Chern classes $c_k(Q_{|N}) \in H^{2k}(N, \mathbb{C})$ of the normal bundle of a leaf of a holomorphic foliation vanish.

3.36. PROPOSITION. *The relative Chern classes of a holomorphic foliation all vanish.*

4. Monge–Ampère foliations. In this section we examine the condition under which a complex foliation $\mathfrak{F}$ arises from local solutions of the complex Monge–Ampère equation

$$(\partial\bar{\partial}u)^{q+1} = 0, \quad (\partial\bar{\partial}u)^q \neq 0, \tag{4.1}$$

where u is a real-valued, plurisubharmonic function. (Here $(\partial\bar{\partial}u)^k$ denotes the wedge product of k-copies of $\partial\bar{\partial}u$.) The foliation $\mathfrak{F}$ is said to be (*locally*) *Monge–Ampère* if in a neighborhood of each point of M there is a solution u to (4.1) such that the tangential distribution of $\mathfrak{F}$ is given by the formula

$$L = \{X \in TM \mid i(X)\partial\bar{\partial}u = 0\}. \tag{4.2}$$

Recall that a real form σ of type $(1, 1)$ is said to be *nonnegative* if for every vector $X \in T_{(1,0)}M$ the inequality

$$\sigma(X, \bar{X}) \geq 0 \tag{4.3}$$

holds. Since every real closed nonnegative form σ can be locally expressed in the form $i\partial\bar{\partial}u$, we are led to the next definition.

4.4. DEFINITION. A complex foliation $\mathfrak{F}$ is said to be (*locally*) *tangentially Monge–Ampère* if (in a neighborhood of each point of M) there is a nonnegative, real form σ of type $(1, 1)$ such that the equalities $L = \{X \in TM \mid i(X)\sigma = 0\}$ and

$$d_{\mathfrak{F}}\sigma = 0 \tag{4.5}$$

both hold. If the stronger condition, $d\sigma = 0$, holds then the foliation $\mathfrak{F}$ is said to be (*locally*) *Monge–Ampère*.

4.6. REMARK. Condition (4.5) is equivalent to the requirement that the Lie derivatives $L_X\sigma$ vanish for all $X \in \Gamma(M, L)$; that is, that the form σ be covariant with respect to the *smooth* Bott connection.

For σ as above and θ a local framing of Q^* by $(1, 0)$-forms, there is a unique, positive definite Hermitian matrix $\mathbf{U} = (U_{\alpha\bar{\beta}})$ defined by the identity

$$\sigma = (i/2)U_{\alpha\bar{\beta}}\theta^\alpha \wedge \overline{\theta^\beta}, \tag{4.6}$$

which in turn defines a Hermitian inner product

$$h = U_{\alpha\bar{\beta}}\theta^\alpha \otimes \overline{\theta^\beta} \tag{4.7}$$

on the normal bundle Q. Condition (4.5) now has a nice geometric interpretation in terms of the complex Bott connection and the antiholomorphic torsion tensor.

4.9. PROPOSITION. *Let h be a Hermitian inner product on the normal bundle of $\mathfrak{F}$ and let σ be the associated normal* (1,1)*-form. Then the form σ is $d_{\mathfrak{F}}$-closed if and only if both of the following conditions hold:*

$$\nabla h = 0 \tag{4.10}$$

and

$$\tau \bigstar \sigma = 0, \tag{4.11}$$

where

$$\tau \bigstar \sigma \equiv (i/2)\Lambda^\gamma_{\bar{\alpha}} U_{\gamma\bar{\beta}} \otimes \overline{\theta^\alpha \wedge \theta^\beta}. \tag{4.12}$$

Proof. Expand $-2id_{\mathfrak{F}}\sigma$ using the structure equation (3.29) and its complex conjugate as follows:

$$\begin{aligned}-2id_{\mathfrak{F}}\sigma &= d_{\mathfrak{F}}U_{\alpha\bar{\beta}} \wedge \theta^\alpha \wedge \overline{\theta^\beta} + U_{\alpha\bar{\beta}} \wedge d_{\mathfrak{F}}\theta^\alpha \wedge \overline{\theta^\beta} - U_{\alpha\bar{\beta}} \wedge \theta^\alpha \wedge d_{\mathfrak{F}}\overline{\theta^\beta} \\ &= (d_{\mathfrak{F}}U_{\alpha\bar{\beta}} - U_{\gamma\bar{\beta}}\omega^\gamma_\alpha - U_{\alpha\bar{\gamma}}\overline{\omega^\gamma_\beta}) \wedge \theta^\alpha \wedge \overline{\theta^\beta} - \Lambda^\gamma_{\bar{\alpha}} U_{\gamma\bar{\beta}} \wedge \overline{\theta^\alpha \wedge \theta^\beta} - U_{\alpha\bar{\gamma}}\overline{\Lambda^\gamma_{\bar{\beta}}} \wedge \theta^\alpha \wedge \theta^\beta.\end{aligned}$$

This can be rewritten in the invariant form

$$d_{\mathfrak{F}}\sigma = \nabla\sigma - \tau \bigstar \sigma + \overline{\tau \bigstar \sigma}, \tag{4.13}$$

and since no two terms have the same type it follows that $d_{\mathfrak{F}}\sigma = 0$ if and only if $\nabla\sigma = 0$ and $\tau \bigstar \sigma = 0$. Finally, because $\nabla\sigma = 0$ if and only if $\nabla h = 0$, the result follows. □

4.14. DEFINITION. A complex foliation $\mathfrak{F}$ equipped with a Hermitian inner product h satisfying (4.10) is called a *Hermitian foliation*.

4.15. REMARK. In the case where $\mathfrak{F}$ is Monge–Ampère, Bedford and Burns [1] showed that if $j: N \hookrightarrow M$ is a leaf then the curvature matrix Ω'_N of the unique Hermitian connection of type (1,0) on the normal bundle $Q_{|N} \to N$ satisfies the identity

$$\Omega'_N = -j^*(\Lambda \wedge \bar{\Lambda}). \tag{4.16}$$

But by virtue of Proposition 3.25(b), if $\mathfrak{F}$ is Hermitian then the complex Bott partial connection also restricts to the canonical connection on each leaf of $\mathfrak{F}$. In this way Proposition 3.25 generalizes the results of [1].

Condition (4.11) is a symmetry condition on the torsion tensor τ. For suppose that (4.11) is satisfied, and choose a Hermitian framing $\boldsymbol{\theta}$ for Q^* of type (1,0) with respect to which the equation $U_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$ holds, where $\delta_{\alpha\bar{\beta}}$ denotes the Kronecker delta; then condition (4.11) can be written in the form

$$\Lambda^\alpha_{\bar{\beta}} - \Lambda^\beta_{\bar{\alpha}} = 0. \tag{4.17}$$

This leads to the following definition.

