REPRESENTATIONS OF THE MAUTNER GROUP AND COCYCLES OF AN IRRATIONAL ROTATION

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1. Introduction. The five-dimensional Lie group known as the Mautner group is the smallest connected Lie group whose unitary dual remains unknown. Its representation theory is linked to the cohomology of functions on the circle under an irrational rotation. In this paper, we use cocycles to produce a five parameter family of representations of the Mautner group. This family completes all the known families in a natural way and extends them to reveal the dependence of the representation theory of the group on the angle defining it.

The Mautner group M is ordinarily defined to be the set $\mathbf{C}\times\mathbf{C}\times\mathbf{R}$ together with the multiplication rule

$$(z,w,t)(z',w',t')=(z+e(t/2\pi)z',w+e(t)w',t+t'),$$

where $e(x)=e^{2\pi ix}$; that is, M is the semidirect product of two-dimensional complex space with the real line, where the real number t acts on $\mathbf{C}^2$ by the matrix

$$\begin{bmatrix} e(t/2\pi) & 0 \\ 0 & e(t) \end{bmatrix}.$$

If α and β are nonzero real numbers, a similar semidirect product $M_{\alpha,\beta}$ can be defined, where the real number t acts this time by the matrix

$$\begin{bmatrix} e(\alpha t) & 0 \\ 0 & e(\beta t) \end{bmatrix}.$$

If the quotient $\theta=\beta/\alpha$ is irrational then $M_{\alpha,\beta}$ exhibits the "Winding Line" phenomenon in its structure, so that all these groups seem to be analogous from the point of view of ergodic theory. In addition, the primitive ideal spaces of the $M_{\alpha,\beta}$'s, for β/α irrational, are all identical: $\mathrm{Prim}(M_{\alpha,\beta})$ is the union of the real line (characters) with the set of nondegenerate tori $S_\rho\times S_r$ in $\mathbf{C}^2$ (normal factors).

It would be reasonable to expect the representation theory of all these groups to be alike as well. For any irrational β/α, it is known that $M_{\alpha,\beta}$ is not of type I, so that its unitary dual (equivalence classes of irreducible unitary representations) cannot be parameterized in a smooth way. Further, the known parameterized families of representations of the ordinary Mautner group (see [2], [6], [3]) can be easily transferred to an arbitrary $M_{\alpha,\beta}$. However, we have found by extending these families that the representation theory of $M_{\alpha,\beta}$ depends substantially on the number theoretic properties of the quotient β/α. We present here a five- (real) parameter family of formulae defining irreducible unitary representations of the group $M_{\alpha,\beta}$ for all α and β with β/α irrational. The unitary

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equivalence among these representations is described by means of an explicit (non-smooth) relation on the parameters; this relation differs depending on the continued fraction expansion of the number β/α.

For clarity, we begin with the following observation.

1.1. PROPOSITION. *$M_{\alpha,\beta}$ is isomorphic to $M_{\alpha',\beta'}$ if and only if $\beta/\alpha=\beta'/\alpha'$.*

Proof. The "if" part is easy. Conversely, if $M_{\alpha,\beta}$ is isomorphic to $M_{\alpha',\beta'}$, then so are their Lie algebras. The generator of the subgroup **R** in $M_{\alpha,\beta}$, and its image under the isomorphism into the Lie algebra of $M_{\alpha',\beta'}$, must have the same eigenvalue in their adjoint representations. A simple linear algebra computation now gives the proposition. □

In view of the preceding, we may restrict our attention to the groups $M_{1,\theta}$, which we simply denote by M_θ. The ordinary Mautner group M is then $M_{2\pi}$.

We produce our family of representations, as in [2], by studying a particular Little group D_θ known as the discrete Mautner group (see §2). We use Mackey–Ramsay theory to reduce the problem of finding irreducible representations of D_θ (and thus M_θ) to that of analyzing the irreducible cocycles of rotation by $-\theta$. In Section 3, we use the results of [8] to establish the cohomology relations among the elements in a certain parameterized family of cocycles, and hence to establish the unitary equivalence among the corresponding representations. All the cocycles in Section 3 are one-dimensional. In Section 4, we use these one-dimensional results to obtain the cohomology relations among a parameterized family of irreducible n-dimensional cocycles.

