AN INVARIANT FOR UNITARY REPRESENTATIONS OF NILPOTENT LIE GROUPS

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1. Introduction. In this paper, we define an invariant for the irreducible unitary representations of a simply connected nilpotent Lie group G. The invariant $i(\rho)$ for a representation ρ is an element of $H^*(\mathcal{G})$, the real cohomology of the Lie algebra $\mathcal{G}$ of G. The cohomology class $i(\rho)$ is constructed using the coadjoint orbit $\mathcal{O}$ corresponding to ρ and has degree $\dim(\mathcal{O})+1$. If we view $\dim(\mathcal{O})$ as a primary invariant, then $i(\rho)$ is a more subtle secondary invariant that can be used to distinguish between representations whose orbits have the same dimension.

The definition of $i(\rho)$ in terms of orbits is given in Section 2. The remaining sections address two central questions concerning $i(\rho)$. Firstly, is the invariant computable in examples and can it be non-zero? Secondly, what information does the invariant contain about a representation—that is, what does it measure?

Examples are presented in Section 4. These show that the invariant is relatively easy to compute and is frequently non-zero. The second question is more difficult and provides a direction for further research. Here we present three results along these lines. In Section 3, we show that if two representations differ by a multiplicative character then the invariants for these representations coincide. In Section 5, we prove that for groups with one-dimensional center, $i(\rho)$ is non-zero for representations ρ that are square integrable modulo the center. In Section 6, we show that for a class of groups (the 3-step groups with one-dimensional center), the invariant vanishes for certain representations.

In Section 7, we discuss some unsolved problems concerning $i(\rho)$.

2. Definition of the invariant. We begin with the symplectic structure for coadjoint orbits. Throughout, G will denote a Lie group with Lie algebra $\mathcal{G}$. We write $\mathcal{O}_f$ for the orbit of $f \in \mathcal{G}^*$ under the coadjoint action of G on $\mathcal{G}^*$. We have a projection map

$$\pi_f : G \to \mathcal{O}_f, \quad \pi_f(g) = \mathrm{Ad}^*(g)f. \tag{2.1}$$

If w is the 2-form on $\mathcal{O}_f$ constructed in [8], then we have

$$\pi_f^*(w) = -\mathcal{d}f. \tag{2.2}$$

Here we view f as a left invariant 1-form on G, and $\mathcal{d}f$ is the exterior derivative of f in the de Rham complex $(\Omega(G), \mathcal{d})$.

The left invariant forms on G are a subcomplex of $\Omega(G)$ which can be identified with the exterior algebra $\Lambda(\mathcal{G}^*)$. The cohomology of this complex is denoted by $H^*(\mathcal{G})$ and agrees with the algebraic notion of Lie algebra cohomology with trivial (real) coefficients [3].

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Let $2q$ be the dimension of $\mathcal{O}$, which is necessarily even.

2.3. LEMMA. *For* $f \in \mathcal{O}$, $f \wedge (df)^q \in \Lambda^{2q+1}(\mathcal{G}^*)$ *is a closed form.*

Proof.

$$\begin{aligned} d(f \wedge (df)^q) &= (df)^{q+1} \\ &= (-\pi_f^*(w))^{q+1} \\ &= (-1)^{q+1}\pi_f^*(w^{q+1}). \end{aligned}$$

Here $w^{q+1} = 0$ since it is a $(2q+2)$-form on a manifold $\mathcal{O}$ of dimension $2q$. □

One obtains a cohomology class $[f \wedge (df)^q] \in H^{2q+1}(\mathcal{G})$. When G is connected, this depends only on $\mathcal{O}$.

