LAMINATIONS, FINITELY GENERATED PERFECT GROUPS, AND ACYCLIC MAPS

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1. Introduction. Let (M, N_1, N_2) denote a $(n+1)$-dimensional cobordism; that is, M is a compact, connected $(n+1)$-manifold with two boundary components N_1 and N_2. We investigate the conditions under which (M, N_1, N_2) admits a *lamination,* by which we mean an upper semicontinuous decomposition G of M into closed n-manifolds with $N_k \in G$ $(k=1,2)$. We also consider a closely related question: Given two closed n-manifolds N_1 and N_2, when does there exist a laminated cobordism (M, N_1, N_2)?

Homological equivalence of N_1 and N_2 is a necessary condition for the existence of a laminated cobordism (M, N_1, N_2); in his initial work [7] Daverman proved that then $H_*(M, N_k)=0$ $(k=1,2)$. We show it not sufficient by presenting an example (Example 3.1) of a cobordism (M, N_1, N_2) satisfying this homology condition and such that there is no laminated cobordism (M', N_1, N_2).

On the other hand, a well-known sufficient condition for the existence of a lamination $(n \neq 3)$ is that (M, N_1, N_2) be an h-cobordism (each inclusion $i_k: N_k \to M$ is a homotopy equivalence), since then $M - N_2$ is homeomorphic to $N_1 \times [0,1)$. Other types of laminations exist, however; in the presence of wildness the decomposition elements can have varying homotopy types [7, Example 5.3]. Our chief interest centers on cobordisms (M, N_1, N_2) for which $i_2: N_2 \to M$ is a homotopy equivalence but $i_1: N_1 \to M$ is not. Under this assumption on i_2, it is easy to verify that $H_*(M, N_1)=0$ and that the kernel of $i_{1\#}: \pi_1(N_1) \to \pi_1(M)$ is perfect. If, in addition, kernel$(i_{1\#})$ is the normal closure of a finitely generated perfect group, then as our main result we demonstrate how to impose a lamination on (M, N_1, N_2); in particular, we obtain M, up to attachment of a h-cobordism, as the mapping cylinder of an acyclic map from N_1 to an n-manifold homotopy equivalent to N_2 (Theorem 5.2).

2. Technical lemmas. This section provides a listing of some utilitarian facts about the manifolds admitting laminations.

DEFINITION 2.1. A *laminated cobordism* is a cobordism (M, N_1, N_2), where M is a compact $(n+1)$-manifold having boundary components N_1 and N_2, together with an usc decomposition G of M into closed n-manifolds such that $N_1, N_2 \in G$.

First we state two results from previous work.

LEMMA 2.2 [7, Corollary 6.3]. *In a laminated cobordism (M, N_1, N_2) the inclusion-induced $H_*(g) \to H_*(M)$ is an isomorphism for each $g \in G$.*

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LEMMA 2.3 [8, Lemma 3.1]. *Let (M, N_1, N_2) be a laminated cobordism, $g \in G$, and C_k the closure of the component of $M-g$ that contains N_k $(k=1,2)$. Then the inclusion-induced $\pi_1(g) \to \pi_1(C_k)$ is a surjection.*

The next lemma contains additional information about laminated manifolds. As a notational matter, $[K, K]$ denotes the commutator subgroup of a given group K. If $K=[K,K]$, K is said to be *perfect*.

LEMMA 2.4. *Let (M, N_1, N_2) be a laminated cobordism. For $k=1,2$, denote by $i_{k\#}: \pi_1(N_k) \to \pi_1(M)$ the inclusion-induced homomorphism. Then:*
(a) kernel$(i_{k\#})$ *is a perfect, normal subgroup of* $\pi_1(N_k)$, *and*
(b) (M, N_1, N_2) *is an h-cobordism if both $i_{1\#}$ and $i_{2\#}$ are isomorphisms.*

