LIFTING OF OPERATORS AND PRESCRIBED NUMBERS OF NEGATIVE SQUARES

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1. Introduction. The problem we are interested in can be formulated in full generality as follows.

Let $\mathcal{H}_i$ be Hilbert spaces and let $J_i \in \mathcal{L}(\mathcal{H}_i)$ be symmetries ($J \in \mathcal{L}(\mathcal{H})$ is a symmetry if $J = J^* = J^{-1}$), $i = 1, 2$. Consider also $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that the number of negative squares of the self-adjoint operator $J_1 - T^*J_2T$ is a given cardinal κ—see Section 2 for the terminology. (We denote this situation by writing $\kappa^-(J_1 - T^*J_2T) = \kappa$.) For other Hilbert spaces $\mathcal{H}_i'$ and symmetries $J_i' \in \mathcal{L}(\mathcal{H}_i')$, consider $\tilde{\mathcal{H}}_i = \mathcal{H}_i \oplus \mathcal{H}_i'$ and $\tilde{J}_i = J_i \oplus J_i'$, $i = 1, 2$. If $\tilde{\kappa}$ is another given cardinal, then

$$(*) \qquad \begin{cases} \text{Give a description of all operators } \tilde{T} \in \mathcal{L}(\tilde{\mathcal{H}}_1, \tilde{\mathcal{H}}_2) \text{ such that} \\ P^{\tilde{\mathcal{H}}_2}_{\mathcal{H}_2} \tilde{T} \mid \mathcal{H}_1 = T, \quad \text{and} \quad \kappa^-(\tilde{J}_1 - \tilde{T}^*\tilde{J}_2\tilde{T}) = \tilde{\kappa}. \end{cases}$$

(For a closed subspace $\mathcal{G}$ of a Hilbert space $\mathcal{H}$, $P^{\mathcal{H}}_{\mathcal{G}}$ stands for the orthogonal projection onto $\mathcal{G}$.)

The "definite" case of Problem (*) (i.e., all the symmetries involved equal the identity and $\kappa = \tilde{\kappa} = 0$, making T and $\tilde{T}$ contractions) is a well-known problem in dilation theory. A full solution of it (which includes the description of the defect spaces of $\tilde{T}$) and its (long) history can be found in [3]. The methods involved proved to be useful for the geometric approach to dilation theory and to some extrapolation problems.

The passing to the "indefinite" case has strong motivations and many efforts have been made along this line both in extrapolation problems and in dilation theory (see, as samples from a very large list, [16], [10], [1], [5], [12], [4], [8]). This makes our Problem (*) quite natural. On the other hand, it is transparent that Problem (*) involves linear operators on indefinite inner product spaces. In this setting the formulations become simpler, and the "invariant" part (i.e., that independent from the chosen symmetries) can be pointed out. Of course, the usual difficulties of the "indefinite" case (e.g., the lack of an adequate substitute for the square root of "positive" operators) will show up.

In this paper we adapt the methods of [3] for giving a solution for Problem (*) in the Pontryagin case and when $\tilde{\kappa}$ has the least admissible value (see Theorem 5.3). Let us note that the existence problem for $\tilde{\kappa}$ bigger than the least admissible value can be easily deduced from there, but the description of all solutions involves (as suggested by Section 2) some new parameters and, on the other hand, some parameters may be unbounded.

For proving the main result, we need several facts which are presented in Sections 2–4. We begin by recalling the necessary Krein space terminology, and with

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a discussion on indefinite factorizations of self-adjoint operators (to replace the factorizations of positive operators in Hilbert space). Note that Corollary 2.6 deals with the best situation which will be used later. In Section 3 we analyze the connections between the ranks of negativity and positivity for $I-T^{\#}T$ and $I-TT^{\#}$, where T is a Krein space operator and $T^{\#}$ is its J-adjoint. These connections will imply some restrictions on κ and $\tilde{\kappa}$. Section 4 contains the main technical point (Proposition 4.1) which provides a substitute for the defect relations from the definite case (i.e., $TD_T=D_{T^*}T$ for a Hilbert space contraction; see [18]). Some "link operators" are constructed and their properties are established. These give the possibility of introducing (Corollary 4.5) the "elementary rotation" in this indefinite case. Our solution (in the above-mentioned case) for Problem (*) is given in Section 5.

Some applications are given in [9]; other applications and connections with previous results will be presented elsewhere.

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2. Indefinite factorizations of self-adjoint operators. We will use without specification some elementary notions and facts from Krein space theory; these can be found in [6] or [11]. Let us recall some of them here, merely for fixing the notations.

A Krein space $\mathcal{K}$ can be thought as a Hilbert space endowed with a supplementary indefinite inner product $[\cdot,\cdot]$ given by

$$[x,y]=(Jx,y),\quad x,y\in\mathcal{K}, \tag{2.1}$$

where $J\in\mathcal{L}(\mathcal{K})$ is a symmetry (called a fundamental symmetry—f.s. for short—of the Krein space $(\mathcal{K},[\cdot,\cdot])$). There are many (equivalent) ways of turning the Krein space $(\mathcal{K},[\cdot,\cdot])$ into a Hilbert space, depending on the chosen f.s. Topological notions on $\mathcal{K}$ are associated to the Hilbert space structure of $\mathcal{K}$; the same is true for the notation $\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$, where $\mathcal{K}_1,\mathcal{K}_2$ are Krein spaces. Positivity, negativity, and neutrality of vectors are referred to as the indefinite inner product being (respectively) ≥ 0, ≤ 0, $=0$. The J-orthogonal of $\mathcal{G}\subset\mathcal{K}$ (denoted $\mathcal{G}^{[\perp]}$) is its orthogonal with respect to $[\cdot,\cdot]$. A subspace $\mathcal{G}\subset\mathcal{K}$ is called nondegenerate if its isotropic part $(=G\cap G^{[\perp]})$ is trivial. If $T\in\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$, its J-adjoint $T^{\#}$ is defined to be $J_1T^*J_2$, where J_i is a f.s. of $\mathcal{K}_i$, $i=1,2$. A (possible unbounded) operator T is J-isometric if $[Tx,Ty]=[x,y]$ for every $x,y\in\mathcal{D}(T)$, the domain of T; a continuous surjective J-isometry is called a J-unitary operator. The operator $T\in\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$ is called J-contractive (more precisely (J_1,J_2)-contractive) if $J_1-T^*J_2T\geq 0$; that is, if $I-T^{\#}T$ is a J_1-positive operator.

If J is any f.s. of the Krein space $\mathcal{K}$, and $J=P_+-P_-$ is its Jordan decomposition, then $\dim P_+(\mathcal{K})(=\kappa^+(\mathcal{K}))$ and $\dim P_-(\mathcal{K})(=\kappa^-(\mathcal{K}))$ are independent of the chosen J. If one of these numbers is finite (we always choose in this case $\kappa^-(\mathcal{K})<\infty$) then $\mathcal{K}$ is called a Pontryagin space.