2.4. LEMMA. *Let G be connected and $f, f' \in \mathcal{O}$. Then*

$$[f \wedge (df)^q] = [f' \wedge (df')^q].$$

Proof. Since f and f' belong to the same orbit, we can write $f' = \mathrm{Ad}^*(g)f$ for some $g \in G$. We now require a few facts about Lie algebra cohomology for which we refer the reader to [3]. The representation Ad^* extends multiplicatively to all of $\Lambda(\mathcal{G}^*)$ and each $\mathrm{Ad}^*(g)$ commutes with the exterior derivative. One obtains then

$$[f' \wedge (df')^q] = \mathrm{Ad}^*(g)_*([f \wedge (df)^q]),$$

where $\mathrm{Ad}^*(g)_* : H^{2q+1}(\mathcal{G}) \to H^{2q+1}(\mathcal{G})$ is the map induced by $\mathrm{Ad}^*(g)$ in cohomology. When G is connected, $\mathrm{Ad}^*(g)_*$ is the identity map for every $g \in G$. □

Simple examples show the necessity of the connectivity assumption in Lemma 2.4. From now on, we will assume that all our Lie groups are connected and make the following definition.

2.5. DEFINITION. Let $\mathcal{O} \subset \mathcal{G}^*$ be a coadjoint orbit of dimension $2q$. Then the invariant of $\mathcal{O}$ is

$$\iota(\mathcal{O}) = [f \wedge (df)^q] \in H^{2q+1}(\mathcal{G}),$$

where f is any point in $\mathcal{O}$.

One motivation for this seemingly ad hoc definition comes from foliation theory. The definition of $\iota(\mathcal{O})$ is similar to that for the secondary characteristic classes of a foliation [2]. In fact, $\iota(\mathcal{O})$ can be viewed as a Lie algebraic version of a characteristic class for (exact) transversely symplectic foliations arising in [1].

In what follows we will be referring to connected, simply connected nilpotent Lie groups as "nilpotent groups." If G is a nilpotent group with Lie algebra $\mathcal{G}$, we will denote the exponential map by $\exp: \mathcal{G} \to G$. We will refer to coadjoint orbits as "orbits." All representations of G are assumed to be unitary. There is a one-one correspondence between the orbits in $\mathcal{G}^*$ and (equivalence classes of) irreducible representations of G [5]. If ρ is such a representation, then we write $\iota(\rho)$ for $\iota(\mathcal{O}_\rho)$, where $\mathcal{O}_\rho$ is the orbit corresponding to ρ.

The invariant satisfies a useful naturality property.

2.6. LEMMA. *Let* $\phi: G_1 \to G_2$ *be a homomorphism of nilpotent groups and* $\rho: G_2 \to U(H)$ *be an irreducible representation.*

(a) *If* $\rho \circ \phi$ *is irreducible then* $i(\rho \circ \phi) = \phi^*(i(\rho))$ *(where* $\phi^*: H^*(\mathcal{G}_2) \to H^*(\mathcal{G}_1)$ *is induced by* ϕ*).*

(b) *If* $\rho \circ \phi$ *is not irreducible then* $\phi^*(i(\rho)) = 0$.

Proof. Suppose $f \in \mathcal{O}_\rho$. So

$$i(\rho) = [f \wedge (df)^q] \quad \text{and} \quad \phi^*(i(\rho)) = [\phi^*(f) \wedge (d(\phi^*(f)))^q],$$

where $2q = \dim(\mathcal{O}_\rho)$. When $\rho \circ \phi$ is irreducible, $\phi^*: \mathcal{G}_2^* \to \mathcal{G}_1^*$ maps $\mathcal{O}_\rho$ diffeomorphically to $\mathcal{O}_{\rho\circ\phi}$. This proves (a) since, in this case, $\dim(\mathcal{O}_\rho) = \dim(\mathcal{O}_{\rho\circ\phi})$ and $\phi^*(f) \in \mathcal{O}_{\rho\circ\phi}$. If $\rho \circ \phi$ is not irreducible then $\phi^*(\mathcal{O}_\rho)$ is a union of lower-dimensional orbits. In this case we have $(d(\phi^*(f)))^q = 0$. □

3. Characters. When an orbit $\mathcal{O}$ consists of a single point $\{f\}$, the invariant becomes $i(\mathcal{O}) = [f] \in H^1(\mathcal{G})$. Note that f is fixed by every $\mathrm{Ad}^*(g)$ so that $f([\mathcal{G}, \mathcal{G}]) = 0$ and hence $df = 0$. There are no non-zero exact left invariant 1-forms on G, so $H^1(\mathcal{G}) = \mathrm{Ker}(d: \mathcal{G}^* \to \Lambda^2(\mathcal{G}^*))$. We see that $[f] = \{f\}$; the invariant for a one-point orbit is simply the orbit itself viewed as a cohomology class. In the nilpotent case, one has the following result for $\hat{G}$, the set of characters of G.