Proof. Let $\tilde{M}$ be the universal cover of M. Then Bd $\tilde{M}=\tilde{N}_1 \cup \tilde{N}_2$, where $\tilde{N}_k$ represents the cover of N_k corresponding to kernel$(i_{k\#})$, a normal subgroup of $\pi_1(N_k)$. The given lamination on M lifts to a lamination of sorts $\tilde{G}$ on $\tilde{M}$. Lemma 2.3 ensures that the elements of $\tilde{G}$ are connected, but the decomposition G may fail to be usc, since in general its elements are non-compact. Nevertheless, we still have $H_*(\tilde{M}, \tilde{N}_k)=0$ [7, proof of Proposition 8.1]. Due to the simple-connectivity of $\tilde{M}$, $H_1(\tilde{N}_k)=H_1(\tilde{M})=0$. But $H_1(\tilde{N}_k) \cong \pi_1(\tilde{N}_k)/[\pi_1(\tilde{N}_k), \pi_1(\tilde{N}_k)]$ and, thus, kernel$(i_{k\#}) \cong \pi_1(\tilde{N}_k)$ is perfect, which proves (a). For (b), it follows that $\pi_1(\tilde{N}_k)=1$. By Whitehead's theorem [21, Theorem 1], $\pi_i(\tilde{M}, \tilde{N}_k)=0$ and $\pi_i(M, N_k)=0$ for $i>1$, so the inclusions $N_k \to M$ are homotopy equivalences. □

The following lemma is a special case of a well-known fact (see [14, Lemma 2.0]). We include a proof for completeness.

LEMMA 2.5. *If (M, N_1, N_2) is a cobordism such that $i_2: N_2 \to M$ is a homotopy equivalence, then* (1) $H_*(M, N_1)=0$ *and* (2) kernel$(i_{1\#})$ *is perfect.*

Proof. Let $(\tilde{M}, \tilde{N}_1, \tilde{N}_2)$ be as in the proof of Lemma 2.4. Then $\tilde{i}_2: \tilde{N}_2 \to \tilde{M}$ is a proper homotopy equivalence. By duality, $H_*(\tilde{M}, \tilde{N}_1) \cong H_c^*(\tilde{M}, \tilde{N}_2)$, where H_c^* denotes cohomology with compact supports [10, Proposition 7.2], and $H_c^*(\tilde{M}, \tilde{N}_2)=0$ because $\tilde{i}_2$ is a proper homotopy equivalence. In particular, $H_1(\tilde{N}_1)=0$. As in the proof of Lemma 2.4, kernel$(i_{1\#})$ is perfect. This proves (2); (1) follows from standard duality arguments. □

3. Homology cobordant manifolds that are not laminated cobordant. In this section we produce an example showing that the existence of a homology cobordism (M, N_1, N_2) is not sufficient to guarantee the existence of a laminated cobordism (M', N_1, N_2) having the same boundary components.

EXAMPLE 3.1: a cobordism (M, N_1, N_2) satisfying $H_*(M, N_k)=0$ for $k=1,2$ such that there is no laminated cobordism (M', N_1, N_2).

Assume $n \geq 5$. We construct (M, N_1, N_2) in the $(n+1)$-sphere, S^{n+1}. Let $S^n \subset S^{n+1}$ be an equator and $S^n \times [-3,3] \subset S^{n+1}$ a bicollar on $S^n = S^n \times \{0\}$. Let $f_1: S^{n-2} \times B^2 \to S^n$ be an unknotted PL embedding, $f_2: S^{n-2} \times B^2 \to f_1(S^{n-2} \times \operatorname{Int} B^2)$ a PL (possibly knotted) embedding inducing isomorphisms on integral homology, and C_k the closure of the complement of $f_k(S^{n-2}\times B^2)$ in S^n $(k=1,2)$. Note that C_1 is PL homeomorphic to $S^1\times B^{n-1}$ and, by duality, the inclusion $C_1\to C_2$ induces homology isomorphisms.

