NOTE ON A PAPER OF P. PHILIPPON

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In Proposition 3.3 of [6], Philippon has sharpened previous results of Masser and Wüstholz on the degrees of the isolated components of ideals generated by polynomials of known degree over a field of characteristic 0. Here we propose to shorten and, we hope, render the scheme of that proof more transparent by (a) returning to the systematic use of localization (as in [2], [3], [4]) and (b) by defining a convolution to state the multihomogeneous Bezout theorem and to evaluate the highest homogeneous term of the Hilbert polynomial. The first half of Philippon's proposition would also fit neatly into our framework, but a concise version was given in [1] already.

We need some notation. A multihomogeneous ideal in $R=k[\mathbf{x}_1,\dots,\mathbf{x}_p]$ is an ideal I generated by polynomials which are simultaneously homogeneous in each of the p sets of variables $\mathbf{x}_i=(x_{i0},\dots,x_{iN_i})$ separately. If I is also prime and no $\mathbf{x}_i\subset I$, then I is called *relevant*. For $d\geq 0$, let

$$\mathfrak{N}(d)=\{\mathbf{j}=(j_1,\dots,j_p)\in\mathbf{Z}^p_{\geq 0}: j_1+\cdots+j_p=d\}.$$

For $\mathbf{D}=(D_1,\dots,D_p)$ and $\boldsymbol{\delta}=(\delta_{\mathbf{j}})_{\mathbf{j}\in\mathfrak{N}(d)}$, define the convolution $\boldsymbol{\pi}=\boldsymbol{\delta}*\mathbf{D}$ by the formula $\pi_{\mathbf{j}}=\sum_{i=1}^p D_i\delta_{\mathbf{j}+\mathbf{e}_i}$ for each $\mathbf{j}\in\mathfrak{N}(d-1)$, where $\mathbf{e}_1,\dots,\mathbf{e}_p$ are the standard basis of $\mathbf{Z}^p$. We note that if $\dim I=d\geq 0$, then I has a degree $\boldsymbol{\delta}(I)$ with components $\delta_{\mathbf{j}}(I)\in\mathbf{Z}_{\geq 0}$ for every $\mathbf{j}\in\mathfrak{N}(d)$ and some $\delta_{\mathbf{j}}(I)$ positive. Moreover when $d\geq 1$, the multihomogeneous Bezout theorem (see, e.g., Lemma A5 of [4]) states that if P is multihomogeneous of degree $\mathbf{D}$ and P lies in no associated prime of I, then (I,P) is multihomogeneous of dimension $d-1$ and its degree can be expressed in our notation as $\boldsymbol{\delta}(I,P)=\boldsymbol{\delta}(I)*\mathbf{D}$.

For a fixed multihomogeneous ideal $\mathfrak{A}$, Philippon works in the open set U of the maximal spectrum M consisting of those maximal (relevant) ideals, $\mathfrak{M}\in M$, such that $\mathfrak{M}\not\supset\mathfrak{A}$. We say that a multihomogeneous ideal J is *U-perfect* if for every $\mathfrak{M}\in U$, the ring $R_{\mathfrak{M}}/JR_{\mathfrak{M}}$ is Cohen–Macaulay. Then, if $\mathfrak{P}$ is a multihomogeneous prime ideal in such an $\mathfrak{M}$, $R_{\mathfrak{P}}/JR_{\mathfrak{P}}$ is also Cohen–Macaulay. For any multihomogeneous ideal I, a primary component contained in an $\mathfrak{M}\in M$ will be called a primary *U-component* of I, and similarly for associated prime components. Philippon introduced a function $S_UH(I;\mathbf{D})$, which can be written in our notation as

$$S_UH(I;\mathbf{D})=\sum_{k=0}^{d}\delta_k(I_{(k)})*\underbrace{\mathbf{D}*\cdots*\mathbf{D}}_{k\text{ times}}=\sum_{k=0}^{d}\boldsymbol{\delta}(I_{(k)})*\mathbf{D}^k,$$

where $I_{(k)}$ is the intersection of all isolated U-components of I of dimension k. The *U-degree* (*U-dimension*) of an ideal is the degree (dimension) of the inter-

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section of its U-components. We say that I is *U-unmixed* if all its U-components have the same dimension. The first-time reader is invited to ignore all U prefixes, that is, to restrict his attention to the usual global case $U = M$.