4.18. DEFINITION. A complex foliation is called *symmetric* (resp. *antisymmetric*) if in the neighborhood of each point there is a local framing θ with respect to which the torsion matrix Λ is symmetric (resp. antisymmetric), and framings with respect to which Λ is symmetric (resp. antisymmetric) are called *symmetric* (resp. *antisymmetric*) framings.

4.19. PROLONGATION OF THE EQUATION $d_{\mathfrak{F}}\sigma = 0$. We now want to present a set of necessary and sufficient conditions for a foliation to be locally tangentially Monge–Ampère. The idea is to repeatedly take Lie derivatives of the symmetry condition (4.11) with respect to vector fields tangent to $\mathfrak{F}$, using the invariance condition $\nabla\sigma = 0$ to turn the resulting differential equations for the form σ into a set of purely *algebraic* conditions. We show that at a generic point, there are only a finite number of such independent conditions which, together with the conditions that (i) σ be nonnegative and (ii) $\sigma^q \neq 0$, are necessary and sufficient for the local solvability of the system of equation $d_{\mathfrak{F}}\sigma = 0$.

Begin by letting $X_{(1)}, \dots, X_{(p)}$ be a local framing of $L_{(1,0)}$ chosen so that the conditions $[X_{(j)}, X_{(k)}] = [X_{(j)}, \overline{X_{(k)}}] = 0$ for $1 \le j,\ k \le p$ hold, and choose a coframing θ so that the connection form ω is of type $(1,0)$. (Such frames are constructed in paragraph 1.4.) If f is a smooth function on M and $J = (j_1, j_2, \dots, j_k)$ is a multi-index, we will write $f_{,J} = X_{(j_1)}(X_{(j_2)}(\dots(X_{(j_k)}0\dots)))$ and denote the inner product $i(X_{(j)})\Lambda^{\alpha}_{\bar\beta}$ by $\Lambda^{\alpha}_{\bar\beta j}$. Finally, set $\Gamma^{\alpha}_{\beta j} = \omega^{\alpha}_{\beta}(X_{(j)})$ and $\Lambda^{\alpha}_{\bar\beta j} = i(X_{(j)})\Lambda^{\alpha}_{\bar\beta}$. With these conventions we have the following lemma, which is nothing but a local coordinate version of Proposition 4.9.

4.20. LEMMA. *The complex foliation $\mathfrak{F}$ is (locally) tangentially Monge–Ampère if and only if there is a matrix $\mathbf{U} = (U_{\alpha\bar\beta})$ which is a solution of the system of equations*

$$U_{\alpha\bar\beta,j} = U_{\gamma\bar\beta}\Gamma^{\gamma}_{\alpha j}, \tag{4.21}$$

$$U_{\gamma\bar\alpha}\Lambda^{\gamma}_{\bar\beta j} - U_{\gamma\bar\beta}\Lambda^{\gamma}_{\bar\alpha j} = 0, \tag{4.22}$$

$$U_{\alpha\bar\beta} - \overline{U_{\beta\bar\alpha}} = 0, \tag{4.23}$$

and which satisfies the open condition that it be positive definite.

Differentiating equation (4.22) with respect to the vector field $X_{(k)}$ will yield another equation that must be satisfied by the form σ. After differentiation use the identity (4.21) to obtain the *prolonged equation*

$$U_{\gamma\bar\alpha}\{\Lambda^{\gamma}_{\bar\beta j,k} + \Gamma^{\gamma}_{\sigma k}\Lambda^{\sigma}_{\bar\beta j}\} - U_{\gamma\bar\beta}\{\Lambda^{\gamma}_{\bar\alpha j,k} + \Gamma^{\gamma}_{\sigma k}\Lambda^{\sigma}_{\bar\alpha j}\} = 0, \tag{4.24}$$

which, upon setting $A^{\alpha}_{\bar\beta j,k} = \Lambda^{\alpha}_{\bar\beta j,k} + \Gamma^{\alpha}_{\gamma k}\Lambda^{\gamma}_{\bar\beta j}$, assumes the form

$$U_{\gamma\bar\alpha}A^{\gamma}_{\bar\beta j,k} - U_{\gamma\bar\beta}A^{\gamma}_{\bar\alpha j,k} = 0. \tag{4.25}$$

4.26. REMARK. Differentiating (4.22) with respect to the vector fields $\overline{X_{(k)}}$ will not yield a new equation, because (4.21), (4.22), (4.23), and (4.25) imply the identity

$$(U_{\gamma\bar{\alpha}}\Lambda^{\gamma}_{\bar{\beta}j}-U_{\gamma\bar{\beta}}\Lambda^{\gamma}_{\bar{\alpha}j})_{,\bar{k}}=0.$$

To see this, note that the structure equation (3.28) is equivalent to the equations

$$\begin{aligned}\Lambda^{\alpha}_{\bar{\beta}j,\bar{k}}&=\Lambda^{\alpha}_{\bar{\gamma}k}\overline{\Gamma^{\gamma}_{\beta k}},\\ \Lambda^{\alpha}_{\bar{\beta}j,k}-\Lambda^{\alpha}_{\bar{\beta}k,j}&=(\Gamma^{\alpha}_{\gamma j}\Lambda^{\gamma}_{\bar{\beta}k}-\Gamma^{\alpha}_{\gamma k}\Lambda^{\gamma}_{\bar{\beta}j}).\end{aligned} \tag{4.27}$$

Equations (4.27) and (4.22) can be used to calculate as follows:

$$\begin{aligned}(U_{\gamma\bar{\alpha}}\Lambda^{\gamma}_{\bar{\beta}j}-U_{\gamma\bar{\beta}}\Lambda^{\gamma}_{\bar{\alpha}j})_{,\bar{k}}&=(U_{\gamma\bar{\sigma}}\overline{\Gamma^{\sigma}_{\alpha k}}\Lambda^{\gamma}_{\bar{\beta}j}+U_{\gamma\bar{\alpha}}\Lambda^{\gamma}_{\bar{\beta}j,\bar{k}})-(U_{\gamma\bar{\sigma}}\overline{\Gamma^{\sigma}_{\beta k}}\Lambda^{\gamma}_{\bar{\alpha}j}+U_{\gamma\bar{\beta}}\Lambda^{\gamma}_{\bar{\alpha}j,\bar{k}})\\&=(U_{\gamma\bar{\sigma}}\overline{\Gamma^{\sigma}_{\alpha k}}\Lambda^{\gamma}_{\bar{\beta}j}+U_{\gamma\bar{\alpha}}\Lambda^{\gamma}_{\bar{\sigma}j}\overline{\Gamma^{\sigma}_{\beta k}})-(U_{\gamma\bar{\sigma}}\overline{\Gamma^{\sigma}_{\beta k}}\Lambda^{\gamma}_{\bar{\alpha}j}+U_{\gamma\bar{\beta}}\Lambda^{\gamma}_{\bar{\sigma}j}\overline{\Gamma^{\sigma}_{\alpha k}})\\&=(U_{\gamma\bar{\beta}}\overline{\Gamma^{\sigma}_{\alpha k}}\Lambda^{\gamma}_{\bar{\sigma}j}+U_{\gamma\bar{\alpha}}\Lambda^{\gamma}_{\bar{\sigma}j}\overline{\Gamma^{\sigma}_{\beta k}})-(U_{\gamma\bar{\alpha}}\overline{\Gamma^{\sigma}_{\beta k}}\Lambda^{\gamma}_{\bar{\sigma}j}+U_{\gamma\bar{\beta}}\Lambda^{\gamma}_{\bar{\sigma}j}\overline{\Gamma^{\sigma}_{\alpha k}})\\&=0.\end{aligned}$$