In [2] there was given a four-parameter family of irreducible representations of $M_{2\pi}$. Later, Kawakami [6] showed that the integer parameter d of [2] could be extended to all rational numbers a/b with the same equivalence holding among the parameters, thus including also the case $d=1/2$ which followed from earlier work of Brown [4]. In [3], higher-dimensional cocycles were given which generalized the one-dimensional results of [2] in a natural way, but which did not include Kawakami's rationally parameterized family. The one-dimensional cocycles in this paper extend Kawakami's parameter a/b to all real numbers and introduce yet another independent real parameter. The higher-dimensional cocycles in this paper do the same with the integer parameter of [3], thus bringing together and completing all the known families of representations of the Mautner group.

We are aware that constructing larger and larger parameterized families of irreducible representations does not inevitably lead to a description of all irreducible representations of M_θ. We are aware, in fact, that many experts feel that such a complete description will be impossible to give. However, these formulae are interesting and suggestive in their own right, especially because of the connection they reveal between the equivalence of representations and the number theory of the rotation angle.

2. The correspondence between representations of M_θ and cocycles of the irrational rotation $-\theta$. In this section we describe the Mackey–Ramsay theory as it applies to the group M_θ. Because our goal is to indicate explicit formulae for representations of this group, which we shall do by using explicit formulae for cocycles of the irrational rotation $-\theta$, we want to make the correspondence between these two entities equally explicit.

Fix an irrational number θ, and let N denote the closed normal subgroup of M_θ consisting of the triples $(z, 0, 0)$ for z in $\mathbf{C}$. Then N is regularly imbedded in M_θ (see [7]) and the M_θ-orbits in $\hat{N}$ are just the circles in the complex plane $\hat{\mathbf{C}}$. For each nonzero element ϕ of $\hat{N}$, the stability subgroup H of M_θ for ϕ consists of the triples (z, w, n), z and w in $\mathbf{C}$ and n an integer. Hence, the Little group associated to ϕ is the discrete Mautner group D_θ, that is, the set of pairs (w, n) with w in $\mathbf{C}$ and n in Z with multiplication given by

$$(w,n)(w',n') = (w + e(n\theta)w', n+n').$$

2.1. THEOREM (Mackey). *Let V be a unitary representation of D_θ, let ρ be a positive number, and define the representation $S = S^{(\rho,V)}$ of H by $S_{(z,w,n)} = e(\langle \rho, z\rangle)V_{(w,n)}$, where $\langle\ ,\ \rangle$ denotes the real inner product in $\mathbf{C}$. Define $U^{(\rho,V)}$ to be the induced representation $\mathrm{IND}_H^{(M_\theta)}(S^{(\rho,V)})$. Then:*

(i) *$U^{(\rho,V)}$ is irreducible if and only if V is irreducible.*
(ii) *$U^{(\rho,V)}$ is equivalent to $U^{(\rho',V')}$ if and only if $\rho = \rho'$ and V is equivalent to V'.*
(iii) *Every irreducible representation of M_θ which is not trivial on N is equivalent to some $U^{(\rho,V)}$.*

Proof. This is just the Mackey procedure. See [7]. □

2.2. DEFINITION. Let K be a Hilbert space and let $U(K)$ denote the group of unitary operators on K. A "*$U(K)$-valued cocycle of the irrational rotation $-\theta$*" is a map R of $[0,1)\times Z$ into $U(K)$ which satisfies the *cocycle identity* $R(x, m+n) = R(x,m)R((x-m\theta), n)$, where addition is mod 1 in the first variable. The *dimension* of the cocycle is the dimension of the Hilbert space K, and the *coefficient group* for the cocycle is the group $U(K)$. An *intertwining* between two cocycles R and R' having the same coefficient group $U(K)$ is a map A of $[0,1)$ into the bounded operators on K satisfying the *intertwining equation* $A(x)R(x,m) = R'(x,m)A(x-m\theta)$ for all x in $[0,1)$ and m in Z. Two cocycles are called *cohomologous* if they have the same coefficient group and there exists a unitary operator-valued intertwining between them. (When necessary, we will emphasize the dependence on θ by using the term *θ-cohomologous*.) A single cocycle is called a *coboundary* if it is cohomologous to a cocycle which is constantly the identity; it is called *irreducible* if the only intertwinings between it and itself are the constant scalar-operator-valued functions.