3.1. LEMMA. *For* G *nilpotent,* $i: \hat{G} \to H^1(\mathcal{G})$ *is an isomorphism.*

Proof. The characters $\chi \in \hat{G}$ correspond to one-point orbits $\{f\} \subset \mathcal{G}^*$. In fact, one has $\chi(\exp x) = e^{if(x)}$ for $x \in \mathcal{G}$ [5]. If χ_0, χ_1 are characters corresponding to $\{f_0\}$ and $\{f_1\}$, then this formula shows that $\chi_0\chi_1$ corresponds to $\{f_0 + f_1\}$. This shows that the bijection $i: \hat{G} \to H^1(\mathcal{G})$ is an isomorphism between the multiplicative group $\hat{G}$ and the additive group $H^1(\mathcal{G})$. □

An irreducible representation of a nilpotent group G is either a character or infinite-dimensional [5]. If $\rho: G \to U(H)$ is a representation and $\chi \in \hat{G}$, then we obtain a new representation $\chi\rho$ by pointwise multiplication: $(\chi\rho)(g) = \chi(g)\rho(g)$.

3.2. THEOREM. *Let* G *be nilpotent,* $\rho: G \to U(H)$ *an infinite-dimensional irreducible representation, and* $\chi \in \hat{G}$ *a character. Then*

$$i(\chi\rho) = i(\rho).$$

Proof. Suppose that $\mathcal{O} \subset \mathcal{G}^*$ is the orbit corresponding to ρ and that $\{f\}$ corresponds to χ. It is a well-known fact that $\chi\rho$ is given by the orbit $\mathcal{O} + f = \{h + f \mid h \in \mathcal{O}\}$. This orbit is diffeomorphic to $\mathcal{O}$ in an obvious way.

Let $h \in \mathcal{O}$. Then

$$i(\chi\rho) = [(h+f) \wedge d(h+f)^q] = [h \wedge (dh)^q + f \wedge (dh)^q] \quad \text{as } df = 0.$$

Here $2q = \dim(\mathcal{O})$ is positive since ρ is infinite-dimensional. Using $q \geq 1$ and $df = 0$, we see that

$$f \wedge (dh)^q = d(-f \wedge h \wedge (dh)^{q-1}).$$

Hence $i(\chi\rho) = [h \wedge (dh)^q] = i(\rho)$. □

4. Examples. In this section we explicitly compute our invariant for a number of examples.

4.1. *The Heisenberg group.* Let H_n be the $(2n+1)$-dimensional Heisenberg group with Lie algebra $\mathcal{H}_n$. $\mathcal{H}_n$ has basis $X_1, \dots, X_n, Y_1, \dots, Y_n, Z$ with non-zero brackets $[X_i, Y_j] = \delta_{ij} Z$. $\mathcal{H}_n^*$ has dual basis $\mu_1, \dots, \mu_n, \nu_1, \dots, \nu_n, \lambda$, with coordinates (a, b, c) where $a, b \in \mathbf{R}^n$ and $c \in \mathbf{R}$.

The co-adjoint orbits in $\mathcal{H}_n^*$ consist of single points $(a, b, 0)$ (discussed in §3), and $2n$-dimensional hyperplanes $\{(a, b, c): a, b \in \mathbf{R}^n\}$ with $c \neq 0$. An orbit of the latter type contains an element $f = c\lambda$, $c \neq 0$. We have

$$\begin{aligned} df &= c\,d\lambda \\ &= c\left(\sum_{i=1}^{n} \nu_i \wedge \mu_i\right), \end{aligned}$$

so that

$$i(\mathcal{O}_f) = n!\, c^{n+1}[\lambda \wedge \nu_1 \wedge \mu_1 \wedge \cdots \wedge \nu_n \wedge \mu_n],$$

which is non-zero since the volume form $\lambda \wedge \nu_1 \wedge \mu_1 \wedge \cdots \wedge \nu_n \wedge \mu_n$ is not an exact form.