Inductively obtain prolonged conditions on σ by assuming that the quantities $A^{\alpha}_{\bar{\beta}j,K}$ have been constructed such that the condition

$$U_{\gamma\bar{\alpha}}A^{\gamma}_{\bar{\beta}j,K}-U_{\gamma\bar{\beta}}A^{\gamma}_{\bar{\alpha}j,K}=0 \tag{4.28}$$

holds for every multi-index $K=(k_1,k_2,\ldots,k_s)$ of length $s\le r$. Differentiating with respect to $X_{(k)}$ then yields the new conditions

$$U_{\gamma\bar{\alpha}}A^{\gamma}_{\bar{\beta}j,(K,k)}-U_{\gamma\bar{\beta}}A^{\gamma}_{\bar{\alpha}j,(K,k)}=0, \tag{4.29}$$

where

$$A^{\alpha}_{\bar{\beta}j,(K,k)}\equiv(A^{\alpha}_{\bar{\beta}j,K})_{,k}+\Gamma^{\alpha}_{\sigma k}A^{\sigma}_{\bar{\beta}j,K}. \tag{4.30}$$

Conditions (4.29) are clearly necessary for $\mathfrak{F}$ to be tangentially Monge-Ampère; in fact, they were shown in [2] to be necessary conditions for $\mathfrak{F}$ to be Monge-Ampère. We will show that near a generic point of M they are also sufficient for $\mathfrak{F}$ to be tangentially Monge-Ampère.

It will be helpful to reformulate the condition that $\mathfrak{F}$ be tangentially Monge-Ampère. Denote by $\pi: E\to M$ the real subbundle of $Q^{(1,0)}$ consisting of real, normal forms of type (1, 1); that is,

$$E=\{(i/2)U_{\alpha\bar{\beta}}\theta^{\alpha}\wedge\overline{\theta^{\beta}}\mid U_{\alpha\bar{\beta}}=\overline{U_{\beta\bar{\alpha}}}\},$$

and observe that, by virtue of the easily checked identity $\nabla\bar{\sigma}=\overline{\nabla\sigma}$, the complex Bott connection restricts to a partial connection on E. Denote by $V\to M$ the subset of E defined as follows:

$$V=\{U_{\alpha\bar{\beta}}\theta^{\alpha}\wedge\overline{\theta^{\beta}}\in E\mid U_{\gamma\bar{\alpha}}\Lambda^{\gamma}_{\bar{\beta}j}-U_{\gamma\bar{\beta}}\Lambda^{\gamma}_{\bar{\alpha}j}=0,\ \forall\alpha,\beta,j\}, \tag{4.34}$$

and observe that *V has locally constant rank on an open dense subset of M*. Since we are interested in analyzing $\mathfrak{F}$ at a generic point, we may assume that V is a subbundle of E of constant rank r. In general, the connection ∇ does not leave the subbundle $V\subseteq E$ invariant. We want to find conditions for the existence of a covariant constant section of V. To this end define V_∞ to be the largest subset of V satisfying the invariance condition

$$\nabla_X s\in\Gamma(M,V_\infty)\quad \forall X\in\Gamma(M,L)\ \text{and}\ \forall s\in\Gamma(M,V_\infty). \tag{4.35}$$

Again, by restricting to an open dense subset of M if necessary, we may assume that V_∞ is a subbundle of V of rank r_∞. The next lemma shows that, at a generic point of M, there are covariant constant sections of V if and only if V_∞ is not empty; consequently, *the foliation $\mathfrak{F}$ is tangentially Monge–Ampère if and only if the intersection of the subbundle V_∞ with the open set of positive definite Hermitian inner products on Q is nonempty.*

4.36. LEMMA. *The subbundle $V \subseteq E$ satisfies the partial flatness condition*

$$R(X, Y)s = 0$$

for $X, Y \in \Gamma(M, L)$ and $s \in \Gamma(M, V)$. Moreover, if V_∞ has constant rank $r_\infty > 0$ in a neighborhood U of the point $x_0 \in M$ then the connection ∇ restricts to a partially flat connection on V_∞, and if s_0 is contained in the fiber of V_∞ at x_0 then there is a section $s \in \Gamma(U', V_\infty)$, $U' \subset U$ a neighborhood of x_0, such that $s(x_0) = s_0$ and the equality $\nabla s = 0$ holds on U'.

Proof. Denote the curvature matrix of $Q^{(1,1)}$ relative to the local framing $\theta^\alpha \wedge \overline{\theta^\beta}$ by $\Omega^{\alpha\bar\beta}_{\gamma\bar\sigma}$. Then, from the structure equations for ∇ we have the identity

$$\Omega^{\alpha\bar\beta}_{\gamma\bar\sigma} \equiv \Omega^\alpha_\gamma \delta^{\bar\beta}_{\bar\sigma} + \delta^\alpha_\gamma \overline{\Omega^\beta_\sigma} = \Lambda^\alpha_{\bar\mu} \wedge \overline{\Lambda^\mu_{\bar\gamma}} \delta^{\bar\beta}_{\bar\sigma} + \delta^\alpha_\gamma \overline{\Lambda^\beta_{\bar\mu}} \wedge \Lambda^\mu_{\bar\sigma}.$$

