LOCAL INDEX THEOREM FOR FAMILIES

Harold Donnelly

1. Introduction. The index theorem for families of elliptic operators was proved by Atiyah and Singer [1] using global topological methods. Recently, Quillen [7] provided the formalism for a heat equation proof. This program has been carried through by Bismut [3] and Berline and Vergne [2]. A strong local theorem is obtained by the evaluation of certain heat equation asymptotics.

The purpose of the present paper is to give an alternative proof of the local index theorem for families. We develop the method of Getzler [5], who proved the local index theorem for a single elliptic operator. This solution to the problem appears to be more direct than other approaches. Hopefully, it should facilitate further developments.

Here is a brief outline of our paper. In Section 2, we derive certain curvature identities for connections on fiber bundles. Similar identities seem to be understood in [3]. Section 3 gives the precise statement of the local form of the index theorem for families of elliptic operators. One uses the superconnection formalism of Bismut and Quillen. The most substantial part of the work is Section 4. We use Getzler's ϵ-rescaling to prove the index theorem for families. In addition to the arguments of [5], certain judicious conjugations must be chosen to remove the new singularities in the rescaled differential operator and to bring the operator into standard form.

The author thanks E. Miller and T. Spencer for helpful conversations during the development of this work.

2. Curvature identities. Let M be a compact connected differentiable manifold. Suppose that M is the total space of a fiber bundle $F \to M \to B$, where the fiber F and base B are compact connected manifolds. A connection on this fiber bundle provides a splitting $TM = T^HM \oplus TF$ of the tangent space to M. We identify vector fields $X \in TB$ with their horizontal lifts $X \in T^HM$.

Suppose a Riemannian metric is given for B. Let ∇^B denote the Levi-Civita connection on TB. The metric of TB lifts to a smooth inner product on T^HM. Define T^HM and TF to be orthogonal. A smooth inner product along the fibers TF then gives a Riemannian metric for M. The corresponding Levi-Civita connection will be denoted by ∇^M.

The connection ∇^M need not preserve the splitting of TM into horizontal and vertical subspaces. Therefore, we define a second connection ∇ on TM. Let $P_F : TM \to TF$ be the orthogonal projection. As in [3], there is a unique connection ∇ on TM satisfying the following properties:

Received December 5, 1986.

Partially supported by NSF Grant #DMS-8610730 (1).

Michigan Math. J. 35 (1988).

(i) if $Y \in TB$, $Z \in TB$, $\nabla_Y Z = \nabla_Y^B Z$;
(ii) if $Y \in TF$, $Z \in TB$, $\nabla_Y Z = 0$;
(iii) if $Y \in TM$, $Z \in TF$, $\nabla_Y Z = P_F(\nabla_Y^M Z)$.

Clearly, ∇ preserves the metric of M and the splitting $TM = T^H M \oplus TF$. The torsion tensor of ∇ will be denoted by T. One has an auxiliary tensor field $S = \nabla^M - \nabla$.

Let R^M be the curvature tensor of the Levi-Civita connection ∇^M. Define R to be the curvature for ∇. One has the following basic identity.

PROPOSITION 2.1. *For any X, Y, Z, $W \in TM$,*

$$\langle R(X,Y)Z, W\rangle = \langle R^M(X,Y)Z, W\rangle - \langle(\nabla_X S)_Y Z, W\rangle + \langle(\nabla_Y S)_X Z, W\rangle - \langle S_X Z, S_Y W\rangle + \langle S_Y Z, S_X W\rangle - \langle S_{T(X,Y)} Z, W\rangle.$$

Proof. By definition

$$\langle R(X,Y)Z, W\rangle = \langle(\nabla_X^M - S_X)(\nabla_Y^M - S_Y)Z - (\nabla_Y^M - S_Y)(\nabla_X^M - S_X)Z - (\nabla_{[X,Y]}^M - S_{[X,Y]})Z, W\rangle.$$

Expanding the right-hand side gives

$$\begin{aligned}\langle R(X,Y)Z, W\rangle &= \langle R^M(X,Y)Z, W\rangle - \langle \nabla_X^M S_Y Z, W\rangle - \langle S_X \nabla_Y^M Z, W\rangle + \langle S_X S_Y Z, W\rangle \\ &\quad + \langle S_Y \nabla_X^M Z, W\rangle + \langle \nabla_Y^M S_X Z, W\rangle - \langle S_Y S_X Z, W\rangle + \langle S_{[X,Y]} Z, W\rangle.\end{aligned}$$