If μ is a measure on $[0,1)$, we may formulate the measurable versions of these definitions. Cocycles and intertwinings will be required to be measurable maps, and the cocycle identity and intertwining equation must only hold μ almost everywhere.

Given a measure μ on $[0,1)$ whose null sets are preserved under rotation by θ, let $d_\mu(x,m)$ be the associated Radon–Nikodym function:

$$\int_0^1 f(x)\,d\mu(x) = \int_0^1 f(x-m\theta)\,d_\mu(x,m)\,d\mu(x).$$

2.3. THEOREM (Ramsay). *Let μ be an ergodic measure on* $[0,1)$ *whose null sets are preserved under rotation by θ; let R be a μ-measurable, $U(K)$-valued cocycle of the irrational rotation $-\theta$; and let r be a positive real number. Define the representation $V \equiv V^{(r,R,\mu)}$ of D_θ, acting in $L^2(S^1, K, \mu)$, by*

$$[V_{(w,m)}^{(r,R,\mu)} f](x) = (d_\mu(x,m))^{1/2} e(\langle re(x), w\rangle) R(x,m) f(x-m\theta).$$

Then:

(i) *$V^{(r,R,\mu)}$ is irreducible if and only if R is irreducible.*

(ii) *$V^{(r,R,\mu)}$ is equivalent to $V^{(r',R',\mu')}$ if and only if $r=r'$, R and R' are cohomologous, and μ and μ' are equivalent.*

(iii) *Every irreducible representation of D_θ, which is not trivial on the normal subgroup of pairs $(w, 0)$ for w in $\mathbf{C}$, is equivalent to some $V^{(r,R,\mu)}$.*

Proof. See [9]. □

3. A parameterized family of cocycles and their equivalence relations. We present here a three-parameter family of one-dimensional, Lebesgue measurable cocycles of the irrational rotation $-\theta$. If θ has bounded partial quotients in its continued fraction expansion, we are able to describe analytically the cohomology among these cocycles in terms of the parameters. Thus Theorems 2.1 and 2.3 (with μ = Lebesgue measure) can be used to construct a five- (real) parameter family of irreducible representations of M_θ in which the unitary equivalence is given in terms of the parameters. For θ with unbounded partial quotients, we can describe the cohomology completely only for two of the parameters. For the third parameter, we show that there are uncountably many more equivalence relations than in the bounded partial quotients case, and that these equivalences depend on the continued fraction expansion of θ.

To define a cocycle R, it is sufficient to specify the function $R(x,1)$ on $[0,1)$; the cocycle identity then gives

$$R(x,n) = \begin{cases} R(x,1)R(x-\theta,1)\cdots R(x-(n-1)\theta,1) & \text{if } n>0 \\ 1 & \text{if } n=0 \\ R^{-1}(x+\theta,1)R^{-1}(x+2\theta,1)\cdots R^{-1}(x-n\theta,1) & \text{if } n<0. \end{cases}$$

The cocycle identity can also be applied to the intertwining equation to reduce questions of cohomology and irreducibility to questions concerning only the function $R(x,1)$. Using the ergodicity of an irrational rotation, we see that all one-dimensional cocycles are irreducible. Two one-dimensional cocycles R and R' are cohomologous if and only if their quotient is a coboundary, that is, if there exists a nonzero Lebesgue measurable function g mapping $[0,1)$ to the unit circle such that

$$g(x) = g(x-\theta)R'(x,1)/R(x,1).$$

We call two functions cohomologous if the cocycles they define are cohomologous.

For $x \in \mathbf{R}$, define the function e_s by $e_s(x) = e(sx)$. For f a function on $[0,1)$ and $t \in \mathbf{R}$, let ${}^t f$ denote the translate of f by t (mod 1) so that ${}^t f(x) = f(x+t)$. With this notation, our parameterized family of cocycles is given by the following.