So if $s = (i/2) U_{\alpha\bar\beta} \theta^\alpha \wedge \overline{\theta^\beta}$ is a section of V, then

$$\begin{aligned} R_\nabla(X_{(j)}, \overline{X_{(k)}})s &= \Omega^{\alpha\bar\beta}_{\gamma\bar\sigma}(X_{(j)}, \overline{X_{(k)}})s \\ &= (i/2)(\Lambda^\alpha_{\bar\mu j} \overline{\Lambda^\mu_{\bar\gamma k}} U_{\alpha\bar\sigma} - \Lambda^\mu_{\bar\sigma j} \overline{\Lambda^\beta_{\bar\mu k}} U_{\gamma\bar\beta}) \theta^\gamma \wedge \overline{\theta^\sigma} \\ &= (i/2)(\Lambda^\beta_{\bar\sigma j} \overline{\Lambda^\mu_{\bar\gamma k}} U_{\beta\bar\mu} - \Lambda^\mu_{\bar\sigma j} \overline{\Lambda^\beta_{\bar\mu k} U_{\beta\bar\gamma}}) \theta^\gamma \wedge \overline{\theta^\sigma} \\ &= (i/2)(\Lambda^\beta_{\bar\sigma j} \overline{\Lambda^\mu_{\bar\gamma k}} U_{\beta\bar\mu} - \Lambda^\mu_{\bar\sigma j} \overline{\Lambda^\beta_{\bar\gamma k} U_{\beta\bar\mu}}) \theta^\gamma \wedge \overline{\theta^\sigma} \\ &= (i/2)(\Lambda^\beta_{\bar\sigma j} \overline{\Lambda^\mu_{\bar\gamma k}} U_{\beta\bar\mu} - \Lambda^\mu_{\bar\sigma j} \overline{\Lambda^\beta_{\bar\gamma k}} U_{\mu\bar\beta}) \theta^\gamma \wedge \overline{\theta^\sigma} \\ &= 0. \end{aligned}$$

By the results of the previous paragraph, the partial connection ∇ restricts to define a partially flat partial connection on V_∞. To obtain the section s, extend s_0 in any way to a smooth section along a real submanifold Σ of dimension $2q$ transverse to $\mathfrak{F}$ and containing the point x_0. The process of parallel translation along the leaves of $\mathfrak{F}$ can now be used to extend s' to a covariant constant section s with the required properties. □

Before proceeding it will be convenient to simplify the notation by re-indexing. Let $e^1, e^2, \ldots, e^m$, $m = q^2$, be a local framing of E, let the indices a, b, and c range between 1 and m, and write the (partial) covariant derivative in the form

$$\nabla e^a = \Gamma^a_{bj}[dw^j] \otimes e^b + \Gamma^a_{b\bar j}[d\bar w^j] \otimes e^b. \tag{4.37}$$

Then there are smooth functions P^a_t, $t = 1, 2, \ldots, m-r$, such that conditions (4.34) for a section $s = s_a e^a \in E$ to be a section of the bundle V can be rewritten in the form

$$s_a P^a_t = 0, \quad t = 1, 2, \ldots, m-r. \tag{4.38}$$

Suppose now that s is a section of V with $\nabla s=0$ (so that in fact s is a section of V_∞). Then the identities

(4.39) $$X_{(k)}(s_a)+\Gamma^b_{ak}s_a=0, \qquad \overline{X_{(k)}}(s_a)+\Gamma^b_{a\bar{k}}s_a=0$$

hold, and differentiating equation (4.38) with respect to the vector fields $X_{(k)}$ and $\overline{X_{(k)}}$ respectively gives the equations

$$X_{(k)}(s_a)P^a_t+s_aX_{(k)}(P^a_t)=0, \qquad \overline{X_{(k)}}(s_a)P^a_t+s_a\overline{X_{(k)}}(P^a_t)=0,$$

which by virtue of (4.39) can be rewritten in the form

(4.40) $$s_aP^a_{t,k}=0, \qquad s_aP^a_{t,\bar{k}}=0,$$

where

(4.41) $$P^a_{t,k}\equiv X_{(k)}(P^a_t)-\Gamma^a_{bk}P^b_t, \qquad P^a_{t,\bar{k}}\equiv\overline{X_{(k)}}(P^a_t)-\Gamma^a_{b\bar{k}}P^b_t.$$

More generally, for each multi-index

$$K=(k_1,k_2,\dots,k_j), \quad k_i\in\{1,2,\dots,p,\bar{1},\bar{2},\dots,\bar{p}\},$$

inductively define functions $P^a_{t,K}$ by the formulas

(4.42) $$P^a_{t,(K,k)}\equiv X_{(k)}(P^a_{t,K})-\Gamma^a_{bk}P^b_{t,K}, \qquad P^a_{t,(K,\bar{k})}\equiv\overline{X_{(k)}}(P^a_{t,K})-\Gamma^a_{b\bar{k}}P^b_{t,K}$$

and set $P^a_{t,0}\equiv P^a_t$. Then any section s of V with $\nabla s=0$ satisfies the additional set of linear homogeneous equations

(4.43) $$s_aP^a_{t,K}=0,$$

which are the defining equations for V_∞ as the next lemma shows.

4.44. LEMMA.

(a) *Suppose that the system of equations* (4.43) *has constant rank r in a neighborhood of the point x_0. Then the equality*

$$V_\infty=\{s=s_ae^a\mid s_aP^a_{j,K}=0,\, 0\le|K|\le r\}$$

holds. Here $|(k_1,k_2,\dots,k_t)|$ denotes the length t of the multi-index, and $P^a_{j,K}\equiv P^a_j$ when $|K|=0$.

(b) *If $s=s_ae^a$ is a section of V then the expressions $s_aP^a_{t,K}$, for $K=(k_1,k_2,\dots,k_{r'})$, are completely symmetric in $k_1,k_2,\dots,k_{r'}$.*

Proof. (a) Begin by considering the restriction on the length $|K|$ of the multi-index K. Suppose that for some integer r' the rank of the sets of equations

(4.45) $$s_aP^a_{j,K}=0, \quad \forall 0\le|K|\le r'$$

and

(4.46) $$s_aP^a_{j,K}=0, \quad \forall 0\le|K|\le r'+1$$

are equal. Then it is easy to see that the rank of the full system of equations (4.43) with no restriction on the length of the index K is also r'. Hence, because the rank of V is r and V is defined by equations (4.43) with $|K|=0$, at most we need consider multi-indices of length r.

The inclusion $\subseteq$ is easily demonstrated, for by Lemma 4.36 if s_0 is a point in V_∞ then it extends to a covariant constant section $s \in \Gamma(M, V)$, and by construction conditions (4.41) are satisfied by the section s.

To prove the reverse inclusion begin by letting $s = s_a e^a$ be a section of the bundle V satisfying equations (4.41), and differentiate the equation $s_a P^a_{t,K} = 0$ with respect to each of the vector fields $X_{(k)}$ and $\overline{X_{(k)}}$ to obtain the identities

$$X_{(k)}(s_a)P^a_{t,K} = -s_a X_{(k)}(P^a_{t,K}), \qquad \overline{X_{(k)}}(s_a)P^a_{t,K} = -s_a \overline{X_{(k)}}(P^a_{t,K}).$$

But then

$$(X_{(k)}(s_a) + \Gamma^b_{ak}s_a)P^a_{t,K} = -(X_{(k)}(P^a_{t,K}) - \Gamma^a_{bk}P^b_{t,K})s_a = -P^a_{t,(K,k)} = 0,$$

showing that $\nabla_{X_{(k)}}s$ satisfies (4.41). A similar calculation shows that $\nabla_{\overline{X_{(k)}}}s$ also satisfies (4.41).