Since $\nabla^M = \nabla + S$, we may write

$$\begin{aligned}\langle R(X,Y)Z, W\rangle &= \langle R^M(X,Y)Z, W\rangle - \langle \nabla_X S_Y Z, W\rangle + \langle \nabla_Y S_X Z, W\rangle - \langle S_X \nabla_Y Z, W\rangle \\ &\quad + \langle S_Y \nabla_X Z, W\rangle + \langle S_Y S_X Z, W\rangle - \langle S_X S_Y Z, W\rangle + \langle S_{[X,Y]} Z, W\rangle.\end{aligned} \tag{2.2}$$

The extension of a connection must satisfy

$$\langle \nabla_X S_Y Z, W\rangle = \langle(\nabla_X S)_Y Z, W\rangle + \langle S_{\nabla_X Y} Z, W\rangle + \langle S_Y \nabla_X Z, W\rangle$$

and the analogous formula with X and Y interchanged.

We substitute these formulas in (2.2). Using the definition of the torsion

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

yields

$$\begin{aligned}\langle R(X,Y)Z, W\rangle &= \langle R^M(X,Y)Z, W\rangle - \langle(\nabla_X S)_Y Z, W\rangle + \langle(\nabla_Y S)_X Z, W\rangle \\ &\quad + \langle S_Y S_X Z, W\rangle - \langle S_X S_Y Z, W\rangle - \langle S_{T(X,Y)} Z, W\rangle.\end{aligned}$$

The endomorphism S_X is skew adjoint since S is the difference of two metric preserving connections. Proposition 2.1 follows. $\square$

Our primary concern is the curvature of ∇ restricted to the tangent bundle TF along the fibers. A more restricted choice of vector fields gives the following.

PROPOSITION 2.3. *If $X \in TB$, $Y \in TM$ and $W, Z \in TF$, then*

$$\langle R(X,Y)Z, W\rangle = \langle (\nabla_Z S)_W X, Y\rangle - \langle (\nabla_W S)_Z X, Y\rangle + \langle S_Z X, S_W Y - S_Y W\rangle + \langle S_W X, S_Y Z - S_Z Y\rangle.$$

Proof. If U is any vector field and V is in TF, the definition of S implies that $S_U V$ is horizontal. Using Proposition 2.1, we obtain

$$\langle R(X,Y)Z, W\rangle = \langle R^M(X,Y)Z, W\rangle - \langle (\nabla_X S)_Y Z, W\rangle + \langle (\nabla_Y S)_X Z, W\rangle - \langle S_X Z, S_Y W\rangle + \langle S_Y Z, S_X W\rangle. \tag{2.4}$$

The definition of ∇ gives $\langle R(Z,W)X, Y\rangle = 0$ since horizontal vector fields have zero covariant derivatives in vertical directions. Also $T(Z,W) = 0$, since $\nabla_Z W$ coincides with the Levi-Civita connection of the fiber, for $Z, W \in TF$. Applying Proposition 2.1,

$$\langle R^M(Z,W)X, Y\rangle = \langle (\nabla_Z S)_W X, Y\rangle - \langle (\nabla_W S)_Z X, Y\rangle + \langle S_Z X, S_W Y\rangle - \langle S_W X, S_Z Y\rangle.$$

A basic symmetry for the curvature of a Levi-Civita connection is

$$\langle R^M(X,Y)Z, W\rangle = \langle R^M(Z,W)X, Y\rangle.$$

Substitution into (2.4) yields

$$\begin{aligned}\langle R(X,Y)Z, W\rangle &= \langle (\nabla_Z S)_W X, Y\rangle - \langle (\nabla_W S)_Z X, Y\rangle - \langle (\nabla_X S)_Y Z, W\rangle + \langle (\nabla_Y S)_X Z, W\rangle \\ &\quad + \langle S_Z X, S_W Y\rangle - \langle S_X Z, S_Y W\rangle + \langle S_Y Z, S_X W\rangle - \langle S_W X, S_Z Y\rangle.\end{aligned} \tag{2.5}$$

Using the property, of S, which was mentioned at the beginning of our proof:

$$\begin{aligned}\langle (\nabla_Y S)_X Z, W\rangle &= \langle \nabla_Y S_X Z, W\rangle - \langle S_{\nabla_Y X} Z, W\rangle - \langle S_X \nabla_Y Z, W\rangle \\ &= \langle \nabla_Y S_X Z, W\rangle.\end{aligned}$$

Because ∇ is metric-preserving,

$$\langle (\nabla_Y S)_X Z, W\rangle = Y\langle S_X Z, W\rangle - \langle S_X Z, \nabla_Y W\rangle = 0.$$

Similarly, the corresponding term with X and Y reversed also vanishes.