In $S^n\times[-3,3]\subset S^{n+1}$ define

$$N_1=(f_1(S^{n-2}\times \mathrm{Bd}\, B^2)\times[-1,1])\cup(C_1\times\{-1,1\})$$

and

$$N_2=(f_2(S^{n-2}\times \mathrm{Bd}\, B^2)\times[-2,2])\cup(C_2\times\{-2,2\}).$$

Now let M denote the closure of the component of $S^{n+1}-(N_1\cup N_2)$ bounded by both N_1 and N_2. For $k=1,2$ observe that N_k is the double of C_k along its boundary and, consequently, N_1 is homeomorphic to $S^1\times S^{n-1}$. Also, M is the closure of $(C_2\times[-2,2])-(C_1\times[-1,1])$. Of course the inclusion $C_1\times[-1,1]\to C_2\times[-2,2]$ induces homology isomorphisms, so excision shows $H_*(M,N_1)=0$; by duality again, $H_*(M,N_2)=0$.

The example arises by letting f_2 be the knotted embedding of Stallings [15, Theorem V] for which $\pi_1(C_2)\cong Z$ and $\pi_2(C_2)\neq 0$. Then $\pi_1(N_2)\cong Z$ as well and $\pi_2(N_2)\neq 0$, since C_2 is a retract of N_2.

CLAIM. There exists no laminated cobordism (M',N_1,N_2).

Suppose otherwise. By Lemma 2.3, $i_{k\#}:\pi_1(N_k)\to\pi_1(M')$ is a surjection $(k=1,2)$. Since $\pi_1(N_k)\cong Z$ contains no non-trivial perfect subgroups, Lemma 2.4(a) attests that $i_{k\#}$ is also an injection. Thus, Lemma 2.4(b) indicates that (M',N_1,N_2) is an h-cobordism. In particular N_1 and N_2 are homotopy equivalent, which is patently absurd, because $\pi_2(N_1)\cong\pi_2(S^1\times S^{n-1})\cong 0$ and $\pi_2(N_2)\neq 0$.

4. Extended mapping cylinders of acyclic maps. Here the goal is to construct laminated cobordisms as mapping cylinders of certain acyclic maps.

A compact subset K of an ANR X is *strongly Z-acyclic* if each neighborhood U of X contains another neighborhood V of X such that the inclusion-induced $H_*(V;Z)\to H_*(U;Z)$ is trivial. A map $f:X\to Y$ between ANR's is *acyclic* if $f^{-1}(y)$ is strongly Z-acyclic for each $y\in Y$. A compactum K is *nearly 1-movable* if the following holds for some (and hence for every) embedding of K in an ANR X:

> Each neighborhood U of K contains another neighborhood V of K such that for every loop $L:\mathrm{Bd}\, B^2\to V$ and for every neighborhood W of K there exists a finite collection of pairwise disjoint disks $\{B_i\}$ in $\mathrm{Int}\, B^2$ and there exists an extension L' of L to $L':(B^2-\bigcup \mathrm{Int}\, B_i,\bigcup \mathrm{Bd}\, B_i)\to(U,W)$.

Less formally, this amounts to the assertion that every loop in V is homotopic in U to a product of conjugates of loops in W. Finally, given a map $f:X\to Y$, we define the *extended mapping cylinder of f*, $M_e(f)$, to be $X\times[-1,0]\cup_{\tilde f} Y\times[0,1]$, where $X\times\{0\}$ and $Y\times\{0\}$ are identified via the map $\tilde f(x,0)=(f(x),0)$. Note that this is simply the standard mapping cylinder with a collar attached to Y.

The following should be transparent.

PROPOSITION 4.1. *If* $f: N_1 \to N_2$ *is a map such that* $M_e(f)$ *is a compact* $(n+1)$*-manifold with boundary, then* $(M_e(f), N_1 \times \{-1\}, N_2 \times \{1\})$ *is a laminated cobordism.*

The next result records conditions under which extended mapping cylinders are manifolds.