(b) To prove the symmetry condition, compute as follows:

$$\begin{aligned} P^a_{t,(K,j,k)} &= X_{(k)}(P^a_{t,(K,j)}) - \Gamma^a_{bk}P^b_{t,(K,j)} \\ &= X_{(k)}X_{(j)}(P^a_{t,K}) - X_{(k)}(\Gamma^a_{bj}P^b_{t,K}) - \Gamma^a_{bk}(X_{(j)}(P^b_{t,K}) + \Gamma^b_{cj}P^c_{t,K}) \\ &= X_{(k)}X_{(j)}(P^a_{t,K}) - (\Gamma^a_{bj}X_{(k)}(P^b_{t,K}) + \Gamma^a_{bk}X_{(j)}(P^b_{t,K})) \\ &\quad - (X_{(k)}(\Gamma^a_{cj}) + \Gamma^a_{bk}\Gamma^b_{cj})P^c_{t,K}. \end{aligned}$$

Now use the fact that the vector fields $X_{(k)}$ and $\overline{X_{(k)}}$ $(1 \le k \le p)$ commute (see (1.16) and (1.17)) to arrive at the identity

$$P^a_{t,(K,j,k)} - P^a_{t,(K,k,j)} = -R^a_{cjk}P^c_{t,K}.$$

But by Lemma 4.36, $s_a R^a_{bjk} \equiv 0$ for $s = s_a e^a \in \Gamma(M, V)$. Similar calculations with the pairs of vector fields $X_{(j)}, \overline{X_{(k)}}$ and $\overline{X_{(k)}}, \overline{X_{(j)}}$ show that

$$s_a P^a_{t,(K,j,\bar{k})} = s_a P^a_{t,(K,\bar{k},j)} \quad \text{and} \quad s_a P^a_{t,(K,\bar{j},\bar{k})} = s_a P^a_{t,(K,\bar{k},\bar{j})},$$

thereby proving part (b). □

The final result of the above computations is summarized in the next theorem.

4.47. THEOREM. *Let $A^\alpha_{\bar\beta j,K}$ be as above. Suppose that the system of equations*

$$\begin{cases} U_{\gamma\bar\alpha}A^\gamma_{\bar\beta j,K} - U_{\gamma\bar\beta}A^\gamma_{\bar\alpha j,K} = 0 \\ U_{\alpha\bar\beta} - U_{\beta\bar\alpha} \qquad\qquad\quad = 0 \end{cases}$$

for $K = (k_1, k_2, \ldots, k_s)$, $0 \le s \le 2q$, has constant rank in a neighborhood of a point $x_0 \in M$. Then $\mathfrak{F}$ is tangentially Monge–Ampère in a neighborhood of x_0 if and only if there is a solution $U_{\alpha\bar\beta}$ which is positive definite.

Proof. Observe first that the quantities $A^\alpha_{\bar\beta j,K}$ are precisely the quantities $P^a_{j,K}$ in the case where the framing $e^1, e^2, \ldots$ is taken to be $i(\theta^\alpha \wedge \overline{\theta^\beta} + \theta^\beta \wedge \overline{\theta^\alpha})$, $1 \le \alpha \le \beta \le q$, $(\theta^\alpha \wedge \overline{\theta^\beta} - \theta^\beta \wedge \overline{\theta^\alpha})$, $1 \le \alpha < \beta \le q$. The theorem then follows from the above lemma and Remark 4.26, which by virtue of the symmetry condition shows that multi-indices involving conjugate indices need not be considered. □

4.48 REMARK. If one is interested in Lorentz metrics then the positive definiteness condition in Theorem 4.47 can be replaced by a condition that the matrix $U_{\alpha\bar{\beta}}$ have the proper signature.

We now wish to consider the problem of determining sufficient conditions for a complex foliation $\mathfrak{F}$ to be Monge–Ampère. Let Σ be a closed complex submanifold of M transverse to the leaves of $\mathfrak{F}$. Note that the bundle V_∞ restricts to Σ and that a positive section of the restricted bundle corresponds to a Hermitian metric, h_Σ, on Σ satisfying the algebraic conditions of Theorem 4.47. Call such a metric *admissible.*

4.49 THEOREM. *Assume that the bundle V_∞ has constant rank in a neighborhood of Σ. Then $\mathfrak{F}$ is Monge–Ampère in a neighborhood of Σ if and only if it supports an admissible Kähler metric.*

Proof. Let σ_0 be the Kähler form of an admissible metric. It defines a section of the restriction to Σ of the flat bundle V_∞. Extend the form by parallel translation to a neighborhood of Σ; call the extended form σ. We need only show that σ is closed. But by Remark 4.6, if X is any vector field tangent to $\mathfrak{F}$ then the Lie derivative $i(X)d\sigma$ vanishes, and since Lie differentiation commutes with exterior differentiation the form $d\sigma$ is locally constant along the leaves of $\mathfrak{F}$. But clearly $d\sigma = 0$ at all points of Σ. It follows that σ is closed in a neighborhood of Σ. □

4.50. REMARK. The problem of determining the existence of admissible Kähler metrics remains open and will be addressed in future work using the techniques of exterior differential systems.

4.51. REMARK. Note that in the special case of codimension-1 foliations the conditions of Theorems 4.48 and 4.49 are empty (all 1×1 matrices are symmetric and every $(1,1)$-form on a complex curve is closed). Hence, in this case we obtain the well-known result that every codimension-1 complex foliation is (locally) Monge–Ampère.

5. The first Chern class of symmetric foliations. In Section 3 we observed that the first relative Chern class $c_1(\mathfrak{F}) \in H^2(M, C^\infty_{\mathfrak{F}})$ of a holomorphic foliation vanishes. In this section we will prove that under certain conditions (e.g., if the leaves of $\mathfrak{F}$ are compact and Kähler) the converse holds. The fundamental result is contained in the next lemma.