Equation (2.5) reduces to

$$\langle R(X,Y)Z, W\rangle = \langle (\nabla_Z S)_W X, Y\rangle - \langle (\nabla_W S)_Z X, Y\rangle + \langle S_Z X, S_W Y\rangle - \langle S_X Z, S_Y W\rangle + \langle S_Y Z, S_X W\rangle - \langle S_W X, S_Z Y\rangle. \tag{2.6}$$

According to [3, p. 99], $T(U,V)$ has values in the vertical space TF, for any vectors U, V. Thus, using the definition of S and T,

$$\langle S_Z X - S_X Z, S_Y W\rangle = \langle T(X,Z), S_Y W\rangle = 0,$$

$$\langle S_X W - S_W X, S_Y Z\rangle = \langle T(W,X), S_Y Z\rangle = 0.$$

Proposition 2.3 follows by substitution into formula (2.6). □

We now restrict the values of Y. It is easy to deduce the following.

PROPOSITION 2.7.
(i) *If* $X \in TB$ *and* $Y, W, Z \in TF$, *then*

$$\langle R(X,Y)Z, W\rangle = \langle (\nabla_Z S)_W X, Y\rangle - \langle (\nabla_W S)_Z X, Y\rangle$$

(ii) *If* $X, Y \in TB$ *and* $W, Z \in TF$, *then*

$$\begin{aligned}\langle R(X,Y)Z, W\rangle &= \langle (\nabla_Z S)_W X, Y\rangle - \langle (\nabla_W S)_Z X, Y\rangle \\ &\quad + \langle P_F S_Z X, P_F S_W Y\rangle - \langle P_F S_W X, P_F S_Z Y\rangle.\end{aligned}$$

Proof. (i) If $Y, W, Z \in TF$, one has $S_W Y - S_Y W = T(Y, W) = 0$ and $S_Y Z - S_Z Y = T(Z, Y) = 0$. The last two terms in Proposition 2.3 vanish.

(ii) If $Y \in TB$ and $W, Z \in TF$, then $S_W Y - S_Y W = T(Y, W)$ lies in TF. However, $S_Y W$ is horizontal and therefore $S_W Y - S_Y W = P_F S_W Y$. Similarly, $S_Z Y - S_Y Z = P_F S_Z Y$. The result follows from Proposition 2.3.

It will be necessary to rewrite Proposition 2.3 in classical tensor notation. Let f_α be an orthonormal frame field for TB and e_i an orthonormal frame field for TF. In general, Greek indices α, β will refer to the base and Latin indices i, j, k will refer to the fiber. In the frame field f_α, e_i for TM, one has

$$\begin{aligned} R_{\alpha ijk} &= S_{ji\alpha,k} - S_{ki\alpha,j} \\ R_{\alpha\beta ij} &= S_{i\beta\alpha,j} - S_{j\beta\alpha,i} + S_{ik\alpha}S_{jk\beta} - S_{jk\alpha}S_{ik\beta}. \end{aligned} \tag{2.8}$$

The equations (2.8) follow from Proposition 2.7 and the fact that S is skew symmetric in its second and third indices.

Apparently, curvature identities similar to those above are implicit in [3]. It seems worthwhile to have a clear statement and proof of these basic facts.

3. Asymptotic expansion. Suppose that the bundle TF along the fibers is spin. One has the corresponding spin bundles $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$. The exterior algebra $\Lambda(B)$ lifts from B to form a bundle over M. The connection ∇ induces connections on $\mathcal{S}$, $\Lambda(B)$, and the twisted tensor product $\mathcal{S} \otimes \Lambda(B)$. Assume that $\xi \to M$ is a Hermitian vector bundle with unitary connection A. The curvature of A will be denoted by L. Naturally, one has an induced connection on sections of $\mathcal{S} \otimes \xi \otimes \Lambda(B)$.

Let $n = 2l$ be the dimension of the fibers F. The Clifford algebra along the fibers is generated by $e_1, e_2, \ldots, e_n$. We think of $\mathrm{Cliff}(n)$ as an exterior algebra $\Lambda(n)$ with a new multiplication $\circ$. The Clifford multiplication is defined by $v \circ a = v \wedge a + v \lrcorner a$ for $v \in R^n$ and $a \in \Lambda(n)$. Here $\wedge$ is exterior and $\lrcorner$ is interior multiplication.