3.1. DEFINITION. For real numbers λ, s, and t, define the cocycle $R^{(\lambda,s,t)}$ by the formula

$$R^{(\lambda,s,t)}(x,1)=e(\lambda)\,{}^t e_s(x).$$

If we let $f_{t,s}$ be the function on $[0,1)$ defined by

$$f_{t,s}(x)=\begin{cases}1 & \text{for } x\in[0,1-\{t\}) \\ e(s) & \text{for } x\in[1-\{t\},1),\end{cases}$$

then we have (recalling that arithmetic in the argument of these functions is mod 1)

$${}^t e_s=(e(s\{t\}))(f_{t,-s})(e_s),$$

where $\{\ \}$ = fractional part. This form is the key to using the results of [8] to determine the cohomology among the $R^{(\lambda,s,t)}$. Our first step is the following.

3.2. THEOREM. *Let s, s', λ and λ' be arbitrary real numbers. Then $R^{(\lambda,s,0)}$ is cohomologous to $R^{(\lambda',s',0)}$ if and only if $s=s'$ and $\lambda-\lambda'=p+q\theta$ for some integers p and q.*

Proof. The "if" direction is established easily by the equation $e_q(x)=e(q\theta)e_q(x-\theta)$. Conversely, we see that $e(\lambda)e_s$ is cohomologous to $e(\lambda')e_{s'}$ implies that $e_{s-s'}$ is cohomologous to a scalar, and thus (since cohomology is preserved by translation) to any translate of itself. Thus for an arbitrary real t, we have that $e_{s-s'}$ is cohomologous to ${}^t e_{s-s'}=(e((s-s')\{t\}))(e_{s-s'})(f_{t,s'-s})$. But then $f_{t,s'-s}$ is cohomologous to a scalar for all t. If $s'-s$ is not an integer, this is impossible by [8] (or [10] and [11]); if $s'-s$ is a nonzero integer it is impossible since the scalar $e((s-s')\{t\})$ depends on t. Thus $s=s'$, and so $\lambda-\lambda'$ must satisfy $g(x)=e(\lambda-\lambda')g(x-\theta)$ for some nonzero measurable g. An elementary Fourier series argument gives $\lambda-\lambda'=p+q\theta$. □

This result, which holds for all θ, yields a two-parameter family of cocycles whose cohomology is entirely known. Theorems 2.1 and 2.3 can be used to construct a four-parameter family of irreducible representations of M_θ for which all the equivalence relations are known. The parameter s extends the parameter d of [2] and the rational parameter a/b of [6] to all real numbers. Thus for arbitrary θ, we have completed the known parameterized family of representations in a natural way.

Now we add the third parameter t. The value of studying this parameter is not so much in extending the family of representations as in what it reveals about the relationship between the number theory of θ and the representation theory of M_θ. (The third parameter will also be needed to establish irreducibility of the higher-dimensional cocycles in the next section.)

This relationship will be expressed in terms of the continued fraction expansion of θ. We let θ have continued fraction expansion $[a_1,a_2,\dots]$ with convergents $[a_1,a_2,\dots,a_k]=m_k/n_k$. The a_k values are called the partial quotients of θ; if there exists a constant c depending only on θ such that $a_k<c$ for all k, we say θ has bounded partial quotients.

3.3. THEOREM. *If θ has bounded partial quotients, then $R^{(\lambda,s,t)}$ is cohomologous to $R^{(\lambda',s',t')}$ if and only if $s=s'$ and either*
(i) *$t-t'=j+k\theta$ and $\lambda-\lambda'=p+q\theta$; or*
(ii) *s is an integer, t and t' are arbitrary, and $\lambda-\lambda'=p+q\theta-s(t-t')$,*
for some integers j, k, p and q.

Proof. Again the "if" direction is easy: $e(\lambda-\lambda')\,{}^{t}e_s\,{}^{t'}e_{-s}$ is a coboundary if and only if its translate $e(\lambda-\lambda')\,{}^{t-t'}e_s e_{-s}$ is. Using the comment after Definition 3.1, the latter can be written in the form $e(\lambda-\lambda'+s\{t-t'\})f_{t-t',s}$. This is a coboundary by Corollary 2.3 of [8] if (i) holds and by an elementary Fourier series argument if (ii) holds.