Let us come to the main topic of this section. Consider $A \in \mathcal{L}(\mathcal{H})$, a self-adjoint operator on the Hilbert space $\mathcal{H}$. Denote by sgn the function signum defined by

$$\left\{\begin{array}{l} \operatorname{sgn}: \mathbf{R} \to \{-1, 0, 1\} \\ \operatorname{sgn}(t) = \begin{cases} -1 & t<0 \\ 0 & t=0 \\ 1 & t>0. \end{cases} \end{array}\right. \tag{2.2}$$

Put $S_A = \operatorname{sgn}(A)$; then S_A is the self-adjoint partial isometry which appears in the polar decomposition of A. Thus:

$$\begin{gathered} \ker(S_A) = \ker(A) \\ S_A(\mathcal{H}) = \overline{\mathcal{R}(A)} \\ A = S_A|A|, \end{gathered} \tag{2.3}$$

where $\mathcal{R}(A)$ stands for the range of A. The signature numbers of A are defined as follows:

$$\begin{gathered} \kappa^-(A) = \dim\ker(I + S_A) \\ \kappa^+(A) = \dim\ker(I - S_A) \\ \kappa^0(A) = \dim\ker S_A. \end{gathered} \tag{2.4}$$

Recall that $\kappa^-(A)$ $(\kappa^+(A))$ is also the number of negative (positive) squares of the quadratic form (Ax, x), $x \in \mathcal{H}$ (see, e.g., [11]). Denote by $\mathcal{H}_A$ the Krein space obtained from $\overline{\mathcal{R}(A)}$ and its symmetry S_A. The fundamental decomposition of $\mathcal{H}_A$ associated with S_A is

$$\mathcal{H}_A = \mathcal{H}_A^+[+]\mathcal{H}_A^-, \tag{2.5}$$

where $\mathcal{H}_A^+ = \ker(I - S_A)$ and $\mathcal{H}_A^- = \ker(I + S_A)$. It follows that

$$\left\{\begin{array}{l} \kappa^+(\mathcal{H}_A) = \kappa^+(A) = \kappa^+(S_A), \\ \kappa^-(\mathcal{H}_A) = \kappa^-(A) = \kappa^-(S_A). \end{array}\right. \tag{2.6}$$

Note that $\mathcal{R}(|A|^{1/2})$ is invariant by S_A and dense in $\mathcal{H}_A$. Denote this space by $\mathcal{R}'(A)$.

Our analysis of indefinite factorizations of A is contained in the next proposition; its first part can be found in [7] (for the matrix case see [16]).

2.1. PROPOSITION. *Let $\mathcal{H}$ be a Hilbert space, $A \in \mathcal{L}(\mathcal{H})$, $A = A^*$, and let $\mathcal{K}$ be a Krein space with a f.s. J. Then*

(i) *There exists $B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that*

$$A = B^*JB (= B^\# B) \tag{2.7}$$

if and only if

$$\left\{\begin{array}{l}\kappa^{+}(\mathcal{K})\geq\kappa^{+}(A),\\ \kappa^{-}(\mathcal{K})\geq\kappa^{-}(A).\end{array}\right. \tag{2.8}$$

(ii) *Suppose* (2.8) *is fulfilled; then* $B\in\mathcal{L}(\mathcal{H},\mathcal{K})$ *verifies* (2.7) *if and only if*

$$B=(C|A|^{1/2}\,|_{\mathcal{H}_A}\quad X) \tag{2.9}$$

(*with respect to* $\mathcal{H}=\mathcal{H}_A\oplus\ker A$), *where* $C\colon \mathcal{R}'(A)(\subset\mathcal{H}_A)\to\mathcal{K}$ *is J-isometric such that* $(C|A|^{1/2}\,|_{\mathcal{H}_A})\in\mathcal{L}(\mathcal{H}_A,\mathcal{K})$, *and* $X\in\mathcal{L}(\ker A,\mathcal{K})$ *such that* $X^{\#}X\ (=X^{*}JX)=0$ *and* $\mathcal{R}(X)\subset(\mathcal{R}(C))^{[\perp]}$.

Proof. (i) Suppose that there exists a $B\in\mathcal{L}(\mathcal{H},\mathcal{K})$ which verifies (2.7). Then

$$0<(Ax,x)=(B^{*}JBx,x)=[Bx,Bx], \tag{2.10}$$

for every $x\in\mathcal{H}_A^{+}\setminus\{0\}$. In particular, $B\,|_{\mathcal{H}_A^{+}}$ is injective and $B(\mathcal{H}_A^{+})$ is a nonnegative subspace of $\mathcal{K}$. Hence $\kappa^{+}(A)=\kappa^{+}(\mathcal{H}_A)\leq\kappa^{+}(\mathcal{K})$. The second relation of (2.8) follows similarly.

Conversely, suppose (2.8). Then it follows immediately that there exists a J-isometry $C\in\mathcal{L}(\mathcal{H}_A,\mathcal{K})$ such that

$$C^{*}JC=S_A\,|_{\mathcal{H}_A}. \tag{2.11}$$

Taking $B=C|A|^{1/2}\in\mathcal{L}(\mathcal{H},\mathcal{K})$, we obtain (using (2.11))

$$B^{*}JB=|A|^{1/2}C^{*}JC|A|^{1/2}=|A|^{1/2}S_A|A|^{1/2}=A, \tag{2.12}$$

so B verifies (2.7).

(ii) If $B\in\mathcal{L}(\mathcal{H},\mathcal{K})$ verifies (2.7), then define the linear mapping

$$\left\{\begin{array}{l}C\colon \mathcal{R}'(A)\to\mathcal{K},\\ C|A|^{1/2}x=Bx,\quad x\in\mathcal{H}_A.\end{array}\right. \tag{2.13}$$

The mapping C is well defined, since $|A|^{1/2}$ is injective on $\overline{\mathcal{R}(A)}$. Define also $X\in\mathcal{L}(\ker A,\mathcal{K})$ by:

$$Xx=Bx,\quad x\in\ker A. \tag{2.14}$$

The relation (2.9) follows from (2.13) and (2.14). Let us verify the properties of C and X. For $x,y\in\mathcal{H}_A$,

$$\begin{aligned}[C|A|^{1/2}x,C|A|^{1/2}y]&=(JBx,By)=(Ax,y)\\ &=(S_A|A|^{1/2}x,|A|^{1/2}y)=[|A|^{1/2}x,|A|^{1/2}y],\end{aligned} \tag{2.15}$$

which proves that C is J-isometric. The fact that $C|A|^{1/2}\,|_{\mathcal{H}_A}\in\mathcal{L}(\mathcal{H}_A,\mathcal{K})$ follows from (2.13). Now, for any $x\in\mathcal{H}_A$ and $y\in\ker A$,

$$[C|A|^{1/2}x,Xy]=(JBx,By)=(Ax,y)=0, \tag{2.16}$$

so $\mathcal{R}(X)\subset(\mathcal{R}(C))^{[\perp]}$. Finally

$$(X^{\#}Xx,y)=[Xx,Xy]=(JBx,By)=(Ax,y)=0 \tag{2.17}$$

for every $x,y\in\ker A$, which shows that $X^{\#}X=0$.