In [3], Bismut introduced a remarkable superconnection. The curvature I of Bismut's superconnection is a second-order differential operator on sections of $\mathcal{S} \otimes \xi \otimes \Lambda(B)$ over F. Let I_t be the corresponding operator when the metric on TF is scaled by the factor t^{-1}. In local coordinates on the fiber, one has

$$
\begin{aligned}
I_t = &-tg^{ij}\left(\partial_i + \frac{1}{4}\Gamma_{iab}e_a\wedge e_b\circ + A_i + \frac{1}{2\sqrt{t}}S_{il\alpha}e_l\circ f_\alpha\wedge + \frac{1}{4t}S_{i\beta\gamma}f_\beta\wedge f_\gamma\wedge\right)\\
&\times\left(\partial_j + \frac{1}{4}\Gamma_{jab}e_a\wedge e_b\circ + A_j + \frac{1}{2\sqrt{t}}S_{jl\alpha}e_l\circ f_\alpha\wedge + \frac{1}{4t}S_{j\beta\gamma}f_\beta\wedge f_\gamma\wedge\right)\\
&+tg^{ij}\Gamma^k_{ij}\left(\partial_k + \frac{1}{4}\Gamma_{kab}e_a\wedge e_b\circ + A_k + \frac{1}{2\sqrt{t}}S_{kl\alpha}e_l\circ f_\alpha\wedge + \frac{1}{4t}S_{k\beta\gamma}f_\beta\wedge f_\gamma\wedge\right)\\
&+\frac{1}{4}tK - \frac{1}{2}te_i\wedge e_j\circ L_{ij} - \frac{1}{2}f_\alpha\wedge f_\beta\wedge L_{\alpha\beta} - \sqrt{t}e_i\circ f_\alpha\wedge L_{i\alpha}.
\end{aligned}
\tag{3.1}
$$

Here $\partial_i = \partial/\partial x_i$, Γ are the Christoffel symbols, and K is the scalar curvature of the fiber.

Consider the heat equation problem on sections of $\mathcal{S}\otimes\xi\otimes\Lambda(B)$ over F:

$$\left(\frac{\partial}{\partial s} + I_t\right)g(x,s) = 0,$$
$$g(x,0) = g(x).$$

The theory of parabolic equations provides a fundamental solution, which is smooth for $s>0$: $\exp(-sI_t)(x,y)$. Our concern is with the value $s=1$.

The first issue is to establish existence of an asymptotic expansion for

$$\exp(-I_t)(x,x) \quad \text{as } t\downarrow 0.$$

Some care is required because of the singularities in the coefficients of I_t. Here, the exterior algebra $\Lambda(B)$ plays a crucial role. One has the following.

PROPOSITION 3.2. *For some positive integer $p\geq n$,*

$$\exp(-I_t)(x,x) \sim t^{-p/2}\sum_{i=0}^{\infty} t^i E_i(x,x)$$

for endomorphisms E_i of the fiber $\mathcal{S}_x\otimes\xi\otimes\Lambda(B)$.

Proof. Let J_t be the operator obtained when the Dirac Laplacian of $\mathcal{S}\otimes\xi$ is extended trivially to $\mathcal{S}\otimes\xi\otimes\Lambda(B)$. The extension is well defined since ∇ annihilates horizontal lifts from B. In local coordinates, one has

$$
\begin{aligned}
J_t = &-tg^{ij}(\partial_i + \tfrac{1}{4}\Gamma_{iab}e_a\wedge e_b\circ + A_i)(\partial_j + \tfrac{1}{4}\Gamma_{jab}e_a\wedge e_b\circ + A_j)\\
&+tg^{ij}\Gamma^k_{ij}(\partial_k + \tfrac{1}{4}\Gamma_{kab}e_a\wedge e_b\circ + A_k) + \tfrac{1}{4}tK - \tfrac{1}{2}te_i\wedge e_j\circ L_{ij}.
\end{aligned}
$$

Since J_t has no singular terms in t, the heat kernel $\exp(-J_t)(x,x)$ has a well known asymptotic expansion [6] with $p=n$.