Conversely, if $e(\lambda)\,{}^{t}e_s$ is cohomologous to $e(\lambda')\,{}^{t'}e_{s'}$, then $e(\lambda-\lambda')\,{}^{t-t'}e_s e_{-s'}$ is a coboundary. In the case where both s and s' are integers,

$$e(\lambda-\lambda')\,{}^{t-t'}e_s e_{-s'}=e(\lambda-\lambda'+s(t-t'))e_{s-s'}.$$

Thus Theorem 3.2 implies that $s=s'$ and $\lambda-\lambda'+s(t-t')=p+q\theta$. Now we turn to the case where not both s and s' are integers. Since ${}^{(t-t')}e_s e_{-s'}$ is cohomologous to a scalar, we have for every integer n that ${}^{(t-t')}e_{s+n}e_{-(s'+n)}$ is cohomologous to a scalar, and thus that ${}^{2(t-t')}e_{s+n}\,{}^{(t-t')}e_{-(s'+n)}$ is cohomologous to ${}^{(t-t')}e_{s+n}e_{-(s'+n)}$. From this we get that $f_{t-t',-s-s'-2n}f_{2(t-t'),s+n}$ must be cohomologous to a scalar. Choose n so that $s+n\neq-(s'+n)$. Then by Theorem 3.1 of [8], $t-t'=j+k\theta$ for some integers j and k. Thus, since ${}^{(j+k\theta)}f$ is cohomologous to f for any function f, we have that $e_{s+n}e_{-(s'+n)}$ is cohomologous to a scalar. Now Theorem 3.2 shows that $s=s'$. Finally, that $e(\lambda-\lambda')\,{}^{(j+k\theta)}e_s e_{-s}$ is a coboundary implies that $e(\lambda-\lambda')$ is a coboundary and thus that $\lambda-\lambda'=p+q\theta$. □

Now we show that in the unbounded partial quotient case, the cohomology among the cocycles $R^{(\lambda,s,t)}$ is quite different. In this case we show that there are uncountably many more equivalence relations than in the bounded partial quotient case. These are given in the following theorem. The full picture of the cohomology is still unknown.

3.4. THEOREM. *For arbitrary θ, $R^{(\lambda,s,t)}$ is cohomologous to $R^{(\lambda',s,t')}$ if*
(i) *$t-t'$ can be written in the form $\sum_{k=0}^{\infty}b_k n_k\theta$ (mod 1), where n_k are the denominators for the convergents of θ, b_k an integer with $|b_k|<a_{k+1}$ satisfying $\sum_{k=0}^{\infty}|b_k|n_k\|n_k\theta\|<\infty$ and $\sum_{k=0}^{\infty}\|b_k s\|<\infty$; and*
(ii) *$\lambda-\lambda'=\sum_{k=0}^{\infty}\underline{b_k s}\,\underline{n_k\theta}+p+q\theta$. (Here $\|x\|=$ the distance from x to the closest integer, and $\underline{x}=x-$ the closest integer.)*

Proof. We must show that under conditions (i) and (ii), $e(\lambda-\lambda')f_{t-t',-s}$ is cohomologous to 1. This follows from Theorem 2.5 of [8]. □

If θ has bounded partial quotients, condition (i) implies that $t-t'=j+k\theta$. However, if θ has unbounded partial quotients, there are an uncountable number of new cohomology relations given by this theorem, even with s fixed at $s=1/2$ (see [11]).

4. Higher-dimensional cocycles. Next, we use the results of Section 3 for one-dimensional cocycles to construct a family of n-dimensional cocycles whose cohomology relative to Lebesgue measure we can describe in terms of parameters. Our formulae generalize those in [3] in the same way that the ones in Section 3 generalize those in [2].

Let n be an integer >1. As in the previous section, we will define our $U(\mathbf{C}^n)$-valued cocycle R by specifying the function $U(x)=R(x,1)$ (recall that R can be retrieved from U by the formula $R(x,k)=U(x)U(x-\theta)\cdots U(x-(k-1)\theta)$ for $k>0$ and an analogous formula for $k<0$). Again, the intertwining equation reduces to its statement about $R(x,1)$, so that the irreducibility and cohomology of the cocycles R can be determined by studying the solvability of the equation $A(x)U(x)=U'(x)A(x-\theta)$. Because the terms in this equation do not commute for $n>1$, determining solvability is in general much more difficult than in the one-dimensional case. We construct here a family of U's for which this kind of analysis can be carried out.