THEOREM 4.2. *Let* $f: N_1 \to N_2$ *be a surjective map between closed n-manifolds* $(n > 3)$ *such that each preimage* $f^{-1}(y)$ *is nearly* 1*-movable and of dimension at most* $n-2$. *Then the extended mapping cylinder* $M_e(f)$ *is a manifold if and only if f is acyclic.*

Proof. Assume $M_e(f)$ is a manifold. Let V be an open ball neighborhood of a point $y \in N_2$. Then $M_e(f \mid f^{-1}(V))$ is also a manifold and the inclusion

$$f^{-1}(V) \times [-1, 0) \to (f^{-1}(V) \times [-1, 0)) \cup (V \times \{0\})$$

induces homology isomorphisms [7, Corollary 6.3]. Thus, $f^{-1}(V)$ has the homology of an open n-ball and so f is acyclic.

Now suppose f is acyclic. According to a result of Borsuk [2, Theorem 9.1, p. 116], $M_e(f)$ is an ANR. That it is a generalized n-manifold follows from the Vietoris–Begle mapping theorem [1] and the observation that $M_e(f)$ is the acyclic image of $N_1 \times [-1, 1]$. Applying Edwards' cell-like approximation theorem [11] and a resolution theorem (either [4, Theorem 7.2] or the more general result of Quinn [17; 18], which does apply because $M_e(f)$ obviously contains Euclidean patches) in order to prove $M_e(f)$ is a manifold, we need only show that it satisfies the disjoint disks property. Indeed, verifying the following local version of the disjoint disks property is sufficient ([3, p. 107]).

> For each neighborhood U of $y \in N_2$ there exists a smaller neighborhood V of y such that any two disjoint loops L_1, L_2: Bd $B^2 \to f^{-1}(V) \times [-1, 0)$ can be extended to $L_2', L_2': B^2 \to f^{-1}(U) \times [-1, 0) \cup U \times \{0\}$ having disjoint images.

Fix $y \in U \subset N_2$ and choose V with $y \in V \subset U$ such that V contracts in U and $f^{-1}(V)$ satisfies the hypothesis of nearly 1-movability for $f^{-1}(y)$ in $f^{-1}(U)$. As an easy consequence, L_1 is null-homotopic in $(f^{-1}(U) \times [-1, 0)) \cup \{(y, 0)\}$. Since dimension$(f^{-1}(y)) \leq n-2$, we can assume that, after slight adjustment,

$$L_2(\text{Bd}\, B^2) \cap f^{-1}(y) \times [-1, 0) = \varnothing$$

and we can then extend L_2 to a map

$$L_2^*: (\text{Bd}\, B^2 \times [0, 1], \text{Bd}\, B^2 \times \{1\}) \to (f^{-1}(V) \times [-1, 0) \cup V \times \{0\}, V \times \{0\})$$

whose image misses $L_1(B^2)$. Finally, we obtain L_2' by contracting $L_2^* \mid \text{Bd}\, B^2 \times \{1\}$ in $U \times \{0\}$ missing $(y, 0)$.

REMARK. The requirement $\dim f^{-1}(y) \leq n-2$ in Theorem 4.2 is necessary, since a map $f: N_1 \to N_2$ collapsing out the spine of a noncontractible homology cell leads to a non-manifold mapping cylinder $M_e(f)$. Whether the point inverses of an acyclic map $f: N_1 \to N_2$ must be nearly 1-movable is a previously identified open question [9, p. 300].

We shall exploit Theorem 4.2 in constructing laminated cobordisms (M, N_1, N_2) for which the inclusion $i_2: N_2 \to M$ is a homotopy equivalence. For any such cobordism, Lemma 2.5 ensures that the kernel of $i_{1\#}: \pi_1(N_1) \to \pi_1(M)$ is a perfect normal subgroup of $\pi_1(N_1)$. Since $i_{1\#}$ is a surjection of finitely presented groups, its kernel is the normal closure of a finite set [20, Lemma 3.11]. The main theorem of this section deals with the special case when this kernel is the normal closure of a finitely generated perfect group.