Conversely, suppose B is as in (2.9). Then if $x=x_1+x_2$ and $y=y_1+y_2$, with $x, y \in \mathcal{H}$, $x_1, y_1 \in \mathcal{H}_A$, and $x_2, y_2 \in \ker A$, we have:

$$
\begin{aligned}
(B^*JBx, y) &= (JBx, By) = (JC|A|^{1/2}x_1 + JXx_2, C|A|^{1/2}x_1 + Xx_2) \\
&= [C|A|^{1/2}x_1, C|A|^{1/2}x_1] + [C|A|^{1/2}x_1, Xx_2] \\
&\quad + [Xx_2, C|A|^{1/2}x_1] + [Xx_2, Xx_2] \\
&= [|A|^{1/2}x_1, |A|^{1/2}x_1],
\end{aligned} \tag{2.18}
$$

where we used that $\mathcal{R}(X) \subset (\mathcal{R}(C))^{[\perp]}$, that C is J-isometric, and that $X^\# X = 0$. Thus

$$(B^*JBx, y) = [|A|^{1/2}x_1, |A|^{1/2}x_1] = (S_A|A|^{1/2}x_1, |A|^{1/2}x_1) = (Ax, y), \tag{2.19}$$

which implies that $B^*JB = A$. □

2.2. REMARK. From the preceding construction it follows easily that $\mathcal{R}(X)$ is exactly the isotropic part of $\mathcal{R}(B)$.

In the rest of this section we keep the notations of Proposition 2.1. In the next corollaries we present some supplementary facts on the formula (2.9).

2.3. COROLLARY. *If $\overline{B\mathcal{H}_A}$ is a nondegenerate subspace of $\mathcal{K}$, then the corresponding C is closable.*

Proof. Because $\overline{B\mathcal{H}_A} = \overline{C|A|^{1/2}\mathcal{H}}$ is nondegenerate, it follows that

$$C|A|^{1/2}\mathcal{H} + (C|A|^{1/2}\mathcal{H})^{[\perp]}$$

is dense in $\mathcal{K}$. Now, for $x \in \mathcal{H}$, $y = C|A|^{1/2}h$, $h \in \mathcal{H}$, and $z \in (C|A|^{1/2}\mathcal{H})^{[\perp]}$, we have:

$$[C|A|^{1/2}x, y+z] = [C|A|^{1/2}x, C|A|^{1/2}h] = [|A|^{1/2}x, |A|^{1/2}h], \tag{2.20}$$

so $y+z \in \mathcal{D}(C^\#)$ and $C^\#(y+z) = |A|^{1/2}h$. In particular, $C^\#$ is densely defined; thus C is closable. □

Since the closure of a J-isometric operator is J-isometric, Corollary 2.3 implies that if $\overline{(B\mathcal{H}_A)}$ is nondegenerate then C can be chosen *closed.*

2.4. COROLLARY. *Suppose that $\mathcal{H}_A$ is a Pontryagin space. Then C is bounded if and only if $\overline{B\mathcal{H}_A}$ is a regular subspace of $\mathcal{K}$.*

Proof. We choose $\kappa^-(\mathcal{H}_A) < \infty$. From (2.13) it follows that

$$\mathcal{D}(C) = |A|^{1/2}\mathcal{H}_A = |A|^{1/2}\mathcal{H}_A^+ + |A|^{1/2}\mathcal{H}_A^- = \mathcal{H}_A^- + |A|^{1/2}\mathcal{H}_A^+, \tag{2.21}$$

where $|A|^{1/2}\mathcal{H}_A^+ \subset \mathcal{H}_A^+$ is a uniformly positive subspace of $\mathcal{H}_A$ (see [6, Theorem V.5.6]). From [6, Theorem VI.3.5], the isometry C is bounded if and only if $C\mathcal{H}_A^-$ and $C|A|^{1/2}\mathcal{H}_A^+$ are uniformly definite. Because $C\mathcal{H}_A^-$ is finite dimensional, it follows that C is bounded if and only if $C|A|^{1/2}\mathcal{H}_A^+$ is uniformly positive. Now

$$\overline{B\mathcal{H}_A} = \overline{\mathcal{R}(C)} = C\mathcal{H}_A^- [+] \overline{C|A|^{1/2}\mathcal{H}_A^+}. \tag{2.22}$$

Then $\overline{B\mathcal{H}_A}$ is regular if and only if $\overline{C|A|^{1/2}\mathcal{H}_A^+}$ is uniformly positive (see [6, Theorem V.8.2]). □

2.5. COROLLARY. *Suppose that A has closed range. Then the J-isometry C is bounded.*

Proof. Follows immediately from the fact that $|A|^{1/2}$ is invertible on $\mathcal{H}_A$ and $C|A|^{1/2}\in\mathcal{L}(\mathcal{H}_A,\mathcal{K})$. □

2.6. COROLLARY. *Suppose that $\kappa^-(A)=\kappa^-(\mathcal{K})<\infty$. Then the formula*

$$B=C|A|^{1/2} \tag{2.23}$$

establishes a one-to-one correspondence between all operators $B\in\mathcal{L}(\mathcal{H},\mathcal{K})$ with $B^*JB=A$ and all J-isometric operators $C\in\mathcal{L}(\mathcal{H}_A,\mathcal{K})$.

Proof. Arguing as in (2.22), it follows that $\mathcal{R}(B)$ contains a negative subspace of maximal dimension, so $\mathcal{R}(B)$ is nondegenerate. Then Remark 2.2 implies that, in the representation (2.9) of B, we have $X=0$. Moreover, from [6, Corollary IX.2.3] it follows that $\overline{\mathcal{R}(B)}$ is regular, so C is bounded by Corollary 2.4. □

3. Connections with the J-adjoint. Let $\mathcal{K}$ be a Krein space (with a f.s. J) and $A\in\mathcal{L}(\mathcal{K})$ a J-self-adjoint operator (i.e., $A=A^\#$). Then JA is self-adjoint, and we define the J-signature numbers of A by:

$$\kappa^j[A]=\kappa^j(JA),\quad j=+,-,0. \tag{3.1}$$

Note that $\kappa^j(A)$, $j=+,-,0$, are respectively the rank of positivity, negativity and isotropy of the quadratic form $[Ax,x]$, $x\in\mathcal{K}$ (see [11]).

In this section we give some simple relations between the J-signature numbers of the defect operators.

3.1. PROPOSITION. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two Krein spaces and $T\in\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$. Then:*

$$\kappa^\pm[I-T^\#T]+\kappa^\pm(\mathcal{K}_2)=\kappa^\pm[I-TT^\#]+\kappa^\pm(\mathcal{K}_1); \tag*{(3.2)$_\pm$}$$

$$\kappa^0[I-T^\#T]=\kappa^0[I-TT^\#]. \tag*{(3.2)$_0$}$$

Proof. We use the known trick from the definite case. Take $\mathcal{K}=\mathcal{K}_1[+]\mathcal{K}_2$ and $\hat{T}\in\mathcal{L}(\mathcal{K})$ defined by

$$\hat{T}=\begin{pmatrix} I & T^\# \\ T & I\end{pmatrix}. \tag{3.3}$$

Note that $\hat{T}$ can be factorized in two dual ways:

$$\hat{T}=\begin{pmatrix} I & 0 \\ T & I\end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & I-TT^\#\end{pmatrix}\begin{pmatrix} I & T^\# \\ 0 & I\end{pmatrix} \tag{3.4}$$

and

$$\hat{T}=\begin{pmatrix} I & T^{\#} \\ 0 & I \end{pmatrix}\begin{pmatrix} I-T^{\#}T & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ T & I \end{pmatrix}. \tag{3.5}$$

The formulas (3.2) follow from the computation of $\kappa^{j}[\hat{T}]$, $j=+,-,0$, using successively (3.4) and (3.5). □