Let I denote the $(n-1)\times(n-1)$ identity matrix, and let P be the $n\times n$ permutation matrix which is represented in block form by

$$P=\begin{bmatrix} 0 & 1 \\ I & 0 \end{bmatrix}.$$

4.1. DEFINITION. For an integer $n>1$ and real numbers λ, s, and t, define the function $U^{(n,\lambda,s,t)}$ of $[0,1)$ into $U(\mathbf{C}^n)$ by $U^{(n,\lambda,s,t)}(x)=PD^{(n,\lambda,s,t)}(x)$, where $D^{(n,\lambda,s,t)}(x)$ is the $n\times n$ diagonal matrix represented in block form by

$$\begin{bmatrix} I & 0 \\ 0 & e(\lambda)\,{}^t e_s(x) \end{bmatrix}.$$

Let $R^{(n,\lambda,s,t)}$ be the n-dimensional cocycle of the irrational rotation $-\theta$ which is determined by $U^{(n,\lambda,s,t)}$.

Remark. The cocycles just defined are obviously of a very special sort. They do, however, generalize the formulae in [3]: our parameter s extends the integer parameter d_2 and our parameter t adds yet another real parameter. It is shown in [3] that the formulae given there (and thus the formulae in this section) include the cocycles discovered in quite different contexts by Bagchi, Mathew, and Nadkarni [1] and by Ismagilov [5].

4.2. THEOREM. *Assume θ has bounded partial quotients. Then:*

(i) *$R^{(n,\lambda,s,t)}$ is irreducible if and only if s is not an integer for which*

$$gcd(s,n)>1.$$

(ii) *Two irreducible cocycles $R^{(n,\lambda,s,t)}$ and $R^{(n,\lambda',s',t')}$ are cohomologous if and only if $s=s'$, and either* (a) *$t-t'=j+k\theta$ and $\lambda-\lambda'=p+qn\theta$, or* (b) *$s\in Z$, t and t' are arbitrary, and $\lambda-\lambda'+s(t-t')=p+q\theta$; where $p,q,j,k\in Z$.*

Proof. Let $V^{(n,\lambda,s,t)}$ be the unitary representation of D_θ which corresponds to the cocycle $R^{(n,\lambda,s,t)}$ as in Theorem 2.3. We shall prove our theorem by analyzing the restriction of $V^{(n,\lambda,s,t)}$ to the subgroup D' of D_θ which consists of the pairs (w,kn) for w in $\mathbf{C}$ and k in Z. We let $R'^{(n,\lambda,s,t)}$ be the cocycle of the irrational rotation $-n\theta$ defined by $R'^{(n,\lambda,s,t)}(x,k)=R^{(n,\lambda,s,t)}(x,kn)$, and let $V'^{(n,\lambda,s,t)}$

denote the unitary representation of $D_{n\theta}$ corresponding to $R'^{(n,\lambda,s,t)}$ as in Theorem 2.3. Now D' is identical with $D_{n\theta}$, and we see that $V^{(n,\lambda,s,t)}|_{D'}$ is precisely $V'^{(n,\lambda,s,t)}$.

We write

$$R^{(n,\lambda,s,t)}(x,kn)=PD(x)P^{n-1}P^2D(x-\theta)P^{n-2}\cdots D(x-(kn-1)\theta),$$

and see by elementary linear algebra that $R^{(n,\lambda,s,t)}(x,kn)$ is a diagonal matrix whose j,j entry is $R'^{(n,\lambda,s,t,j)}(x,k)$, where $R'^{(n,\lambda,s,t,j)}$ is the one-dimensional cocycle of the rotation $-n\theta$ defined by the function

$$R'^{(n,\lambda,s,t,j)}(x,1)=e(\lambda)^{(t-(j-1)\theta)}e_s(x).$$

It follows that $V'^{(n,\lambda,s,t)}$ is the direct sum of n representations $V'^{(n,\lambda,s,t,j)}$, where $V'^{(n,\lambda,s,t,j)}$ is the representation of $D_{n\theta}$ corresponding to the cocycle $R'^{(n,\lambda,s,t,j)}$ in Theorem 2.3. Hence $V^{(n,\lambda,s,t)}|_{D'}$ is the direct sum of the n representations $V'^{(n,\lambda,s,t,j)}$.