From now on, let J be U-perfect and let $I = (J, P_1, \ldots, P_m)$ with $P_1, \ldots, P_m$ multihomogeneous of multidegree at most $\mathbf{D}$; that is, all $\deg_{\mathbf{x}_i} P_j \leq D_i$.

PROPOSITION. *If J is U-unmixed of U-dimension d, then*

$$\sum_{k=0}^{d} \delta(I_{(k)}) * \mathbf{D}^k \leq \delta(J) * \mathbf{D}^d.$$

Proof. Let $d_0 \geq 0$ be the minimal dimension of isolated U-components of I, that is, $d_0 = \min\{k : \delta(I_{(k)}) \neq \mathbf{0}\}$. For $i = d, \ldots, d_0$ let $M_i = R \setminus \bigcup \mathfrak{P}$, where $\mathfrak{P}$ runs through C_i, the set of all isolated prime U-components of I of dimension at most i. Our aim is to construct a sequence of multihomogeneous polynomials $Q_{d-1}, \ldots, Q_{d_0}$ of multihomogeneous degree $\mathbf{D}$ which are linear combinations of the P's (possibly times appropriate monomials) such that

(a) for each $A_i = (J, Q_{d-1}, \ldots, Q_i)$ and $\mathfrak{P} \in C_i$, $(A_i)_{\mathfrak{P}}$ is of the principal class in the localization $R_{\mathfrak{P}}$ and therefore $(A_i)_{M_i}$ and $J_i = (A_i)_{M_i} \cap R$ are unmixed of dimension i; and such that

(b) $$\sum_{k=i+1}^{d} \delta(I_{(k)}) * \mathbf{D}^{k-i} + \delta(J_i) \leq \delta(J) * \mathbf{D}^{d-i},$$

where $\leq$ denotes inequality in *each* component.

Note that when $i = d$ we have equality in (b). Assume now that for a particular i ($d_0 < i \leq d$) we have constructed $Q_{d-1}, \ldots, Q_i$ so that (a) and (b) hold. Then write $J_i = G_i \cap B_i$, where G_i is the intersection of all the (good) primary components of J_i corresponding to isolated prime U-components of I, and B_i is the intersection of all the other primary components of J_i.

We first remark that all the isolated prime U-components of I of dimension i appear associated to a primary component of G_i. For let $\mathfrak{P}$ be such a prime. Then $I_{\mathfrak{P}} \supset (J, Q_{d-1}, \ldots, Q_i)_{\mathfrak{P}} = (A_i)_{\mathfrak{P}}$ and so the $\mathfrak{P}$-primary component $\mathfrak{Q}$ of I contains J_i:

$$\mathfrak{Q} = I_{\mathfrak{P}} \cap R \supset (A_i)_{M_i} \cap R = J_i.$$

Since J_i is of the dimension of $\mathfrak{P}$, $\mathfrak{P}$ is a prime component of J_i, and thus it follows from the preceding inclusion and Lemma A4 of [4] that

$$\delta(I_{(i)}) \leq \delta(G_i).$$

Each prime component of B_i lies properly within an isolated prime U-component of I; so a sufficiently general linear combination Q_{i-1} of the P's (possibly times appropriate monomials to obtain polynomials of degree exactly $\mathbf{D}$) will lie outside all prime components of B_i. Set $A_{i-1} = (J, Q_{d-1}, \ldots, Q_{i-1})$ so that for each isolated U-component $\mathfrak{P} \in C_{i-1}$, $(A_{i-1})_{\mathfrak{P}}$ is of the principal class in $R_{\mathfrak{P}}$ and therefore unmixed of dimension $i-1$. Thus the ideals $(A_{i-1})_{M_{i-1}}$ and J_{i-1} are unmixed of dimension $i-1$, and by Bezout's theorem, since J_{i-1} omits any primary non-U-components of (B_i, Q_{i-1}),