Note that if n is even $i(\mathcal{O}_f)$ is distinct for distinct orbits.

4.2. *A 3-step group.* The simplest 3-step nilpotent group SH is 4-dimensional with one-dimensional center. A general class of 3-step groups is discussed in Section 6. The Lie algebra $\mathcal{SH}$ has basis S, X, Y, Z with non-zero brackets $[S, X] = Y$, $[X, Y] = Z$. Note that the Heisenberg group H is a subgroup of SH. Let c, μ, ν, λ be a basis for $\mathcal{SH}^*$ with coordinates (a, b, c, d).

If $f \in \mathcal{SH}^*$, $f = (a, 0, 0, d)$ with $d \neq 0$, then

$$\mathcal{O}_f = \{(a - \tfrac{1}{2}c^2/d, b, c, d): b, c \in \mathbf{R}\} \quad \text{and} \quad i(\mathcal{O}_f) = d^2[\lambda \wedge \nu \wedge \mu] \neq 0.$$

If $f = (0, 0, c, 0)$ with $c \neq 0$, then

$$\mathcal{O}_f = \{(a, b, c, 0): a, b \in \mathbf{R}\} \quad \text{and} \quad i(\mathcal{O}_f) = 0.$$

All other orbits are single points.

4.3. *A 4-step group.* Let $\mathcal{G}$ be a Lie algebra with basis X, Y_1, Y_2, Y_3, Y_4 and non-zero brackets $[X, Y_i] = Y_{i+1}$, $i = 1, 2, 3$. Then the corresponding group G is a 4-step group with one-dimensional center. Let $\mathcal{G}^*$ have dual basis $\mu, \nu_1, \nu_2, \nu_3, \nu_4$ and coordinates (a, b, c, d, e).

If $f \in \mathcal{G}^*$, $f = (0, b, c, 0, e)$ with $e \neq 0$, then

$$\mathcal{O}_f = \{(a, b + cd/e + \tfrac{1}{6}d^3/e^2, c + \tfrac{1}{2}d^2/e, d, e): a, d \in \mathbf{R}\}.$$

Then we have $i(\mathcal{O}_f) = -e^2[\nu_3 \wedge \nu_4 \wedge \mu] + 2ce[\nu_2 \wedge \nu_3 \wedge \mu]$. Note that if $b = c = 0$, we still have $i(\mathcal{O}_f) \neq 0$.

If $f = (0, b, 0, d, 0)$ with $d \neq 0$, then

$$\mathcal{O}_f = \{(a, b + \tfrac{1}{2}c^2/d, c, d, 0): a, c \in \mathbf{R}\} \quad \text{and} \quad i(\mathcal{O}_f) = -d^2[\nu_2 \wedge \nu_3 \wedge \mu] \neq 0.$$

4.4. *A group with* 2-*dimensional center.* If we look at the group of 4×4 upper-triangular matrices modulo its center, we get a 5-dimensional group T with 2-dimensional center. The Lie algebra $\mathfrak{T}$ has basis X_1, X_2, X_3, Z_1, Z_2 with non-zero brackets $[X_1, X_2]=Z_1$ and $[X_2, X_3]=Z_2$. $\mathfrak{T}^*$ has a dual basis $\mu_1, \mu_2, \mu_3, \lambda_1, \lambda_2$ and coordinates (a, b) where $a\in\mathbf{R}^3$ and $b\in\mathbf{R}^2$.

If $f=(0,0,a_3,b)$ with $b\in\mathbf{R}^2$, $b_1\neq 0$, then

$$\mathcal{O}_f=\{(a_1,a_2,a_3-a_1b_2/b_1,b): a_1,a_2\in\mathbf{R}\} \quad \text{and}$$

$$i(\mathcal{O}_f)=-b_1^2[\lambda_1\wedge\mu_1\wedge\mu_2]-2b_1b_2[\lambda_1\wedge\mu_2\wedge\mu_3]-b_2^2[\lambda_2\wedge\mu_2\wedge\mu_3]\neq 0.$$