3.2. REMARK. If J_i is a f.s. of $\mathcal{K}_i$, $i=1,2$, then the formulas (3.2) become:

$$\kappa^{\pm}(J_1-T^*J_2T)+\kappa^{\pm}(J_2)=\kappa^{\pm}(J_2-TJ_1T^*)+\kappa^{\pm}(J_1) \tag{3.2$'_\pm$}$$

$$\kappa^{0}(J_1-T^*J_2T)=\kappa^{0}(J_2-TJ_1T^*). \tag{3.2$'_0$}$$

3.3. COROLLARY. *Suppose that $\mathcal{K}_1$ and $\mathcal{K}_2$ are Pontryagin spaces and $T\in\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$. Then $\kappa^-(I-T^{\#}T)<\infty$ if and only if $\kappa^-(I-TT^{\#})<\infty$, and in this case*

$$\kappa^-(I-T^{\#}T)-\kappa^-(I-TT^{\#})=\kappa^-(\mathcal{K}_1)-\kappa^-(\mathcal{K}_2). \tag{3.6}$$

3.4. REMARK. In the situation of Corollary 3.3, formula (3.6) can be also written as (see Remark 3.2):

$$\kappa^-(J_1-T^*J_2T)-\kappa^-(J_2-TJ_1T^*)=\kappa^-(J_1)-\kappa^-(J_2). \tag{3.6$'$}$$

For obvious reasons we shall refer to (3.6) or (3.6)′ as the "index formula."

We also record here a well-known fact (see [16]), with the advantage of an elementary proof (see also [8]).

3.5. COROLLARY. *Suppose that $\kappa^-(\mathcal{K}_1)=\kappa^-(\mathcal{K}_2)<\infty$. Then T is a J-contraction (resp. strict J-contraction, uniform J-contraction) if and only if $T^{\#}$ is a J-contraction (resp. strict J-contraction, uniform J-contraction).*

Proof. We only have to note that T being J-contraction means $\kappa^-[I-T^{\#}T]=0$; T being strict J-contraction means $\kappa^-[I-T^{\#}T]=\kappa^0[I-T^{\#}T]=0$; and T being uniform J-contraction means $\kappa^-[I-T^{\#}T]=0$ and $I-T^{\#}T$ is invertible. Then apply Corollary 3.3. □

4. Link operators. Consider $\mathcal{K}_1,\mathcal{K}_2$ two Krein spaces with f.s. (respectively) J_1,J_2. Take $T\in\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$ and define the following operators:

$$J_T=\operatorname{sgn}(J_1-T^*J_2T) \qquad J_{T^*}=\operatorname{sgn}(J_2-TJ_1T^*) \tag{4.1}$$

$$D_T=|J_1-T^*J_2T|^{1/2} \qquad D_{T^*}=|J_2-TJ_1T^*|^{1/2}. \tag{4.2}$$

The following relations follow immediately:

$$J_TD_T=D_TJ_T \qquad J_{T^*}D_{T^*}=D_{T^*}J_T^*; \tag{4.3}$$

$$J_TD_T^2=J_1-T^*J_2T \qquad J_{T^*}D_{T^*}^2=J_2-TJ_1T^*; \tag{4.4}$$

$$\begin{aligned} TJ_1(J_1-T^*J_2T)&=(J_2-TJ_1T^*)J_2T \\ T^*J_2(J_2-TJ_1T^*)&=(J_1-T^*J_2T)J_1T^*. \end{aligned} \tag{4.5}$$

Consider the Krein space $\mathfrak{D}_T$ constructed from $\overline{\mathcal{R}(D_T)}$ with the f.s. J_T; analogously $\mathfrak{D}_{T^*}$ is constructed from $\overline{\mathcal{R}(D_{T^*})}$ with the f.s. J_{T^*}. Note that

$$\kappa^{\pm}(\mathfrak{D}_T)=\kappa^{\pm}(J_T)=\kappa^{\pm}(J_1-T^*J_2T)=\kappa^{\pm}[I-T^{\#}T], \tag{4.6}$$

and similarly for $\mathfrak{D}_{T^*}$. These numbers are invariant to the changing of f.s. J_1 and J_2. The operator D_T (resp. the Krein space $\mathfrak{D}_T$) is called the *defect operator* (resp. the *defect space*) of T. (For the definite case see [18], [10].)

The main technical point of the paper is the next proposition, which provides an "indefinite" analog to the well-known defect relations from the definite case (i.e., $TD_T=D_{T^*}T$).

4.1. PROPOSITION. *With previous notations, there exist uniquely determined operators $L_T\in\mathcal{L}(\mathfrak{D}_T,\mathfrak{D}_{T^*})$ and $L_{T^*}\in\mathcal{L}(\mathfrak{D}_{T^*},\mathfrak{D}_T)$ such that:*

$$D_{T^*}L_T=TJ_1D_T\,|\,\mathfrak{D}_T; \tag{4.7}$$

$$D_TL_{T^*}=T^*J_2D_{T^*}\,|\,\mathfrak{D}_{T^*}. \tag{4.7$_*$}$$

Proof. The proof uses a result due to Krein [13] (see also [15] and [17]) which asserts that a bounded linear operator on a Banach space $\mathfrak{X}$ which is symmetric with respect to a given (definite) continuous inner product on $\mathfrak{X}$ is also bounded with respect to the Hilbert norm on $\mathfrak{X}$ associated to the inner product.

Define on the space $\mathcal{K}_2$ the inner product

$$\langle x,y\rangle=(D_{T^*}^2x,y)\quad\text{for } x,y\in\mathcal{K}_2. \tag{4.8}$$

We will show that the operator $S=J_{T^*}J_2TJ_TJ_1T^*$ is $\langle\cdot,\cdot\rangle$ symmetric. Indeed, using the relations (4.3)–(4.5), we have:

$$\begin{aligned} D_{T^*}^2S&=(J_2-TJ_1T^*)J_2TJ_TJ_1T^*\\ &=TJ_1(J_1-T^*J_2T)J_TJ_1T^*=TJ_1D_T^2J_1T^*\geq 0,\end{aligned} \tag{4.9}$$

and analogously,

$$S^*D_{T^*}^2=TJ_1D_T^2J_1T^*. \tag{4.10}$$

Using the above-mentioned theorem of Krein (and the Schwarz inequality for the inner product $\langle\cdot,\cdot\rangle$), we obtain

$$D_{T^*}^2S\leq\mu D_{T^*}^2 \tag{4.11}$$

for a positive μ. From (4.9) and (4.11) it follows that

$$TJ_1D_T^2J_1T^*\leq\mu D_{T^*}^2, \tag{4.12}$$

which implies the existence of an operator $L_T\in\mathcal{L}(\mathfrak{D}_T,\mathfrak{D}_{T^*})$ verifying (4.7). The uniqueness follows from the injectivity of D_{T^*} on $\mathfrak{D}_{T^*}$.

The statement concerning L_{T^*} follows similarly. □

We refer to L_T and L_{T^*} as the *link operators* associated to T.

4.2. REMARKS. (1) The link operators can be explicitly written when $I-T^{\#}T$ has closed range (equivalently, $I-TT^{\#}$ has closed range—see Proposition 3.1).