5.1. LEMMA. *Let $\mathfrak{F}$ be a symmetric (resp. antisymmetric) foliation. Then the first relative Chern form $C_1(\mathfrak{F})$ is nonpositive (resp. nonnegative) and the foliation is holomorphic precisely at those points where $C_1(\mathfrak{F})$ vanishes.*

Proof. Let θ be a symmetric (resp. antisymmetric) frame for $Q^{(1,0)}$. Then the torsion matrix $\mathbf{A}$ is symmetric (resp. antisymmetric), and it follows from the structure equations for the complex Bott connection that the first Chern form satisfies the identity

(5.2) $$C_1(\mathfrak{F}) \equiv -(i/2\pi)\operatorname{tr}(\Lambda\wedge\bar{\Lambda}) = \mp(i/2\pi)\sum_{\alpha,\beta}\Lambda^\alpha_{\bar\beta}\wedge\overline{\Lambda^\beta_{\bar\alpha}}.$$

Since each of the terms

$$(i/2\pi)\Lambda^\alpha_{\bar\beta}\wedge\overline{\Lambda^\beta_{\bar\alpha}} = (i/2\pi)\Lambda^\alpha_{\bar\beta}\wedge\overline{\Lambda^\alpha_{\bar\beta}}$$

is a nonnegative (1,1)-form so is their sum $\sum (i/2\pi)\Lambda^\alpha_{\bar\beta}\wedge\overline{\Lambda^\beta_{\bar\alpha}}$, whence the claim that the first Chern form is nonpositive (resp. nonnegative).

From the form of (5.2) it is clear that the first Chern class vanishes exactly where the forms $\Lambda^\alpha_{\bar\beta}$ all vanish. But, by Proposition 3.34, the foliation is holomorphic exactly where these forms all vanish. □

5.3. THEOREM.

(a) *Let $\mathfrak{F}$ be a symmetric or antisymmetric foliation of a complex manifold M, let N be a compact leaf, and assume that N is Kähler. Then $\mathfrak{F}$ is holomorphic on N if and only if the Chern class $c_1(Q_N)\in H^2(N,\mathbb{C})$ vanishes.*

(b) *Suppose that M is a compact Kähler manifold and that $\mathfrak{F}$ is a globally tangentially Monge–Ampère foliation of M. Then $\mathfrak{F}$ is holomorphic if and only if the relative Chern class $c_1(\mathfrak{F})\in H^2(M, C^\infty_{\mathfrak{F}})$ vanishes. In particular, if the Chern class $c_1(Q)\in H^2(M,\mathbb{C})$ vanishes then $\mathfrak{F}$ is holomorphic.*

Proof. (a) Let ω_N be the Kähler form on N and consider the top dimensional form

$$\nu = C_1(\mathfrak{F})_N\wedge\omega_N^{p-1},$$

where $C_1(\mathfrak{F})_N$ is the restriction to N of the relative form $C_1(\mathfrak{F})$. Since that partial connection ∇ restricts to an ordinary connection on the bundle $Q_N\to N$, the form $C_1(\mathfrak{F})_N$ is a representative of the first Chern class $c_1(Q_N)\in H^2(N,\mathbb{C})$.

Let $\eta^1,\eta^2,\dots,\eta^p$ be a local Hermitian framing for TN^*. Then

$$\omega_N = (i/2)\textstyle\sum_j \eta^j\wedge\bar\eta^j$$

and the restriction of $\Lambda^\alpha_{\bar\beta}$ to N can be written in the form

(5.4) $$\lambda^\alpha_{\bar\beta\,|N} = A^\alpha_{\bar\beta j}\eta^j,$$

where $A^\alpha_{\bar\beta j}$ are smooth functions satisfying the symmetry condition $A^\alpha_{\bar\beta j} = A^\beta_{\bar\alpha j}$ (resp. the antisymmetry condition $A^\alpha_{\bar\beta j} = -A^\beta_{\bar\alpha j}$). Now compute as follows:

$$\begin{aligned}\nu &= \frac{\mp i}{2\pi}\Big(\sum_{\alpha,\beta} A^\alpha_{\bar\beta j}\overline{A^\alpha_{\bar\beta k}}\eta^j\wedge\overline{\eta^k}\Big)\wedge\omega_N^{p-1}\\ &= \mp\frac{i^p(p-1)!}{2^p\pi}\sum_{\alpha,\beta,j}|A^\alpha_{\bar\beta j}|^2\eta^1\wedge\bar\eta^1\wedge\cdots\wedge\eta^p\wedge\bar\eta^p\\ &= \mp\frac{1}{p\pi}\Big(\sum_{\alpha,\beta,j}|A^\alpha_{\bar\beta j}|^2\Big)\omega_N^p.\end{aligned}$$

It is clear that $\mathfrak{F}$ is holomorphic if and only if the form ν vanishes. But ν is a nonpositive (resp. nonnegative) multiple of the volume form ω_N^p and N is compact. Consequently, the form ν vanishes identically if and only if its cohomology class

$$[\nu]=c_1(Q_N)\cup[\omega_N]^{p-1}\in H^{2p}(N,\mathbb{C})$$

vanishes. Part (a) is now clear.

The proof of part (b) is similar: Let ω_M be the Kähler form of M and let $\sigma\in\Gamma(M,Q^{(1,1)})$ be the $d_{\mathfrak{F}}$-closed normal form defining $\mathfrak{F}$. Observe that the normal (q,q)-form σ^q is d-closed. (It is easily shown that if $\mathfrak{F}$ is a real codimension-$2q$ foliation and ν is a $d_{\mathfrak{F}}$-closed, normal $2q$-form then ν is in fact d-closed.)

Choose a splitting of the cotangent bundle, $TM^*=L^*\otimes Q^*$, and use it to identify relative forms with ordinary forms. In particular, consider the (n,n)-form on M,

$$\nu\equiv c_1(\mathfrak{F})\wedge\omega_M^{p-1}\wedge\sigma^q,$$

and note that ν is independent of the splitting. Next let $\boldsymbol{\theta}$ be a local framing of $Q^{(1,0)}$, unitary with respect to the inner product on Q induced by the form σ, and let $\eta^1,\eta^2,\dots,\eta^n$ be a unitary framing for $TM^{(1,0)}$ chosen so that the forms η^j, for $j>p$, are sections of $Q^{(1,0)}$. With these choices there are formulas

$$\sigma=(i/2)\sum_\alpha\theta^\alpha\wedge\overline{\theta^\alpha}$$

and

$$\Lambda^\alpha_{\bar\beta}=\sum_{k=1}^{p}A^\alpha_{\bar\beta k}\iota^*(\eta^k),$$

where the functions $A^\alpha_{\bar\beta j}$ are symmetric in α and β and where $\iota:L\hookrightarrow TM$ is the inclusion map. A short computation then yields the identity

$$\nu=-\frac{1}{p\pi}\left(\sum_{\alpha,\beta,j}|A^\alpha_{\bar\beta j}|^2\right)\det(\mathbf{U})\frac{p!\,q!}{n!}\omega_M,$$

where $\mathbf{U}=(U_{\alpha\bar\beta})$ is the positive definite Hermitian matrix defined by the formula

$$\sigma=(i/2)U_{\alpha\bar\beta}\eta^{p+\alpha}\wedge\bar\eta^{p+\beta}.$$

Since ν is a nonpositive multiple of the volume form ω_M^n which vanishes if and only if $\mathfrak{F}$ is holomorphic and since M is compact, it follows that $\mathfrak{F}$ is holomorphic if and only if the cohomology class $[\nu]\in H^{2n}(M,\mathbb{C})$ vanishes.