If $f=(a_1,0,0,0,b_2)$ with $b_2\neq 0$, then

$$\mathcal{O}_f=\{(a_1,a_2,a_3,0,b_2): a_2,a_3\in\mathbf{R}\} \quad \text{and} \quad i(\mathcal{O}_f)=-b_2^2[\lambda_2\wedge\mu_2\wedge\mu_3]\neq 0.$$

4.5. *A semi-simple group.* We conclude this section with a different sort of example. Consider the compact semi-simple group $SU(q+1)$ of unitary matrices with determinant 1. Its Lie algebra $su(q+1)$ consists of the skew Hermitian matrices $(A^*=-A)$ of trace zero. The formula

$$f(B)=\frac{1}{i}b_{11},$$

for $B=(b_{j,k})\in su(q+1)$, defines an element of $su(q+1)^*$.

One can check that the stabilizer of f under the coadjoint action is

$$SU(q+1)_f=\left\{\begin{pmatrix} a & \text{---}0\text{---} \\ \begin{matrix}|\\0\\|\end{matrix} & C \end{pmatrix}: C\in U(q),\ a\det(C)=1\right\},$$

which is isomorphic to $U(q)$. The orbit $\mathcal{O}_f$ is diffeomorphic to $SU(q+1)/SU(q+1)_f$ which is the complex projective space $\mathbf{C}P^q$ with its usual symplectic structure. Note that here $\mathcal{O}_f$ is compact, a situation that does not arise for simply connected nilpotent groups [5]. For a discussion of this example we refer the reader to [8].

It is known that $H^*(su(q+1))$ is an exterior algebra $\Lambda(u_2, u_3, \ldots, u_{q+1})$ where the generator u_i has degree $2i-1$ (these are the suspensions of the universal Chern classes). It is shown in [1] that $i(\mathcal{O}_f)$ is a non-zero multiple of u_{q+1} in $H^{2q+1}(su(q+1))$. This shows that an orbit in a non-nilpotent group can have a non-trivial invariant.

5. Square integrable representations. In this section we will prove a general non-vanishing theorem for $i(\rho)$.

5.1. THEOREM. *Let G be nilpotent with a one-dimensional center. If ρ is an irreducible representation of G that is square integrable modulo the center then* $i(\rho)\neq 0$.

Proof. Let $(Z, X_1, X_2, \ldots, X_r)$ be a basis for $\mathcal{G}$ where Z spans the center. The dual basis for $\mathcal{G}^*$ will be denoted $(\lambda, \alpha_1, \alpha_2, \ldots, \alpha_r)$.

Since Z is fixed by the adjoint action of G, any coadjoint orbit is contained in a hyperplane $\{c\lambda+\sum_i a_i\lambda_i \mid a_1,\dots,a_r\in\mathbf{R}\}$ for some fixed $c\in\mathbf{R}$. In [6], it is shown that irreducible square integrable representations ρ correspond to flat orbits $\mathcal{O}_\rho$ of codimension $\dim(Z(G))$. In the present case, this means that $\mathcal{O}_\rho$ is an entire hyperplane as above with $c\neq 0$ (as $\{0\}$ is a one-point orbit, we must have $c\neq 0$). In particular, $\mathcal{O}_\rho=\mathcal{O}_{c\lambda}$ and $r=2q=\dim(\mathcal{O}_\rho)$.

The condition $[Z,\mathcal{G}]=0$ implies that $d\lambda\in\Lambda^2(\alpha_1,\dots,\alpha_{2q})$. So

$$(d\lambda)^q=k\alpha_1\wedge\alpha_2\wedge\cdots\wedge\alpha_{2q}$$

for some $k\in\mathbf{R}$. However, we can also write $(d\lambda)^q=\pi_\lambda^*(w^q)$, where π_λ is the projection map (2.1) and w is the symplectic form on $\mathcal{O}_\lambda$. Since $w^q\neq 0$ and π_λ is a submersion, we conclude that $k\neq 0$.