THEOREM 4.3. *Let N_1 be a closed n-manifold $(n \geq 5)$ such that $\pi_1(N_1)$ contains a finitely generated perfect subgroup P. Then there exists a laminated cobordism (M, N_1, N_2) where M is the extended mapping cylinder of an acyclic map $f: N_1 \to N_2$ between n-manifolds and where $\pi_1(N_2)$ is isomorphic to $\pi_1(N_1)/[P]$.*

Here $[P]$ denotes the normal closure of P in $\pi_1(N_1)$.

Before proving Theorem 4.3, we reproduce some additional group-theoretic nomenclature. A presentation of a group involving k generators and s relations is said to have *deficiency* $s-k$; the *deficiency of the group* is defined to be the minimum deficiency among all its presentations.

PROPOSITION 4.4. *Every finitely generated perfect group P is the homomorphic image of a finitely presented perfect group having deficiency* 0.

Proof. We give Hausmann's proof [13, §2.1]. Take a presentation

$$\langle x_1, \ldots, x_k : r_1, \ldots, r_s \rangle$$

(s possibly infinite) of P. Since P is perfect, each x_i can be written as a commutator c_i, where c_i is regarded as a word in the free group on the generators $x_1, \ldots, x_k$. Let

$$P' = \langle x_1, \ldots, x_k : x_1^{-1}c_1, \ldots, x_k^{-1}c_k \rangle.$$

It is easy to check that P' is perfect and that P is isomorphic to

$$\langle x_1, \ldots, x_k \mid x_1^{-1}c_1, \ldots, x_k^{-1}c_k, r_1, \ldots, r_s \rangle.$$

This completes the proof of Proposition 4.4. □

Proof of Theorem 4.3. Choose P' as in Proposition 4.4, let R^* be a 2-dimensional finite CW-complex associated with the given presentation of P', and let R be a finite simplicial 2-complex homotopy equivalent to R^*.

A straightforward Mayer–Vietoris argument shows R^* (and R) to be homologically trivial: If T denotes the 1-skeleton of R^* and $E_1, \ldots, E_k$ the attached 2-cells, one can see (a) that adjunction of E_i to $T \cup E_1 \cup \cdots \cup E_{i-1}$ must reduce the minimal number of generators required for H_1 by one in order to bring about the obvious $H_1(R^*) \cong 0$, and (b) from an examination of the Mayer–Vietoris sequence, that $H_1(T \cup E_1 \cup \cdots \cup E_i)$ is free of rank $k-i$ while $H_2(T \cup E_1 \cup \cdots \cup E_i)$ is trivial.

Name a PL embedding $h: R \to N_1$ such that $h_\#: \pi_1(R) \to \pi_1(N_1)$ has image equal to P. Let Q be a regular neighborhood of $h(R)$ in N_1. Then Σ^{n-1}, the boundary of Q, is an homology $(n-1)$-sphere (i.e., $H_*(\Sigma^{n-1};Z)=H_*(S^{n-1};Z)$). Because Q has the 2-complex $h(R)$ for a spine and $n\geq 5$, general position ensures that the inclusion-induced homomorphism $\pi_1(\Sigma^{n-1})\to\pi_1(N_1)$ has P as its image.

Now we decompose N_1 into acyclic compacta. Choose a bicollar $\Sigma^{n-1}\times[0,1]$ on Σ^{n-1} and a Cantor set C in $(0,1)$. Let F^{n-1} be the closure of the complement of a PL $(n-1)$-cell in Σ^{n-1} and K be an acyclic 2-complex in F^{n-1} as above with $\pi_1(K)\to\pi_1(F^{n-1})$ surjective ($n=5$ demands more care). Define a decomposition $G(K)$ of N_1 having as its nondegenerate elements the sets $K\times\{c\}$, $c\in C\subset(0,1)$. Topologically these nondegenerate elements of $G(K)$ are all acyclic 2-complexes in $\Sigma^{n-1}\times[0,1]$. Exactly as in [6], the decomposition space $N_2=N_1/G(K)$ is an n-manifold. Furthermore, $\pi_1(N_2)=\pi_1(N_1)/[P]$, essentially because N_2 contains a natural copy of the cone on K. Consequently, for the decomposition map f: $N_1\to N_2$, Theorem 4.2 and Proposition 4.1 demonstrate that $M=M_e(f)$ is a manifold and (M,N_1,N_2) is a laminated cobordism. □

COROLLARY 4.5. *For any homology n-sphere* Σ^n $(n\geq 5)$, *there exists a laminated cobordism* (M,Σ^n,S^n).