We now construct $\exp(-I_t)$ as a perturbation of $\exp(-J_t)$. Duhamel's principal gives:

$$\exp(-I_t) - \exp(-J_t) = \sum_{k=1}^{\infty}\exp(-J_t)(I_t - J_t)\exp(-J_t)\cdots(I_t - J_t)\exp(-J_t). \tag{3.3}$$

The kth term on the right is the k-fold iterate of the usual convolution integral over $(0,1)\times F$. The key point is that each term in $I_t - J_t$ contains at least one exterior variable f_α. So there are only finitely many nonzero terms on the right-hand side of (3.3). Proposition 3.2 now follows from the standard expansion for $\exp(-J_t)(x,y)$ and elementary methods.

The operator I_t commutes with the right action of the exterior algebra $\Lambda(B)$. Therefore, we have $\exp(-I_t)(x,x) \in \mathrm{Hom}(\mathcal{S}_X \otimes \xi_X) \otimes \Lambda(B)$. Taking the supertrace, associated to the splitting $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$, yields $\mathrm{Tr}_s(\exp(-I_t)(x,x))$ in $\Lambda(B)$. According to Proposition 3.2, $\mathrm{Tr}_s(\exp(-I_t)(x,x))$ has an asymptotic expansion as $t \downarrow 0$. Our primary goal is to show that the singular terms vanish and to compute the constant term.

Let Ω be the curvature form of the bundle along the fibers TF. In the orthonormal frame field e_i, f_α, we may write:

$$\Omega_{ij} = \tfrac{1}{2} R_{abij}\, e_a \wedge e_b + R_{\alpha kij}\, f_\alpha \wedge e_k + \tfrac{1}{2} R_{\alpha\beta ij}\, f_\alpha \wedge f_\beta,$$
$$L = \tfrac{1}{2} L_{ab} e_a \wedge e_b + L_{\alpha k}\, f_\alpha \wedge e_k + \tfrac{1}{2} L_{\alpha\beta}\, f_\alpha \wedge f_\beta.$$

In the notation of [5], one has the differential form $\hat{A}(\Omega)\,\mathrm{ch}(L) \in \Lambda(M)$. Now $\Lambda(M) = \Lambda(B) \wedge \Lambda(F)$. Let $[\hat{A}(\Omega)\,\mathrm{ch}(L)]_n \in \Lambda(B)$ denote the coefficient of the volume form $e_1 \wedge e_2 \wedge \cdots \wedge e_n$ in the $\Lambda(B) \wedge \Lambda^n(F)$ component. □

The remainder of this paper is devoted to the proof of the following.

THEOREM 3.4. *For each point $p \in F$,*

$$\lim_{t\to 0} \mathrm{Tr}_s(\exp(-I_t)(p,p)) = (2\pi i)^{-n/2} [\hat{A}(\Omega)\,\mathrm{ch}(L)]_n(p).$$

Different proofs were given earlier by Bismut [3] and Berline and Vergne [2]. As explained in these works, Theorem 3.4 represents a strong local form of the index theorem for families of elliptic operators.

4. Local index for families. We proceed to prove Theorem 3.4. The first step is to transplant the problem from F to R^n. Using the exponential map at our base point $p \in F$, we construct an operator $\bar{I}_t$ which agrees with the normal coordinate expression for I_t near the origin. One may assume that $\bar{I}_t$ coincides with the corresponding Euclidean operator, with $S = 0$, outside a compact set. The analogous statements for the auxiliary bundle ξ hold by parallel translation along radial geodesics. We identify $p \in F$ with the origin 0 in R^n.

Consider the heat equation problem over R^n, for fixed t,

$$\left(\frac{\partial}{\partial s} + \bar{I}_t\right) g(x,s) = 0,$$
$$g(x,0) = g(x).$$

PROPOSITION 4.1. *There exists a unique fundamental solution* $\exp(-s\bar{I}_t)(x,y)$, *which satisfies the decay estimate*

$$|\exp(-s\bar{I}_t)(x,y)| \leq C_1 s^{-n/2} \exp(-C_2|x-y|^2/s)$$

along with analogous estimates for the derivatives in x, y, and s.

Proof. This follows from the construction in [4] of the heat kernel on noncompact manifolds of bounded geometry.

By applying Duhamel's principle in a sufficiently small normal coordinate neighborhood, centered at p, we find that

$$\exp(-\bar{I}_t)(0,0) = \exp(-I_t)(p,p) + O(e^{-C_3/t})$$

Therefore, it suffices to investigate the limit as $t \downarrow 0$ of $\mathrm{Tr}_s(\exp(-\bar{I}_t)(0,0))$.