$$\delta(J_{i-1}) \leq \delta(B_i) * \mathbf{D}.$$

Thus by the two preceding displayed lines and the induction hypothesis,

$$\begin{aligned}\sum_{j=i}^{d} \delta(I_{(k)}) * \mathbf{D}^{k-i+1} + \delta(J_{i-1}) &\leq \left(\sum_{k=i+1}^{d} \delta(I_{(k)}) * \mathbf{D}^{k-i} + \delta(G_i) + \delta(B_i)\right) * \mathbf{D} \\ &= \left(\sum_{k=i+1}^{d} \delta(I_{(k)}) * \mathbf{D}^{k-i} + \delta(J_i)\right) * \mathbf{D} \\ &\leq \delta(J) * \mathbf{D}^{d-(i-1)},\end{aligned}$$

which completes the proof of (b).

Now we obtain the theorem by convolving inequality (b) for $i = d_0$ with $\mathbf{D}$ (d_0 times), noting that $\delta(I_{(i)}) \leq \delta(G_i) \leq \delta(J_i)$ for $i = d_0$ in particular, and recalling that $I_{(k)} = (0)$ and therefore $\delta(I_{(k)}) = \mathbf{0}$ for $k < d_0$. □

When $J = (0)$ and $U = M$, we obtain the following global result, where $\delta(I_{(k)})$ denotes the degree of the isolated components of I in dimension k.

COROLLARY 1. *Let $I = (P_1, \dots, P_m)$ in R and $N = N_1 + \cdots + N_p$. Then*

$$\sum_{k=0}^{N} \delta(I_{(k)}) * \mathbf{D}^k \leq \mathbf{D}^N.$$

To obtain the full result of the second part of Proposition 3.3 of [6], we use the following observation.

LEMMA. *All components of J contained in a fixed $\mathfrak{M} \in M$ have the same dimension.*

Proof. According to the Corollary on p. 258 of [5], all prime components of the zero ideal in the Cohen–Macaulay ring $R_{\mathfrak{M}}/JR_{\mathfrak{M}}$ have the same dimension. But then this is the dimension of their counterparts in $R \cap JR_{\mathfrak{M}}$. □

COROLLARY 2 (Proposition 3.3 of [6]). $S_U H(I; \mathbf{D}) \leq S_U H(J; \mathbf{D})$.

Proof. Let J have U-dimension d and for $i = 0, \dots, j \leq d$, let δ_j^i denote the sum of the U-degrees of the isolated U-components of I of dimension i containing some U-component of J of dimension j. Then by our Lemma,

$$\delta(I_{(i)}) = \sum_{j=i}^{d} \delta_j^i.$$

For fixed j, define U_j by replacing $\mathfrak{U}$ in the definition of U with $\mathfrak{U} \cap D_j$, where D_j is the intersection of all isolated primes of J of dimension other than j. Then J is U_j-unmixed and U_j-perfect of U_j-dimension equal to j. By our Proposition,

$$\sum_{i=0}^{j} \delta_j^i * \mathbf{D}^i \leq \delta(J_{(j)}) * \mathbf{D}^j.$$

Thus, on reversing the order of summation, we find that

$$S_U H(I, \mathbf{D}) = \sum_{i=0}^{d} \delta(I_{(i)}) * \mathbf{D}^i = \sum_{j=0}^{d} \sum_{i=0}^{j} \delta_j^i * \mathbf{D}^i$$
$$\leq \sum_{j=0}^{d} \delta(J_{(j)}) * \mathbf{D}^j = S_U H(J; \mathbf{D}),$$

as desired. □

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