Thus we can write:

$$i(\rho)=kc^{q+1}[\lambda\wedge\alpha_1\wedge\cdots\wedge\alpha_{2q}], \tag{5.2}$$

which is a non-zero multiple of the class of the volume form $\lambda\wedge\alpha_1\wedge\cdots\wedge\alpha_{2q}$. It is easily seen that a volume form always yields a non-zero cohomology class. □

Notice that formula (5.2) shows that $i(\rho)$ is, up to signs, a complete invariant for the irreducible square integrable representations. Indeed, different square integrable representations correspond to different values of c. If q is even (where $2q+1=\dim(G)$), then i is a complete invariant for such representations.

6. 3-step groups. In this section we take a close look at our invariant for 3-step groups with one-dimensional center. As a generalization of Example 4.2, consider the Lie algebra $\mathcal{H}_n$ $(n\geq 2)$ of the $(2n+1)$-dimensional Heisenberg group (Example 4.1). $\mathcal{H}_n$ can be written as a sum $\mathcal{H}_n=\mathcal{W}\oplus\mathcal{Z}$, where $\mathcal{Z}=\langle Z\rangle$ is the center, and $\mathcal{W}=\mathcal{X}\oplus\mathcal{Y}$ with $\mathcal{X}=\langle X_1,\dots,X_n\rangle$, $\mathcal{Y}=\langle Y_1,\dots,Y_n\rangle$.

The Lie algebra bracket defines a symplectic form on $\mathcal{W}$, so we can make sense of the action of the symplectic group Sp_{2n} on $\mathcal{W}$. Let S_n be the subgroup of Sp_{2n} which fixes $\mathcal{Y}$. S_n is isomorphic to the (abelian) group of $(n\times n)$ symmetric matrices, and the semi-direct product SH_n of S_n and H_n is a 3-step nilpotent group with one-dimensional center.

It is shown in [7] that every 3-step nilpotent group with one-dimensional center is a subgroup of SH_n for some n. We use this theorem to prove the following.

6.1. THEOREM. *Let G be a 3-step nilpotent group with one-dimensional center. If $\mathcal{O}$ is a non-generic orbit of maximal dimension in $\mathcal{G}^*$, then $i(\mathcal{O})=0$.*

Proof. First we compute our invariant for non-generic orbits of maximal dimension in $\mathcal{SH}_n^*$. Let $\mathcal{SH}_n$ have basis S_{ij}, X_k, Y_ℓ, Z $(i\geq j$ and $i,j,k,\ell=1,\dots,n)$ with $[S_{ij},X_i]=Y_j$, $[S_{ij},X_j]=Y_i$, $[X_i,Y_j]=\delta_{ij}Z$. $\mathcal{SH}_n^*$ has dual basis $\tau_{ij},\mu_k,\nu_\ell,\lambda$, and coordinates (A,b,c,d) where $A=(A_{ij})\in\mathcal{S}_n$, $b,c\in\mathbf{R}^n$, $d\in\mathbf{R}$.

The orbits of maximal dimension $2n$ in $\mathcal{SH}_n^*$ are of two types. Firstly, if

$$f=(A,0,0,d)\quad\text{with } d\neq 0, \tag{6.2}$$

then $\mathcal{O}_f$ is a parabolic orbit. Secondly, if $f=(A,0,c,0)$ with $c\neq 0$, then $\mathcal{O}_f$ is flat. We compute our invariant for the second type of orbit.

We have $f=\tau+\nu$, where $\tau=\sum_{i\le j} A_{ij}\tau_{ij}$ and $\nu=\sum_{i=1}^n c_i\nu_i$. Then

$$d f = d\nu = \sum_{j=1}^{n} \mu_j\wedge\tau_j,$$

where $\tau_j\in \mathcal{S}^*$. Hence

$$\begin{aligned}(d f)^n &= n!\,\mu_1\wedge\tau_1\wedge\cdots\wedge\mu_n\wedge\tau_n\\ &= \pm n!\,\mu\wedge\bar\tau,\end{aligned}$$

where $\mu=\mu_1\wedge\cdots\wedge\mu_n$ and $\bar\tau=\tau_1\wedge\cdots\wedge\tau_n$.