Indeed, in this case D_T is invertible on $\mathfrak{D}_T$ and D_{T^*} is invertible on $\mathfrak{D}_{T^*}$. Moreover, (4.5) implies that

$$TJ_1\mathfrak{D}_T \subset \mathfrak{D}_{T^*}, \qquad T^*J_2\mathfrak{D}_{T^*} \subset \mathfrak{D}_T. \tag{4.13}$$

Thus, we infer that

$$L_T = D_{T^*}^{-1}TJ_1D_T\,|\,_{\mathfrak{D}_T}; \qquad L_{T^*} = D_T^{-1}T^*J_2D_{T^*}\,|\,_{\mathfrak{D}_{T^*}}. \tag{4.14}$$

In particular, this can be done when $\mathcal{K}_1$ and $\mathcal{K}_2$ are finite dimensional (see [16] and [5] for applications of this situation).

(2) The link operators are also simple if $TJ_1 = J_2T$. Indeed, in this case, we have

$$L_T = J_2T, \qquad L_{T^*} = J_1T^*; \tag{4.15}$$

the relations (4.7) and $(4.7)_*$ become the classical "defect relations" $D_{T^*}T = TD_T$ and $D_TT^* = T^*D_{T^*}$.

4.3. COROLLARY. *With the above notations:*

$$L_{T^*} = L_T^{\#}. \tag{4.16}$$

Proof. From (4.7) and (4.5) we infer:

$$\begin{aligned} D_{T^*}L_TJ_TD_T &= TJ_1D_TJ_TD_T = TJ_1(J_1 - T^*J_2T) \\ &= (J_2 - TJ_1T^*)J_2T = D_{T^*}J_{T^*}D_{T^*}J_2T. \end{aligned} \tag{4.17}$$

Because D_{T^*} is injective on $\mathfrak{D}_{T^*}$, (4.17) implies that

$$L_TJ_TD_T = J_{T^*}D_{T^*}J_2T, \tag{4.18}$$

which is equivalent with

$$D_TJ_TL_T^*J_{T^*} = T^*J_2D_{T^*}. \tag{4.19}$$

Using the uniqueness of L_{T^*}, we have that

$$L_{T^*} = J_TL_T^*J_{T^*}\,|\,_{\mathfrak{D}_{T^*}}, \tag{4.20}$$

which proves the corollary. □

4.4. COROLLARY. *With the above notations:*

$$(J_T - D_TJ_1D_T)\,|\,_{\mathfrak{D}_T} = L_T^*J_{T^*}L_T \tag{4.21}$$

$$(J_{T^*} - D_{T^*}J_2D_{T^*})\,|\,_{\mathfrak{D}_{T^*}} = L_{T^*}^*J_TL_{T^*}. \tag{4.21_*}$$

Proof. Using Corollary 4.3, the relation (4.21) is equivalent with

$$L_{T^*}L_T = (I - J_TD_TJ_1D_T)\,|\,_{\mathfrak{D}_T}. \tag{4.22}$$

For proving (4.22), take $x, y \in \mathcal{K}_1$ and compute

$$\begin{aligned} (L_{T^*}L_TD_Tx, D_Ty) &= (D_TL_{T^*}L_TD_Tx, y) \\ &= (T^*J_2D_{T^*}L_TD_Tx, y) = (T^*J_2TJ_1D_T^2x, y) \end{aligned} \tag{4.23}$$

and

$$\begin{aligned}((I-J_TD_TJ_1D_T)D_Tx, D_Ty) &= (D_T^2x, y)-(J_TD_T^2J_1D_T^2x, y)\\ &= (D_T^2x, y)-((J_1-T^*J_2T)J_1D_T^2x, y)\\ &= (D_T^2x, y)-(D_T^2x, y)+(T^*J_2TJ_1D_T^2x, y).\end{aligned} \tag{4.24}$$

From (4.23) and (4.24) we have

$$(L_{T^*}L_TD_Tx,\ D_Ty)=((I-J_TD_TJ_1D_T)D_Tx, D_Ty),$$

and (4.22) (and hence (4.21)) follows. The relation $(4.21)_*$ is similar. □

4.5. COROLLARY. *The operator:*

$$\begin{cases} R(T): \mathcal{K}_1[+]\mathfrak{D}_{T^*} \to \mathcal{K}_2[+]\mathfrak{D}_T \\ R(T) = \begin{pmatrix} T & D_{T^*} \\ D_T & -J_TL_{T^*} \end{pmatrix} \end{cases} \tag{4.25}$$

is J-unitary.

Proof. Straightforward computations using previous results. □

The operator $R(T)$ is the indefinite version of the "elementary rotation" associated to a Hilbert space contraction. It is expected (see [9]) that $R(T)$ plays in the indefinite case the same fundamental role as its definite analog plays in several manipulations from dilation theory. This can be seen also from [8] where a particular $R(T)$ (with $J_T \geq 0$) is the basic cell in the analysis of a matrix having κ negative squares—in a perfect analogy to the positive definite case.

5. **Main theorem.** In this section we will analyze Problem (*) for Pontryagin spaces. So, let $\mathcal{K}_1$, $\mathcal{K}_2$ be Pontryagin spaces ($\kappa^-(\mathcal{K}_1)<\infty$, $\kappa^-(\mathcal{K}_2)<\infty$) and $T\in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ with $\kappa^-[I-T^{\#}T]$ equals a given cardinal κ. The formula $(3.2)_-$ implies the necessary restriction:

$$\kappa \geq \kappa^-(\mathcal{K}_1)-\kappa^-(\mathcal{K}_2), \tag{5.1}$$

which will be assumed throughout the rest of the paper. Fix J_i $(i=1,2)$ a f.s. of $\mathcal{K}_i$. Consider also Pontryagin spaces $\mathcal{K}_i'$ with f.s. J_i' $(i=1,2)$, and form $\tilde{\mathcal{K}}_i = \mathcal{K}_i[+]\mathcal{K}_i'$ with f.s. $\tilde{J}_i = J_i + J_i'$ $(i=1,2)$. Problem (*) asks for a description of all operators $\tilde{T}\in \mathcal{L}(\tilde{\mathcal{K}}_1, \tilde{\mathcal{K}}_2)$ such that

$$\tilde{T} = \begin{pmatrix} T & A \\ B & X \end{pmatrix} \tag{5.2}$$

with respect to $\tilde{\mathcal{K}}_1 = \mathcal{K}_1[+]\mathcal{K}_1'$ and $\tilde{\mathcal{K}}_2 = \mathcal{K}_2[+]\mathcal{K}_2'$, and $\kappa^-(I-\tilde{T}^{\#}\tilde{T}) = \tilde{\kappa}$, where $\tilde{\kappa}$ is a given cardinal.

As in [3], we consider first the corresponding problem for rows and columns. In this respect we consider the following problems:

$$(*)_r \qquad \begin{cases} \text{Give a description of all operators } T_r \in \mathcal{L}(\tilde{\mathcal{K}}_1, \mathcal{K}_2) \text{ such that} \\ T_r = (T \quad A) \quad \text{and} \quad \kappa^-[I-T_r^{\#}T_r] = \tilde{\kappa}, \end{cases}$$

$$(*)_c \qquad \begin{cases} \text{Give a description of all operators } T_c \in \mathcal{L}(\mathcal{K}_1, \tilde{\mathcal{K}}_2) \text{ such that} \\ T_c = (T \quad B)^t \quad \text{and} \quad \kappa^-[I - T_c^\# T_c] = \tilde{\kappa}; \end{cases}$$

here t stands for the matrix transpose.