The key idea of [5] is to use the rescaling $x \to \epsilon x$, $t \to \epsilon^2 t$, $e_i \to \epsilon^{-1} e_i$. The resulting operator has the coordinate expression:

$$\begin{aligned}
\bar{I}_\epsilon = -tg^{ij}(\epsilon x)\Big(\partial_i + \frac{\epsilon^{-1}}{4}\Gamma_{iab}(\epsilon x)e_a \wedge e_b \circ_\epsilon + \epsilon A_i(\epsilon x) \\
+ \frac{\epsilon^{-1}}{2\sqrt{t}} S_{il\alpha}(\epsilon x) e_l \circ_\epsilon f_\alpha \wedge + \frac{\epsilon^{-1}}{4t} S_{i\beta\gamma}(\epsilon x) f_\beta \wedge f_\gamma \wedge\Big) \qquad (4.2) \\
\times \Big(\partial_j + \frac{\epsilon^{-1}}{4}\Gamma_{jab}(\epsilon x)e_a \wedge e_b \circ_\epsilon + \epsilon A_j(\epsilon x) \\
+ \frac{\epsilon^{-1}}{2\sqrt{t}} S_{jl\alpha}(\epsilon x) e_l \circ_\epsilon f_\alpha \wedge + \frac{\epsilon^{-1}}{4t} S_{j\beta\gamma}(\epsilon x) f_\beta \wedge f_\gamma \wedge\Big) \\
+ tg^{ij}(\epsilon x)\Gamma_{ij}^k(\epsilon x)\Big(\epsilon\partial_k + \frac{1}{4}\Gamma_{kab}(\epsilon x)e_a \wedge e_b \circ_\epsilon + \epsilon^2 A_k(\epsilon x) \\
+ \frac{1}{2\sqrt{t}} S_{kl\alpha}(\epsilon x) e_l \circ_\epsilon f_\alpha \wedge + \frac{1}{4t} S_{k\beta\gamma}(\epsilon x) f_\beta \wedge f_\gamma \wedge\Big) \\
+ \frac{\epsilon^2}{4} tK(\epsilon x) - \frac{t}{2} e_i \wedge e_j \circ_\epsilon L_{ij}(\epsilon x) - \frac{1}{2} f_\alpha \wedge f_\beta \wedge L_{\alpha\beta}(\epsilon x) - \sqrt{t}\, e_i \circ_\epsilon f_\alpha \wedge L_{i\alpha}(\epsilon x).
\end{aligned}$$

The asymptotic expansion in t of Proposition 3.2 for $\exp(-\bar{I}_t)(0,0)$ yields an expansion in ϵ for $\exp(-\bar{I}_\epsilon)(0,0)$. Moreover, one has

$$\lim_{t \to 0} \mathrm{Tr}_s \exp(-\bar{I}_t)(0,0) = \lim_{\epsilon \to 0} \mathrm{Tr}_s \exp(-\bar{I}_\epsilon)(0,0).$$

This means that if either limit exists, then both limits exist and are equal. We will deal directly with the limit in ϵ.

Unfortunately, the tensors $S_{il\alpha}$ and $S_{i\beta\gamma}$ need not vanish at the origin. This means that the coefficients of (4.2) are singular as $\epsilon \downarrow 0$. Define

$$h(x,\epsilon,t) = \exp\Big(\frac{\epsilon^{-1}}{2\sqrt{t}} S_{il\alpha}(0) x_i e_l \wedge f_\alpha + \frac{\epsilon^{-1}}{4t} S_{i\beta\gamma}(0) x_i f_\beta \wedge f_\gamma\Big)$$

Since one is dealing with exterior variables, h is actually of polynomial growth in x.