Now suppose that $c_1(\mathfrak{F})=0$. Then there is a relative 1-form ϕ with $d_{\mathfrak{F}}\phi=C_1(\mathfrak{F})$; because the forms ω_M and σ^q are d-closed, the form ν is exact, as shown by the computation

$$d(\phi\wedge\omega_M^{p-1}\wedge\sigma^q)=d_{\mathfrak{F}}\phi\wedge\omega_M^{p-1}\wedge\sigma^q=\nu.$$

Hence, by the reasoning of the previous paragraph, the vanishing of $c_1(\mathfrak{F})$ implies that $\mathfrak{F}$ is holomorphic.

The last statement of the theorem follows from the fact that the relative first Chern class is images of the Chern class $c_1(Q)\in H(M,\mathbb{C})$ (see Definition 2.16). □

An examination of the above proof shows that, because the Kähler form appears to the $(p-1)$th power in all formulas, for the special case $p=1$ the assumption that M be Kähler can be dropped, yielding the following theorem.

5.5. THEOREM. *Let $\mathfrak{F}$ be a globally tangentially Monge–Ampère foliation of a compact manifold by complex curves. Then $\mathfrak{F}$ is holomorphic if and only if its relative first Chern class $c_1(\mathfrak{F}) \in H^2(M, C^\infty_{\mathfrak{F}})$ vanishes.*

5.6. REMARK. The assumption that $\mathfrak{F}$ be globally tangentially Monge–Ampère is too strong. We need only assume (i) that $\mathfrak{F}$ is symmetric or antisymmetric and (ii) that there exists a closed, nondegenerate normal $2q$-form. A foliation for which condition (ii) is satisfied is called an $\mathrm{SL}(2q)$-foliation, and the obstruction to a foliation being an $\mathrm{SL}(2q)$-foliation is a cohomology class in $H^1(M, \mathcal{C}^\infty_{\mathbb{R},\mathfrak{F}})$, where $\mathcal{C}^\infty_{\mathbb{R},\mathfrak{F}}$ is the sheaf of germs of real-valued functions which are locally constant along the leaves of $\mathfrak{F}$. (To see this observe that forms satisfying (ii) exist locally and are unique up to multiplication by functions which are locally constant along the leaves of $\mathfrak{F}$. The obstruction to consistently fitting together local forms is a Čěch cocycle with values in $\mathcal{C}^\infty_{\mathbb{R},\mathfrak{F}}$.)

6. An intrinsic metric on leaves. Let $j: N' \hookrightarrow M$ be a leaf of a symmetric or antisymmetric foliation $\mathfrak{F}$. We showed in the previous section that the restriction to N' of the first Chern form is a nonpositive (resp. nonnegative) form of type $(1,1)$. Assume that the nondegeneracy condition $C_1(\mathfrak{F})^p \neq 0$ holds on a nonempty open set, $N \subseteq N'$. Then the closed, positive $(1,1)$-form

$$\omega_N \equiv (\mathrm{sgn}(\mathfrak{F})\pi) j^* C_1(\mathfrak{F}) = \mp (i/2) \sum_{\alpha,\beta} \Lambda^\alpha_{\bar\beta} \wedge \overline{\Lambda^\beta_{\bar\alpha}}, \tag{6.1}$$

where $\mathrm{sgn}(\mathfrak{F}) = -1$ for $\mathfrak{F}$ symmetric and $\mathrm{sgn}(\mathfrak{F}) = +1$ for $\mathfrak{F}$ antisymmetric, endows N with the structure of a Kähler manifold. We will denote the underlying Kähler metric by h_N. Since the Kähler structure on N is defined in terms of the complex Bott connection, it is clearly a local biholomorphic invariant of the leaf N'.

Of particular interest is the special case $p = 1$, for in this case the nondegeneracy assumption is satisfied at all points at which the foliation fails to be holomorphic. There are two cases where the induced metric h_N can be shown to have constant curvature.

6.2. THEOREM.

(a) *Let $\mathfrak{F}$ be a complex foliation of a complex two-dimensional manifold M by complex curves. Let N' be a leaf and suppose that the antiholomorphic torsion is nonvanishing on the open set $N \subseteq N'$. Then the metric h_N has constant Gaussian curvature, $K_N = -4$.*

(b) *Let $\mathfrak{F}$ be an antisymmetric complex foliation of a complex three-dimensional manifold M by complex curves and let $N \subset N'$ be as before. Then the metric h_N has constant Gaussian curvature, $K_N = +2$.*

Proof. Recall that the Gaussian curvature can be computed as follows. Choose a unit $(1,0)$-form, say η; then there is a unique connection 1-form ϕ characterized by the conditions

$$d\eta = -\phi \wedge \eta \quad \text{and} \quad \phi + \bar\phi = 0,$$

and the Gaussian curvature is the scalar function K_N defined by the equation

$$d\phi = K_N \frac{\eta \wedge \bar{\eta}}{2}.$$

(a) Let θ be a normal form of type $(1,0)$. Then the structure equations for the complex Bott connection assume the form

$$d_{\mathfrak{F}}\theta = -\omega \wedge \theta - \Lambda \wedge \bar{\theta}, \tag{6.3}$$

$$\Omega = d_{\mathfrak{F}}\omega = -\Lambda \wedge \bar{\Lambda}, \tag{6.4}$$

$$\begin{aligned} d_{\mathfrak{F}}\Lambda &= -\omega \wedge \Lambda - \Lambda \wedge \bar{\omega} \\ &= -\Lambda \wedge \bar{\omega}, \end{aligned} \tag{6.5}$$

where both ω and Λ are relative 1-forms of type $(1,0)$ and we have used the fact that $\omega \wedge \Lambda = 0$ because N has complex dimension equal to 1. Observe that since $\omega_N \equiv (i/2)j^*(\Lambda \wedge \bar{\Lambda})$, the Hermitian metric on N is given by $h_N = j^*\Lambda \otimes j^*\bar{\Lambda}$ and therefore $\eta = j^*\Lambda$ is a unit $(1,0)$-form.

It follows from equation (6.4) that the form $\phi = j^*(\omega - \bar{\omega})$ is the connection 1-form of the canonical Hermitian connection on N relative to this frame. The computation

$$\begin{aligned} d\phi &= dj^*(\omega - \bar{\omega}) \\ &= j^* d_{\mathfrak{F}}(\omega - \bar{\omega}) \\ &= j^*(-\Lambda \wedge \bar{\Lambda} + \bar{\Lambda} \wedge \Lambda) \\ &= -2j^*(\Lambda \wedge \bar{\Lambda}) \\ &= -2\eta \wedge \bar{\eta} \end{aligned}$$

then shows that $K_N = -4$.