Proof. Since $\pi_1(\Sigma^n)$ is a finitely presented perfect group, we can apply Theorem 4.3 with $P=\pi_1(\Sigma^n)$. The resulting manifold N_2 then is a simply connected homology n-sphere, which is equivalent to S^n by the topological Poincaré conjecture [16].

COROLLARY 4.6. *For any two homology n-spheres* Σ_1^n *and* Σ_2^n $(n\geq 5)$, *there exists a laminated cobordism* $(M^*,\Sigma_1^n,\Sigma_2^n)$.

In light of the comments preceding Theorem 4.3, a possible improvement to the theorem would come about if the hypothesis that the perfect group P be finitely generated could be replaced by the weaker assumption that P be the normal closure of a finite set. In fact, for any acyclic map $f: N_1\to N_2$ between closed, orientable n-manifolds, the kernel of $f_\#: \pi_1(N_1)\to\pi_1(N_2)$ is precisely a perfect normal subgroup of $\pi_1(N_1)$ which is the normal closure there of a finite set. To see why, consider the extended mapping cylinder $M_e(f)$. The proof of Theorem 4.2 demonstrates that $M_e(f)$ is a generalized manifold and, of course, the inclusion $N_2\times\{1\}\to M_e(f)$ is a homotopy equivalence. Thus, the duality argument of Lemma 2.5, which applies equally well in generalized manifolds, yields that kernel$(f_\#)$ is perfect.

We summarize these observations in a question regarding acyclic maps.

QUESTION 4.7. Suppose N_1 is a closed n-manifold $(n\geq 5)$ and P is a perfect normal subgroup of $\pi_1(N_1)$ that is the normal closure of a finite set but not the normal closure of a finitely generated perfect subgroup. Does there exist an acyclic map $f: N_1\to N_2$ to a closed n-manifold N_2 for which kernel$(f_\#)=P$?

The next result indicates that if the question has an affirmative answer, then the point inverses must be somewhat pathological.

PROPOSITION 4.8. *Suppose* $f: N_1\to N_2$ *is an acyclic map between closed n-manifolds such that* $f^{-1}(y)$ *is an ANR for each* $y\in N_2$. *Then* kernel$(f_\#)$ *is the normal closure of a finitely generated perfect group.*

Proof. There is a finite collection of pairs (y_i, U_i) such that $y_i \in N_2$, U_i is a neighborhood of $f^{-1}(y_i)$ which deformation retracts to $f^{-1}(y_i)$ in N_1, and $\{U_i\}$ is a cover of N_1. Join each $f^{-1}(y_i)$ to a basepoint x_0 by an arc α_j such that $\alpha_i \cap (\alpha_j \cup f^{-1}(y_j)) = \{x_0\}$ for $i \neq j$. Let $Y = \bigcup_j (f^{-1}(y_j) \cup \alpha_j)$ and let $i: Y \to N$ be the inclusion. It is straightforward to check that $\pi_1(Y)$ is a finitely presented perfect group and kernel$(f_\#)$ is the normal closure in $\pi_1(N_1)$ of the image of $i_\#: \pi_1(Y) \to \pi_1(N)$. □

The hypothesis in Proposition 4.8 that each $f^{-1}(y)$ be an ANR can be weakened to the requirement that each $f^{-1}(y)$ be pointed 1-movable.