Our solution to Problem $(*)_r$ is contained in the following.

5.1. LEMMA. (i) *If Problem* $(*)_r$ *has solutions then it follows that*

$$\tilde{\kappa} \geq \max\{\kappa, \kappa^-(\tilde{\mathcal{K}}_1) - \kappa^-(\mathcal{K}_2)\}. \tag{5.3}$$

(ii) *The formula*

$$T_r = (T \quad D_{T^*}\Gamma) \tag{5.4}$$

establishes a one-to-one correspondence between the set of all solutions to Problem $(*)_r$ *with* $\tilde{\kappa} = \kappa \geq \kappa^-(\tilde{\mathcal{K}}_2) - \kappa^-(\mathcal{K}_2)$ *and the set of all J-contractions* $\Gamma \in \mathcal{L}(\mathcal{K}_1', \mathfrak{D}_{T^*})$. *Moreover, in the above situation, the operators* $U(T_r)$ *and* $U_*(T_r)$ *defined by:*

$$\begin{cases} U(T_r) : \mathfrak{D}_{T_r} \to \mathfrak{D}_T[+]\mathfrak{D}_\Gamma, \\ U(T_r) D_{T_r} = \begin{pmatrix} D_T & -J_T L_{T^*}\Gamma \\ 0 & D_\Gamma \end{pmatrix}, \end{cases} \tag{5.5}$$

and

$$\begin{cases} U_*(T_r) : \mathfrak{D}_{T_r^*} \to \mathfrak{D}_{\Gamma^*}, \\ U^*(T_r) D_{T_r^*} = D_{\Gamma^*} D_{T^*} \end{cases} \tag{5.6}$$

are J-unitary operators.

Proof. (i) As expected, several manipulations with Proposition 2.1 will be necessary. Let T_r be a solution to Problem $(*)_r$. From (3.6)′ it follows that

$$0 \leq \kappa^-(J_2 - T_r \tilde{J}_1 T_r^*) = \tilde{\kappa} - \kappa^-(\tilde{\mathcal{K}}_1) + \kappa^-(\mathcal{K}_2); \tag{5.7}$$

so $\tilde{\kappa} \geq \kappa^-(\tilde{\mathcal{K}}_1) - \kappa^-(\mathcal{K}_2)$. For proving that $\tilde{\kappa} \geq \kappa$, construct a Krein space $\mathcal{K}$ (with a f.s. S) such that $\kappa^-(\mathcal{K}) = \kappa^-(J_2 - T_2 \tilde{J}_1 T_r^*)$ and $\kappa^+(\mathcal{K}) \geq \kappa^+(J_2 - T_r \tilde{J}_1 T_r^*)$. Proposition 2.1(i) implies that there exists an operator $Y \in \mathcal{L}(\mathcal{K}_2, \mathcal{K})$ such that

$$J_2 - T_r \tilde{J}_1 T_r^* = Y^* S Y. \tag{5.8}$$

Matrix computations show that (5.8) is equivalent with

$$J_2 - T J_1 T^* = A J_1' A^* + Y^* S Y = (A \quad Y^*) \begin{pmatrix} J_1' & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} A^* \\ Y \end{pmatrix}. \tag{5.9}$$

Applying again Proposition 2.1(i), for the operator $J_2 - T J_1 T^*$, we obtain that

$$\kappa^-(J_2 - T J_1 T^*) \leq \kappa^-(\mathcal{K}_1') + \kappa^-(\mathcal{K}). \tag{5.10}$$

Using the index formula we have that (5.10) implies $\tilde{\kappa} \geq \kappa$.

(ii) Suppose $\tilde{\kappa} = \kappa \geq \kappa^-(\tilde{\mathcal{K}}_1) - \kappa^-(\mathcal{K}_2)$ and take T_r as a solution to Problem $(*)_r$. Repeating the construction from (i) we obtain the operator $Y \in \mathcal{L}(\mathcal{K}_2, \mathcal{K})$, which verifies (5.9) and, moreover,

$$\kappa^-(J_2 - T J_1 T^*) = \kappa^-(\mathcal{K}_1') + \kappa^-(\mathcal{K}). \tag{5.11}$$

As $\kappa^-(\mathcal{K}_1') + \kappa^-(\mathcal{K}) = \kappa^-(\mathcal{K}_1'[+]\mathcal{K})$, $\mathcal{K}_1'[+]\mathcal{K}$ being the space acted on by the operator $(A \quad Y^*)$, Corollary 2.6 implies that there exists a unique J-isometric operator $\Lambda \in \mathcal{L}(\mathfrak{D}_{T^*}, \mathcal{K}_1'[+]\mathcal{K})$ such that

$$(A^* \quad Y)^t = \Lambda D_{T^*}. \tag{5.12}$$

Represent Λ as $(\Lambda_1 \quad \Lambda_2)^t$ with respect to $\mathcal{K}_1'[+]\mathcal{K}$; then (5.12) implies

$$A = D_{T^*}\Lambda_1^*, \tag{5.13}$$

while the J-isometric condition on Λ implies

$$J_{T^*} - \Lambda_1^* J_1' \Lambda_1 = \Lambda_2^* S \Lambda_2. \tag{5.14}$$

Take $\Gamma = \Lambda_1^*$; then (5.13) implies (5.4). It remains to show that Γ is a J-contraction. For this, note that

$$\kappa^-(\Lambda_2^* S \Lambda_2) \le \kappa^-(S) = \kappa - \kappa^-(\tilde{\mathcal{K}}_1) + \kappa^-(\mathcal{K}_2). \tag{5.15}$$

Writing the index formula for Γ and using (5.14) and (5.15) it follows that

$$\kappa^-(J_1' - \Gamma^* J_{T^*} \Gamma) = 0, \tag{5.16}$$

which says that Γ is a J-contraction.

The identification of defect spaces is obtained as follows. First note that (using (5.8), (5.9), and (5.4))

$$D_{T_r^*} J_{T_r^*} D_{T_r^*} = J_2 - T_r \tilde{J}_1 T_r^* = D_{T^*}(J_{T^*} - \Gamma J_1' \Gamma^*) D_{T^*} = D_{T^*} D_{\Gamma^*} J_{\Gamma^*} D_{\Gamma^*} D_{T^*}, \tag{5.17}$$

which shows that the operator $U_*(T_r)$ defined by (5.6) is a J-isometry. But $\kappa^-(\mathfrak{D}_{T_r^*}) = \kappa^-(\mathfrak{D}_{\Gamma^*}) < \infty$ and $U_*(T_r)$ has dense domain and dense range; thus by Theorem VI.3.5 of [6] it follows that $U_*(T_r)$ is J-unitary. Then we have:

$$D_{T_r} J_{T_r} D_{T_r} = \begin{pmatrix} J_1 - T^* J_2 T & -T^* J_2 D_{T^*} \Gamma \\ -\Gamma^* D_{T^*} J_2 T & J_1' - \Gamma^* D_{T^*} J_2 D_{T^*} \Gamma \end{pmatrix} \tag{5.18}$$

and

$$\begin{aligned} &\begin{pmatrix} D_T & 0 \\ -\Gamma^* L_{T^*}^* J_T & D_\Gamma \end{pmatrix} \begin{pmatrix} J_T & 0 \\ 0 & I_{\mathfrak{D}_\Gamma} \end{pmatrix} \begin{pmatrix} D_T & -J_T L_{T^*} \Gamma \\ 0 & D_\Gamma \end{pmatrix} \\ &\quad = \begin{pmatrix} J_T D_T^2 & -D_T L_{T^*} \Gamma \\ -\Gamma^* L_{T^*}^* D_T & \Gamma^* L_{T^*}^* J_T L_{T^*} \Gamma + D_\Gamma^2 \end{pmatrix}. \end{aligned} \tag{5.19}$$