Fix ϵ and conjugate $\bar{I}_\epsilon$ by h to obtain

$$
\begin{aligned}
J_\epsilon &= h\bar{I}_\epsilon h^{-1} \\
&= -tg^{ij}(\epsilon x)\Big(\partial_i + \frac{\epsilon^{-1}}{4}\Gamma_{iab}(\epsilon x)e_a\wedge e_b + \frac{\epsilon^{-1}}{2\sqrt{t}}(S_{il\alpha}(\epsilon x) - S_{il\alpha}(0))e_l\wedge f_\alpha \\
&\qquad + \frac{\epsilon^{-1}}{4t}(S_{i\beta\gamma}(\epsilon x) - S_{i\beta\gamma}(0))f_\beta\wedge f_\gamma - \frac{1}{4t}S_{il\alpha}(0)S_{kl\beta}(0)x_k f_\alpha\wedge f_\beta\Big) \\
&\quad \times\Big(\partial_j + \frac{\epsilon^{-1}}{4}\Gamma_{jab}(\epsilon x)e_a\wedge e_b + \frac{\epsilon^{-1}}{2\sqrt{t}}(S_{jl\alpha}(\epsilon x) - S_{jl\alpha}(0))e_l\wedge f_\alpha \\
&\qquad + \frac{\epsilon^{-1}}{4t}(S_{j\beta\gamma}(\epsilon x) - S_{j\beta\gamma}(0))f_\beta\wedge f_\gamma - \frac{1}{4t}S_{jl\alpha}(0)S_{kl\beta}(0)x_k f_\alpha\wedge f_\beta\Big) \\
&\quad - \frac{1}{2}te_i\wedge e_j L_{ij}(\epsilon x) - \frac{1}{2}f_\alpha\wedge f_\beta L_{\alpha\beta}(\epsilon x) - \sqrt{t}e_i\wedge f_\alpha L_{i\alpha}(\epsilon x) + r(x,\epsilon).
\end{aligned} \tag{4.3}
$$

The quadratic terms in S arise from applying $\epsilon^{-1}S_{il\alpha}(0)\circ_\epsilon$ to the exponential, since one has $\circ_\epsilon = \wedge + \epsilon^2 \lrcorner$. The symbol $r(x,\epsilon)$ denotes terms which vanish when $\epsilon\to 0$ and will not contribute in the final analysis.

A fundamental solution of the heat equation problem for J_ϵ may be obtained by conjugation:

$$\exp(-sJ_\epsilon)(x,y) = h(x)\exp(-s\bar{I}_\epsilon)(x,y)h^{-1}(y). \tag{4.4}$$

The right-hand side clearly satisfies the heat equation $(\partial/\partial s + J_\epsilon)g(x,s) = 0$ and approaches $\delta_{x,y}$ as $s\to 0$. Since $h(0)=1$ we have, for each fixed ϵ,

$$\mathrm{Tr}_s\exp(-\bar{I}_\epsilon)(0,0) = \mathrm{Tr}_s\exp(-J_\epsilon)(0,0). \tag{4.5}$$

Thus one need only evaluate the limit of the right-hand side as $\epsilon\to 0$.

Our choice of conjugation yields an operator J_ϵ whose coefficients are nonsingular for small ϵ. One has the estimates $\Gamma_{iab}(\epsilon x) = -\frac{1}{2}R_{ijab}(0)\epsilon x_j + r(x,\epsilon^2)$, $S_{il\alpha}(\epsilon x) = S_{il\alpha}(0) + S_{il\alpha,j}\epsilon x_j + r(x,\epsilon^2)$, $S_{i\beta\gamma}(\epsilon x) = S_{i\beta\gamma}(0) + S_{i\beta\gamma,j}(0)\epsilon x_j + r(x,\epsilon^2)$. Fixing x, we obtain

$$J_0 = \lim_{\epsilon\to 0} J_\epsilon = -t\sum_i\left(\partial_i - \frac{1}{4}B_{ij}x_j\right)^2 - t\mathcal{L},$$

where

$$
\begin{aligned}
B_{ij} &= \frac{1}{2}R_{ijab}(0)e_a\wedge e_b - \frac{2}{\sqrt{t}}S_{il\alpha,j}(0)e_l\wedge f_\alpha - \frac{1}{t}[S_{i\beta\gamma,j}(0) - S_{il\beta}(0)S_{jl\gamma}(0)]f_\beta\wedge f_\gamma, \\
\mathcal{L} &= \frac{1}{2}L_{ij}(0)e_i\wedge e_j + \frac{1}{\sqrt{t}}L_{i\alpha}(0)e_i\wedge f_\alpha + \frac{1}{2t}L_{\alpha\beta}(0)f_\alpha\wedge f_\beta.
\end{aligned} \tag{4.6}
$$

There is a fundamental solution $\exp(-sJ_0)(x,y)$ which may be given in closed form. Separate $B_{ij} = C_{ij} + D_{ij}$ into its symmetric and skew symmetric parts. Of course, $2C_{ij}=B_{ij}+B_{ji}$ and $2D_{ij}=B_{ij}-B_{ji}$. Here B, C, D are matrices of differential two-forms. We may state the following.