5. A resolution theorem. Applying the results of Section 4 and some simple-homotopy theory, we restructure certain given laminations as extended mapping cylinders, up to h-cobordisms of acyclic maps. Let (W, M_1, M_2) be a relative cobordism (M_k possibly with boundary) such that W is a compact $(n+1)$-manifold with boundary, $n \geq 5$, the inclusions $M_k \to W$ are homotopy equivalences, and the closure of Bd $W - (M_1 \cup M_2)$ is homeomorphic to Bd $M_k \times I$ $(k = 1, 2)$. The well-known relative s-cobordism theorem states that associated with (W, M_1, M_2) is a torsion element τ of the Whitehead group of $\pi_1(W)$ and that W is a product $M_k \times I$ if and only if $\tau = 0$ (e.g., [19, Chapter 6]). Also well known is that relative h-cobordisms of arbitrary torsion can be constructed by attaching a finite number of 2- and 3-handles to a given product.

These results also provide information about cobordisms (M, N_1, N_2) in which only the one inclusion $N_2 \to M$ is assumed to be a homotopy equivalence.

THEOREM 5.1. *Suppose (M, N_1, N_2) and (M', N_1, N_2') are $(n+1)$-dimensional cobordisms ($n \geq 5$, Bd $N_1 = \varnothing$) such that the inclusions $N_2 \to M$ and $N_2' \to M'$ are homotopy equivalences and the inclusion-induced homomorphisms*

$$\pi_1(N_1) \to \pi_1(M) \quad \textit{and} \quad \pi_1(N_1) \to \pi_1(M')$$

have equal kernels. Then M is homeomorphic to $M' \cup_{N_2'} M''$, where (M'', N_2', N_2'') is an h-cobordism.

Proof. By Lemma 2.5 the inclusions $j: N_1 \to M$ and $j': N_1 \to M'$ induce isomorphisms on n-dimensional homology; hence, a result of Epstein [12], applied to obvious maps $N_1 \to N_2$ and $N_1 \to N_2'$, shows that j, j' induce surjections of fundamental groups. (In case N_2 or N_2' is non-orientable, pass to orientable double covers to achieve the conclusion.) As a result, the natural homomorphisms

$$\pi_1(M) \to \pi_1(M \cup_{N_1} M') \quad \text{and} \quad \pi_1(M') \to \pi_1(M \cup_{N_1} M')$$

are isomorphisms, which implies that the universal covers of M and of M' include naturally in the universal cover of $M \cup_{N_1} M'$. That $(M \cup_{N_1} M', N_2, N_2')$ is an h-cobordism then follows from a duality argument involving its universal cover, like the one set forth in [8, Lemma 3.3].

Let $W = (M \cup_{N_1} M') \times I$. Taking appropriate collars on N_1 and N_2 we can view W as a relative h-cobordism (W, M_1, M_2), where $M_1 = M \times \{0\}$,

$$M_2 = [M' \times \{0\}] \cup [(M \cup_{N_1} M') \times \{1\}] \cup [N_2' \times I],$$

$\operatorname{Bd} M_k$ is the disjoint union of N_1 and N_2, and the closure of $\operatorname{Bd} W-(M_1 \cup M_2)$ is the product of $\operatorname{Bd} M_k$ with an interval. Let τ be the torsion element associated with (W, M_1, M_2). By attaching handles to $N_2 \times I \times I$ we can build a relative h-cobordism (W', M_1', M_2') with torsion $-\tau$, where M_1' is homeomorphic to $N_2 \times I$, $\operatorname{Bd} M_2'$ is homeomorphic to two copies of N_2, and the closure of $\operatorname{Bd} W'-(M_1' \cup M_2')$ is topologically $\operatorname{Bd} M_1' \times I$. Attach (W, M_1, M_2) to (W', M_1', M_2') in the obvious way along copies of $N_2 \times I$. The sum theorem [5, Theorem 23.1] attests that the resulting relative cobordism has trivial torsion and, therefore, is a product. Observe that one end is homeomorphic to M and that the other end is homeomorphic to M' plus an h-cobordism (M'', N_2, N_2'') attached along N_2. □

The primary result of this section follows directly from Theorem 5.1 and Theorem 4.3.