Using $(4.21)_*$, we infer:

$$D_\Gamma^2 + \Gamma^* L_{T^*}^* J_T L_{T^*} \Gamma = J_1' - \Gamma^*(J_{T^*} - L_{T^*}^* J_T L_{T^*})\Gamma = J_1' - \Gamma^* D_{T^*} J_2 D_{T^*} \Gamma. \tag{5.20}$$

From (5.18), (5.19), $(4.7)_*$, and (5.20) it follows that $U(T_r)$ defined by (5.5) is J-isometric. The fact that $U(T_r)$ is J-unitary is obtained similarly as for $U_*(T_r)$.

Conversely, taking a J-contraction $\Gamma \in \mathcal{L}(\mathcal{K}_1', \mathfrak{D}_{T^*})$, we have that

$$\kappa^-(J_{T^*} - \Gamma J \Gamma^*) = \kappa - \kappa^-(\tilde{\mathcal{K}}_1) + \kappa^-(\mathcal{K}_2). \tag{5.21}$$

Taking T_r as in (5.4), the relation (5.17) implies that $\kappa^-(J_{T^*}-\Gamma J_1'\Gamma^*)\geq \kappa^-(J_2-T_r\tilde{J}_1T_r^*)$. The index formula and (5.21) implies then that $\tilde{\kappa}\leq\kappa$. The converse inequality being true by (i), we have that $\tilde{\kappa}=\kappa$.

The fact that the correspondence from (5.4) is one-to-one is clear. □

The solution to Problem $(*)_c$ follows using dual arguments:

5.2. LEMMA. (i) *If Problem* $(*)_c$ *has solutions then it follows that*

$$\tilde{\kappa}\geq\kappa-\kappa^-(\tilde{\mathcal{K}}_2)+\kappa^-(\mathcal{K}_2). \tag{5.22}$$

(ii) *The formula*

$$T_c=(T\quad \Gamma D_T)^t \tag{5.23}$$

establishes a one-to-one correspondence between the set of all solutions to Problem $(*)_c$ *with* $\tilde{\kappa}=\kappa-\kappa^-(\tilde{\mathcal{K}}_2)+\kappa^-(\mathcal{K}_2)$, *and the set of all operators* $\Gamma\in\mathcal{L}(\mathfrak{D}_T,\mathcal{K}_2')$ *such that* $\Gamma^\#$ *is a J-contraction. Moreover, in the above situation, the operators* $U(T_c)$ *and* $U_*(T_c)$ *defined by*

$$\begin{cases} U(T_c):\mathfrak{D}_{T_c}\to\mathfrak{D}_\Gamma,\\ U(T_c)D_{T_c}=D_\Gamma D_T\end{cases} \tag{5.24}$$

and

$$\begin{cases} U_*(T_c):\mathfrak{D}_{T_c^*}\to\mathfrak{D}_{T^*}[+]\mathfrak{D}_{\Gamma^*},\\ U_*(T_c)D_{T_c^*}=\begin{pmatrix} D_{T^*} & -J_{T^*}L_T\Gamma^*\\ 0 & D_{\Gamma^*}\end{pmatrix}\end{cases} \tag{5.25}$$

are J-unitary operators. □

The main result of this paper is the following.

5.3. THEOREM. (i) *If Problem* $(*)$ *has solutions then it follows that*

$$\tilde{\kappa}\geq\max\{\kappa-\kappa^-(\tilde{\mathcal{K}}_2)+\kappa^-(\mathcal{K}_2),\kappa^-(\tilde{\mathcal{K}}_1)-\kappa^-(\tilde{\mathcal{K}}_2)\}. \tag{5.26}$$

(ii) *The formula*

$$\tilde{T}=\begin{pmatrix} T & D_{T^*}\Gamma_1\\ \Gamma_2 D_T & -\Gamma_2L_T^*J_{T^*}\Gamma_1+D_{\Gamma_2^*}\Gamma D_{\Gamma_1}\end{pmatrix} \tag{5.27}$$

establishes a one-to-one correspondence between the set of all solutions to Problem $(*)$ *with* $\tilde{\kappa}=\kappa-\kappa^-(\tilde{\mathcal{K}}_2)+\kappa^-(\mathcal{K}_2)\geq\kappa^-(\tilde{\mathcal{K}}_1)-\kappa^-(\tilde{\mathcal{K}}_2)$ *and the triplets* $\{\Gamma_1,\Gamma_2,\Gamma\}$, *where* $\Gamma_1\in\mathcal{L}(\mathcal{K}_1',\mathfrak{D}_{T^*})$ *and* $\Gamma_2\in\mathcal{L}(\mathfrak{D}_T,\mathcal{K}_2')$ (Γ_1 *and* $\Gamma_1^\#$ *being J-contractions*), *while* $\Gamma\in\mathcal{L}(\mathfrak{D}_{\Gamma_1},\mathfrak{D}_{\Gamma_2^*})$ *is a Hilbert space contraction. Moreover, in the above situation the operators* $U(\tilde{T})$ *and* $U_*(\tilde{T})$ *defined by*

$$\begin{cases} U(\tilde{T}):\mathfrak{D}_{\tilde{T}}\to\mathfrak{D}_{\Gamma_2}[+]\mathfrak{D}_\Gamma,\\ U(\tilde{T})D_{\tilde{T}}=\begin{pmatrix} D_{\Gamma_2}D_T & -(D_{\Gamma_2}L_T^*J_{T^*}\Gamma_1+J_{\Gamma_2}L_{\Gamma_2^*}\Gamma D_{\Gamma_1})\\ 0 & D_\Gamma D_{\Gamma_1}\end{pmatrix}\end{cases} \tag{5.28}$$

and

$$(5.29)\qquad \begin{cases} U_*(\tilde{T}): \mathfrak{D}_{\tilde{T}^*} \to \mathfrak{D}_{\Gamma_1^*}[+]\mathfrak{D}_{\Gamma^*}, \\ U_*(\tilde{T})D_{\tilde{T}^*} = \begin{pmatrix} D_{\Gamma_1^*}D_{T^*} & -(D_{\Gamma_1^*}L_{T^*}^* J_T \Gamma_2^* + J_{\Gamma_1^*}L_{\Gamma_1}\Gamma^* D_{\Gamma_2^*}) \\ 0 & D_{\Gamma^*}D_{\Gamma_2^*} \end{pmatrix} \end{cases}$$

are unitary operators.

Proof. Some details will be omitted because the main idea is the same as in the "definite" case, and the technical difficulties were described above.