The proof of part (b) is similar. Let θ be an antisymmetric framing. Then

$$\mathbf{\Lambda} = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix},$$

where Λ is a relative 1-form of type $(1,0)$. From the structure formula $\mathbf{\Omega} = \mathbf{\Lambda} \wedge \bar{\mathbf{\Lambda}}$ we can compute ω_N:

$$\omega_N = (i/2)j^* \operatorname{tr}(\mathbf{\Omega}) = (i/2)\eta \wedge \bar{\eta},$$

where $\eta = \sqrt{2}j^*\Lambda$ and tr denotes trace. From the structure equation

$$d_{\mathfrak{F}}\mathbf{\Lambda} = -\omega \wedge \mathbf{\Lambda} - \mathbf{\Lambda} \wedge \bar{\omega},$$

we find after a short computation that the connection 1-form relative to η is

$$\phi = (1/2)j^* \operatorname{tr}\{\omega - \bar{\omega}\}.$$

Hence ϕ is the connection 1-form and

$$\begin{aligned} d\phi &= (1/2)j^*\{\operatorname{tr}(-\omega \wedge \omega - \mathbf{\Lambda} \wedge \bar{\mathbf{\Lambda}}) - \operatorname{tr}\overline{(-\omega \wedge \omega - \mathbf{\Lambda} \wedge \bar{\mathbf{\Lambda}})}\} \\ &= (1/2)j^* \operatorname{tr}(-\mathbf{\Lambda} \wedge \bar{\mathbf{\Lambda}} + \bar{\mathbf{\Lambda}} \wedge \mathbf{\Lambda}) \\ &= \eta \wedge \bar{\eta}, \end{aligned}$$

from which the equality $K_N = +2$ follows. $\square$

6.6. EXAMPLE. It follows from the results of the previous section that any foliation of a compact complex two-dimensional manifold by compact holomorphic curves is necessarily holomorphic. (To see this recall that, by a theorem of Edwards, Millett, and Sullivan [7], such a foliation is Hausdorff and is therefore an oriented Riemannian foliation. But the restriction of the normal bundle of a real codimension-2 oriented Riemannian foliation to a leaf is a flat SO(2)-bundle. Since $SO(2)=U(1)$, the result now follows from Theorem 5.3.) This is not true for foliations of complex manifolds of dimension greater than 2, as the following example of Calabi [6] of a foliation of $\mathbb{C}\mathbf{P}^3$ shows.

Recall that the quaternions $\mathbb{H}$ can be identified with $\mathbb{C}^2$ by the map

$$\begin{gathered} \mathbb{C}^2 \to \mathbb{H}, \\ (z_1, z_2) \mapsto z_1 + z_2 \cdot \mathbf{j}, \end{gathered} \tag{6.7}$$

where quaternions are written in the form $a+b\cdot\mathbf{i}+c\cdot\mathbf{j}+d\cdot\mathbf{k}$; a, b, c, and d real; and the complex numbers are embedded in the quaternions via the map $a+bi \mapsto a+b\cdot\mathbf{i}$. Using the relations

$$\mathbf{i}\cdot\mathbf{j}=\mathbf{k}=-\mathbf{j}\cdot\mathbf{i}, \qquad \mathbf{j}\cdot\mathbf{k}=\mathbf{i}=-\mathbf{k}\cdot\mathbf{j}, \qquad \mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-1,$$

one easily checks the formula $z\cdot\mathbf{j}=\mathbf{j}\cdot\bar{z}$ for $z\in\mathbb{C}$, which in turn can be used to prove the identity

$$(\xi+\zeta\cdot\mathbf{j})\cdot(w+z\cdot\mathbf{j})=(\xi z-\zeta\bar{z})+(\xi z+\zeta\bar{w})\cdot\mathbf{j}. \tag{6.8}$$

The map (6.7) induces a diffeomorphism between $\mathbb{C}^4$ and $\mathbb{H}^2$,

$$\pi\colon (z_1, z_2, z_3, z_4) \mapsto (z_1+z_2\cdot\mathbf{j},\, z_3+z_4\cdot\mathbf{j}),$$

and it is easy to see from formula (6.8) that the image of every complex line in $\mathbb{C}^4$ containing the origin is contained in a quaternionic line containing the origin. Therefore, the map π induces a submersion $\hat{\pi}\colon \mathbb{C}\mathbf{P}^3 \to \mathbb{H}\mathbf{P}^1$, which sends each complex line to the unique quaternionic line generated by its image under π. It is not hard to check, again using formula (6.8), that the fibers of $\hat{\pi}$ are of the form

$$\{[\xi z^1-\zeta\bar{z}^2, \xi z^2+\zeta\bar{z}^1, \xi z^3-\zeta\bar{z}^4, \xi z^4+\zeta\bar{z}^3] \mid (\xi,\zeta)\in\mathbb{C}^2\} \tag{6.9}$$

and form a complex foliation $\mathfrak{F}$ of $\mathbb{C}\mathbf{P}^3$ by $\mathbb{C}\mathbf{P}^1$'s.

In affine coordinates $(w, z^1, z^2) \mapsto [1, w, z^1, z^2]$ the tangent bundle L is spanned by the vector field

$$X=\frac{\partial}{\partial w}+\left(\frac{\bar{z}^2-\bar{w}z^1}{1+|w|}\right)\frac{\partial}{\partial z^1}+\left(\frac{\bar{z}^1+\bar{w}z^2}{1+|w|}\right)\frac{\partial}{\partial z^2}$$

and the bundle $Q^{(1,0)}$ is spanned by the 1-forms

$$\begin{cases} \theta^1 = dz^1-\left(\dfrac{\bar{z}^2-\bar{w}z^1}{1+|w|^2}\right)dw, \\[2ex] \theta^2 = dz^2-\left(\dfrac{\bar{z}^1-\bar{w}z^2}{1+|w|^2}\right)dw. \end{cases}$$

Also, relative to this frame, the antiholomorphic torsion matrix Λ assumes the form

$$\Lambda = \begin{pmatrix} 0 & \dfrac{[dw]}{1+|w|^2} \\ -\dfrac{[dw]}{1+|w|^2} & 0 \end{pmatrix}.$$

This foliation is then antisymmetric and, by virtue of the previous theorem, its leaves inherit a metric of constant curvature $+2$. Of course, because the leaves of $\mathfrak{F}$ are spheres and the foliation is homogeneous, this is to be expected. Note, however, that the metric is given by *local* data and its curvature properties are not dependent on the global properties of $\mathfrak{F}$.

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