We now have $f\wedge(d f)^n$ as a linear combination of two terms, $\tau\wedge\mu\wedge\bar\tau$ and $\nu\wedge\mu\wedge\bar\tau$. Let $\hat\mu=\mu_2\wedge\cdots\wedge\mu_n$, $\hat\tau=\tau_2\wedge\cdots\wedge\tau_n$. Then

$$\begin{aligned}d(\nu\wedge\hat\tau\wedge\hat\mu\wedge\tau) &= \mu_1\wedge\tau_1\wedge\hat\tau\wedge\hat\mu\wedge\tau\\ &= \pm\bar\tau\wedge\mu\wedge\tau.\end{aligned}$$

Thus $[\tau\wedge\mu\wedge\bar\tau]=0$. Let $\bar\mu=\sum_{i=1}^n c_i\mu_1\wedge\cdots\wedge\hat\mu_i\wedge\cdots\wedge\mu_n\in\Lambda^{n-1}(\mathcal{SH}_n^*)$. Then

$$\begin{aligned}d(\lambda\wedge\bar\mu\wedge\bar\tau) &= \sum_{i=1}^{n}\nu_i\wedge\mu_i\wedge\bar\mu\wedge\bar\tau\\ &= \sum_{i=1}^{n} c_i\nu_i\wedge\mu\wedge\bar\tau\\ &= \nu\wedge\mu\wedge\bar\tau.\end{aligned}$$

Thus we also have $[\nu\wedge\mu\wedge\bar\tau]=0$, so

$$i(\mathcal{O}_f)=[f\wedge(d f)^n]=0. \tag{6.3}$$

Now suppose G is as in the statement of the theorem. Then there is an injection $J\colon \mathcal{G}\hookrightarrow \mathcal{SH}_n$ for some n, such that $J(\mathcal{G})$ is an ideal in $\mathcal{SH}_n$. The dual map $J^*\colon \mathcal{SH}_n^*\to\mathcal{G}^*$ is Ad*-equivariant.

It is shown in [7] that if $f\in\mathcal{SH}_n^*$ is given by (6.2), then $J^*\colon \mathcal{O}_f\to\mathcal{O}_{J^*f}$ is an isomorphism. (Note that J maps the center of $\mathcal{G}$ onto the center of $\mathcal{SH}_n$.)

If $\mathcal{O}$ is a non-generic orbit in $\mathcal{G}^*$ of maximal dimension $2n$, then J^* is an isomorphism between a non-generic orbit $\mathcal{O}'$ in $\mathcal{SH}_n^*$ and $\mathcal{O}$. We know from (6.3) that $i(\mathcal{O}')=0$, so by Lemma 2.6 we have $i(\mathcal{O})=J^*(i(\mathcal{O}'))=0$. □

NOTE. As $\mathcal{O}$ is *flat,* it corresponds to a representation ρ which is *square integrable* modulo its kernel [6].

7. Further questions. Theorem 5.1 is false for nilpotent groups with center of dimension greater than one. Consider the 2-step group with Lie algebra $\mathcal{G}=\langle X_1,X_2,Y_1,Y_2,Z_1,Z_2\rangle$ and non-zero brackets

$$[X_1,Y_1]=Z_1=[X_2,Y_2]\quad\text{and}\quad [X_1,Y_2]=Z_2.$$

If $\lambda_1 \in \mathcal{G}^*$ is the basis element dual to Z_1, then $\mathcal{O}_{\lambda_1}$ corresponds to a square-integrable representation, but $\iota(\mathcal{O}_{\lambda_1})=0$. We hope to generalize Theorem 5.1 in a different direction as follows.

CONJECTURE. If G is nilpotent with one-dimensional center and $\lambda \in \mathcal{G}^*$ is dual to a basis element for the center of $\mathcal{G}$, then $\iota(\mathcal{O}_\lambda) \neq 0$.

Theorems relating the invariant for an orbit to its differential geometric properties would be of interest. We are encouraged by recent work concerning the geometric meaning of the characteristic classes for a foliation [4].

Observe that by 3.1, every element of $H^1(\mathcal{G})$ is the invariant for some character. How many classes in $H^{2q+1}(\mathcal{G})$ are invariants for irreducible representations? One can also ask how much of $H^*(\mathcal{G})$ is generated by taking products of invariants of irreducible representations.

Finally, it seems natural to try to generalize the invariant itself. One could consider non-irreducible representations, non-nilpotent groups, or related invariants.

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