We will consider a solution $\tilde{T}$ to Problem (*) as a row extension of its first column T_c. The assertion (i) follows by combining Lemma 5.1(i) and Lemma 5.2(i). Indeed, by Lemma 5.1(i),

$$(5.30)\qquad \tilde{\kappa} \geq \max\{\kappa^-[I - T_c^\# T_c], \kappa^-(\tilde{\mathfrak{K}}_1) - \kappa^-(\tilde{\mathfrak{K}}_2)\}.$$

By Lemma 5.2(i)

$$(5.31)\qquad \kappa^-[I - T_c^\# T_c] \geq \kappa - \kappa^-(\tilde{\mathfrak{K}}_2) + \kappa^-(\mathfrak{K}_2).$$

The relations (5.30) and (5.31) implies (5.26).

For proving (ii), suppose first that

$$(5.32)\qquad \tilde{\kappa} = \kappa - \kappa^-(\tilde{\mathfrak{K}}_2) + \kappa^-(\mathfrak{K}_2) \geq \kappa^-(\tilde{\mathfrak{K}}_1) - \kappa^-(\tilde{\mathfrak{K}}_2).$$

By (5.22) it follows that

$$(5.33)\qquad \kappa^-[I - T_c^\# T_c] = \tilde{\kappa}.$$

Thus, we can apply Lemma 5.2(ii) to find an operator $\Gamma_2 \in \mathfrak{L}(\mathfrak{K}_1', \mathfrak{D}_{T^*})$ such that $\Gamma_2^\#$ is a J-contraction and

$$(5.34)\qquad T_c = (T \quad \Gamma_2 D_T)'.$$

From (5.32) and (5.33) it follows that it is possible to apply Lemma 5.2(ii) in the description of $\tilde{T}$ as a row extension of T_c. Thus, there exists a J-contraction $\Delta \in \mathfrak{L}(\mathfrak{K}_1', \mathfrak{D}_{T_c^*})$ with

$$(5.35)\qquad \tilde{T} = (T_c \quad D_{T_c^*}\Delta).$$

Using the J-unitary operator $U_*(T_c)$ from (5.25), define the J-contraction

$$(5.36)\qquad \begin{cases} \Delta' \in \mathfrak{L}(\mathfrak{K}_1', \mathfrak{D}_{T^*}[+]\mathfrak{D}_{\Gamma_2^*}) \\ \Delta' = (J_{T^*}[+]I)U_*(T_c)J_{T_c^*}\Delta. \end{cases}$$

Because $U_*(T_c)$ is J-unitary we have

$$(5.37)\qquad \Delta = U_*^*(T_c)\Delta'.$$

We apply now Lemma 5.2 for the structure of the column operator Δ'; denoting by $\Gamma_1 \in \mathfrak{L}(\mathfrak{K}_1', \mathfrak{D}_{T^*})$ the first component of Δ' and noting that $\kappa^-(\mathfrak{D}_{T^*}[+]\mathfrak{D}_{\Gamma_2^*}) = \kappa^-(\mathfrak{D}_{T^*})$, we have that Γ_1 is a J-contraction and that the condition of Lemma 5.2(ii) is verified. So, there exists an operator $\Gamma \in \mathfrak{L}(\mathfrak{D}_{\Gamma_1}, \mathfrak{D}_{\Gamma_2^*})$ which is a Hilbert space contraction such that

$$(5.38)\qquad \Delta' = (\Gamma_1 \quad \Gamma D_{\Gamma_1})'.$$

From (5.35), (5.37), (5.25), and (5.38) it follows that we have the formula (5.27) for $\tilde{T}$.

The converse assertion from the statement of (ii) can be deduced with the same methods.

The identifications of defect spaces of $\tilde{T}$ need thorough computations with the corresponding identifications from Lemmas 5.1 and 5.2. We indicate only the main part of the computations for obtaining (5.28). This consists in proving that

$$U(T_c)J_{T_c}L_{T_c^*}U_*^*(T_c)=(D_{\Gamma_2}L_T^*J_{T^*} \quad J_{\Gamma_2}L_{\Gamma_2^*}) \tag{5.39}$$

as operators from $D_{T^*}[+]\mathfrak{D}_{\Gamma_2^*}$ into $\mathfrak{D}_{\Gamma_2}$. For this, take $x\in\mathfrak{D}_{\Gamma^*}[+]\mathfrak{D}_{\Gamma_2^*}$ and $k\in\mathcal{K}_1$. Then

$$\begin{aligned}
&(U(T_c)J_{T_c}L_{T_c^*}U_*^*(T_c)x,\, J_{\Gamma_2}D_{\Gamma_2}D_Tk)\\
&=(L_{T_c^*}U_*^*(T_c)x, D_{T_c}k)\\
&=(T_c^*J_2D_{T_c^*}U_*^*(T_c)x, k)\\
&=\left((T^* \quad D_T\Gamma_2^*)\begin{pmatrix} J_2 & 0\\ 0 & J_2'\end{pmatrix}\begin{pmatrix} D_{T^*} & 0\\ -\Gamma_2L_T^*J_{T^*} & D_{\Gamma_2^*}\end{pmatrix}x, k\right)\\
&=((T^*J_2D_{T^*}-D_T\Gamma_2^*J_2'\Gamma_2L_T^*J_{T^*} \quad D_T\Gamma_2^*J_2'\Gamma_2D_{\Gamma_2^*})x, k),
\end{aligned} \tag{5.40}$$

where we used successively (5.24), $(4.7)_*$, and (5.25). But

$$\begin{aligned}
T^*J_2D_{T^*}-D_T\Gamma_2^*J_2'\Gamma_2L_T^*J_{T^*}&=D_TL_{T^*}-D_TJ_TL_T^*J_{T^*}+D_TJ_{\Gamma_2}D_{\Gamma_2}^2L_T^*J_{T^*}\\
&=(D_TD_{\Gamma_2}J_{\Gamma_2})(D_{\Gamma_2}L_T^*J_{T^*}),
\end{aligned} \tag{5.41}$$

where we used successively $(4.7)_*$, the equality $J_T-J_{\Gamma_2}D_{\Gamma_2}^2=\Gamma_2^*J_2'\Gamma_2$, and (4.20). On the other hand, again using $(4.7)_*$, we have that

$$D_T\Gamma_2^*J_2'D_{\Gamma_2^*}=(D_TD_{\Gamma_2}J_{\Gamma_2})(J_{\Gamma_2}L_{\Gamma_2^*}). \tag{5.42}$$

The relations (5.40), (5.41) and (5.42) imply (5.39). □

5.4. REMARKS. (1) Theorem 5.3 contains, as particular cases, the structure of 2×2 Hilbert space block-matrices whose defect operators have a finite number of negative squares, and that of 2×2 J-contractions in Pontryagin spaces (see also [8]).

(2) Theorem 5.3 permits us to say when a given operator on a Pontryagin space has isometric or unitary extensions to a larger space, and to describe all of them (if any). In particular, it follows that all unitary extensions can be obtained from the elementary rotation of the given operator.

(3) Some results from [2] can be extended to the context of Theorem 5.3. In particular, one can prove, using the same methods as in the proof of Theorem 5.3, that the elementary rotation of $\tilde{T}$ has a similar formula as (1.5) in [2] and the same lattice filter representation as in Remark 1.2(e) of [2] (in the figure of that remark, Γ_1 and Γ_2 are interchanged by mistake). The formula for the elementary rotation of $\tilde{T}$ contains both the representation (5.27) and the identifications (5.28) and (5.29)—see (1.6) from [2].

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