PROPOSITION 4.7.

$$\exp(-sJ_0)(x,0)=(4\pi st)^{-n/2}\hat{A}(stD)\exp(C_{ij}x_ix_j/8)$$
$$\times\exp\left[st\mathcal{L}-\frac{1}{4st}\left(\frac{stD/2}{\tanh stD/2}\right)_{ij}x_ix_j\right].$$

Proof. If B_{ij} were skew symmetric, the method of [5] could be applied directly. One reduces to the skew symmetric case by conjugation. Set

$$h(x)=\exp(C_{ij}x_ix_j/8).$$

Since C_{ij} is a differential form, h has at most polynomial growth. A calculation using $C_{ij}=C_{ji}$ gives

$$K_0=h^{-1}J_0h=-t\sum\left(\partial_i-\frac{1}{4}D_{ij}x_j\right)^2-t\mathcal{L}.$$

Since $D_{ij}=-D_{ji}$, the argument of [5] may be transcribed verbatim. This gives

$$\exp(-sK_0)(x,0)=(4\pi st)^{-n/2}\hat{A}(stD)\exp\left[st\mathcal{L}-\frac{1}{4st}\left(\frac{stD/2}{\tanh stD/2}\right)_{ij}x_ix_j\right].$$

Set $\exp(-sJ_0)(x,0)=h(x)\exp(-sK_0)(x,0)$. Clearly $\exp(-sJ_0)(x,0)$ satisfies the heat equation for K_0 and approaches $\delta_{x,0}$ as $s\to 0$. This proves Proposition 4.7. □

We now return to our primary goal for computing the limit in (4.5) as $\epsilon\to 0$. As in [5], one may apply Duhamel's principle and the explicit expression for $\exp(-sJ_0)(x,0)$ to deduce:

$$\lim_{\epsilon\to 0}\exp(-J_\epsilon)(0,0)=\exp(-J_0)(0,0).$$

Using Proposition 4.7 to compute the supertrace of the right-hand side gives

$$\lim_{\epsilon\to 0}\mathrm{Tr}_s\exp(-J_\epsilon)(0,0)=\left(\frac{2}{i}\right)^{n/2}(4\pi t)^{-n/2}[\hat{A}(tD)\,\mathrm{ch}(t\mathcal{L})]_n.$$

According to the explanation in [5], the factor $(2/i)^{n/2}$ arises in computing the supertrace. The n-form component is taken since Clifford multiplication approaches exterior multiplication when $\epsilon\to 0$.

Recall that D is the skew symmetric part of B, given in (4.6). The curvature identities (2.8) show that we may write

$$D_{ij}=\frac{1}{2}R_{abij}(0)e_a\wedge e_b+\frac{1}{\sqrt{t}}R_{\alpha kij}(0)f_\alpha\wedge e_k+\frac{1}{2t}R_{\alpha\beta ij}(0)f_\alpha\wedge f_\beta,$$

$$\mathcal{L}=\frac{1}{2}L_{ab}(0)e_a\wedge e_b+\frac{1}{\sqrt{t}}L_{\alpha i}(0)f_\alpha\wedge e_i+\frac{1}{2t}L_{\alpha\beta}(0)f_\alpha\wedge f_\beta.$$

Theorem 3.4 follows by elementary algebra. □

REFERENCES

1. M. F. Atiyah and I. M. Singer, *The index of elliptic operators* IV, Ann. of Math. (2) 93 (1971), 119–138.
2. N. Berline and M. Vergne, *A proof of Bismut local index theorem for a family of Dirac operators,* Topology 26 (1987), 435–464.
3. J. M. Bismut, *The Atiyah–Singer index theorem for families of Dirac operators: two heat equation proofs,* Invent. Math. 83 (1986), 91–151.
4. H. Donnelly, *Asymptotic expansions for the compact quotients of properly discontinuous group actions,* Illinois J. Math. 23 (1979), 485–496.
5. E. Getzler, *A short proof of the local Atiyah–Singer index theorem,* Topology 25 (1986), 111–117.
6. H. P. McKean and I. M. Singer, *Curvature and the eigenvalues of the Laplacian,* J. Differential Geom. 1 (1967), 43–69.
7. D. Quillen, *Superconnections and the Chern character,* Topology 24 (1985), 89–95.

Department of Mathematics
Purdue University
West Lafayette, IN 47907