THEOREM 5.2. *Suppose (M, N_1, N_2) is an $(n+1)$-dimensional cobordism $(n \geq 5)$ such that $i_2: N_2 \rightarrow M$ is a homotopy equivalence and the kernel of $i_{1\#}$: $\pi_1(N_1) \rightarrow \pi_1(M)$ equals the normal closure in $\pi_1(M)$ of a finitely generated perfect group. Then there exists an acyclic map $f: N_1 \rightarrow N_2'$ to a closed n-manifold N_2' such that M is homeomorphic to $M_e(f) \cup_{N_2' \times \{1\}} M'$, where $(M', N_2' \times \{1\}, N_2)$ is an h-cobordism.*

Proof. Theorem 4.3 gives f and N_2', and Theorem 5.1, applied to (M, N_1, N_2) and $(M_e(f), N_1, N_2' \times \{1\})$, does the rest. □

COROLLARY 5.3. *Under the hypotheses of Theorem 5.2, M admits a lamination G with $N_1, N_2 \in G$.*

We close with a question intimately related to Question 4.7.

QUESTION 5.4. If (M, N_1, N_2) is a cobordism such that $i_2: N_2 \rightarrow M$ is a homotopy equivalence, does M admit a lamination?

REFERENCES

1. E. G. Begle, *The Vietoris mapping theorem for bi-compact spaces,* Ann. of Math. (2) 51 (1950), 534–543.
2. K. Borsuk, *Theory of retracts,* PWN, Warsaw, 1967.
3. J. W. Cannon, *Shrinking cell-like decompositions of manifolds. Codimension three,* Ann. of Math. (2) 110 (1979), 83–112.
4. J. W. Cannon, J. L. Bryant, and R. C. Lacher, *The structure of generalized manifolds having nonmanifold set of trivial dimension.* Geometric Topology (Athens, Ga., 1977), 261–300, Academic Press, New York, 1979.
5. M. M. Cohen, *A course in simple homotopy theory,* Springer, Berlin, 1973.
6. R. J. Daverman, *Embedding phenomena based on decomposition theory: Wild Cantor sets satisfying strong homogeneity properties,* Proc. Amer. Math. Soc. 75 (1979), 177–182.
7. ———, *Decompositions of manifolds into codimension one submanifolds,* Comp. Math. 55 (1985), 185–207.

8. R. J. Daverman and F. C. Tinsley, *Laminated decompositions involving a given submanifold,* Topology Appl. 20 (1985), 107–119.
9. R. J. Daverman and J. J. Walsh, *Acyclic decompositions of manifolds,* Pacific J. Math. 109 (1983), 291–303.
10. A. Dold, *Lectures on algebraic topology,* Springer, Berlin, 1972.
11. R. D. Edwards, *The topology of manifolds and cell-like maps.* Proceedings of the International Congress of Mathematicians (Helsinki, 1978), 111–127, Acad. Sci. Fennica, Helsinki, 1980.
12. D. B. A. Epstein, *The degree of a map,* Proc. London Math. Soc. (3) 16 (1966), 369–383.
13. J. C. Hausmann, *Homological surgery,* Ann. of Math. (2) 104 (1976), 573–584.
14. ———, *Manifolds with a given homology and fundamental group,* Comment. Math. Helv. 53 (1978), 113–134.
15. M. A. Kervaire, *On higher dimensional knots.* Differential and combinatorial topology (A Symposium in honor of Marston Morse), 105–119, Princeton Univ. Press, Princeton, N.J., 1965.
16. M. H. A. Newman, *The engulfing theorem for topological manifolds,* Ann. of Math. (2) 84 (1966), 555–571.
17. F. Quinn, *Resolutions of homology manifolds and the topological characterization of manifolds,* Invent. Math. 72 (1983), 267–284.
18. ———, *An obstruction to the resolution of homology manifolds,* Invent. Math., to appear.
19. C. P. Rourke and B. J. Sanderson, *Introduction to piecewise linear topology,* Springer, Berlin, 1972.
20. L. C. Siebenmann, Ph.D. Thesis, Princeton Univ., 1965.
21. J. H. C. Whitehead, *Combinatorial homotopy: I,* Bull. Amer. Math. Soc. 55 (1949), 213–245.

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