We compute the action of the group D_θ on the element $V'^{(n,\lambda,s,t,j)}$ and find that $V'^{(n,\lambda,s,t,j)}\cdot(w,m)$ is equivalent to $V'^{(n,\lambda,s,t,j-m)}$, which shows that $V^{(n,\lambda,s,t)}|_{D'}$ is concentrated on a D_θ-orbit in $\hat{D}'$. Now, $V'^{(n,\lambda,s,t,i)}$ is equivalent to $V'^{(n,\lambda,s,t,j)}$ if and only if $e(\lambda)^{(t-(i-1)\theta)}e_s$ and $e(\lambda)^{(t-(j-1)\theta)}e_s$ are $n\theta$-cohomologous. By Theorem 3.3 (which applies since the remark following Theorem 2.5 in [8] shows that $n\theta$ has bounded partial quotients if θ does), this is so if and only if $s(j-i)\theta=p+qn\theta$ and s is in Z, or $(j-i)\theta=p+qn\theta$. The latter is impossible for $j\neq i$ since both i and j are between 1 and n. The former holds if and only if $s(j-i)=qn$ and thus $gcd(s,n)>1$. Therefore, the D_θ-orbit of $V'^{(n,\lambda,s,t,1)}$ consists of exactly n points unless s is an integer for which $gcd(s,n)=p>1$. If $gcd(s,n)=p>1$, then the orbit consists of exactly n/p points, and the multiplicity of $V^{(n,\lambda,s,t)}|_{D'}=p$.

If the D_θ-orbit consists of n points, then there is no nontrivial stability subgroup for $V'^{(n,\lambda,s,t,1)}$; $V^{(n,\lambda,s,t)}$ is induced from $V'^{(n,\lambda,s,t,1)}$, and is consequently irreducible by Mackey's theory. Hence $V^{(n,\lambda,s,t)}$ is irreducible if s is not an integer for which $gcd(s,n)>1$.

Conversely, if $V^{(n,\lambda,s,t)}$ is irreducible, then $V^{(n,\lambda,s,t)}$ is equivalent to $\mathrm{IND}_H^{(D_\theta)}S$, where H is the stability subgroup of D_θ for $V'^{(n,\lambda,s,t,1)}$, and where S is a representation of H of the form $S=M\otimes(T\circ\pi)$ with M a Mackey extension of $V'^{(n,\lambda,s,t,1)}$, π the natural map of H onto H/D', and T an irreducible multiplier representation of H/D'. Further, $\dim T$ is the multiplicity of $V^{(n,\lambda,s,t)}|_{D'}$. Since H/D' is a finite cyclic group, we have that $\dim T=1$. If s is an integer for which $gcd(s,n)=p>1$, then we have seen that the multiplicity of $V^{(n,\lambda,s,t)}|_{D'}=p$, so that $V^{(n,\lambda,s,t)}$ is irreducible only if s is not an integer for which $gcd(s,n)>1$. This proves part (i).

Two irreducible cocycles $R^{(n,\lambda,s,t)}$ and $R^{(n,\lambda',s',t')}$ are cohomologous if and only if the corresponding unitary representations $V^{(n,\lambda,s,t)}$ and $V^{(n,\lambda',s',t')}$ are equivalent, and this is so if and only if they restrict to the same n-point orbit in D'. Therefore, $R^{(n,\lambda,s,t)}$ is cohomologous to $R^{(n,\lambda',s',t')}$ if and only if there exists an integer i between 1 and n such that the two one-dimensional cocycles $R'^{(n,\lambda,s,t,i)}$

and $R'^{(n,\lambda',s',t',1)}$ are cohomologous relative to the rotation $-n\theta$. Because $n\theta$ has bounded partial quotients, Theorem 3.3 holds and we have $s=s'$ and either

(a) $t-t'=j+kn\theta+(i-1)\theta$ and $\lambda-\lambda'=p+qn\theta$, which is so if and only if $t-t'=j+k'\theta$ and $\lambda-\lambda'=p+qn\theta$; or

(b) $s\in Z$, t and t' are arbitrary, and $\lambda-\lambda'+s(t-t')=p+qn\theta+s(i-1)\theta$, which is so if and only if $s\in Z$, t and t' are arbitrary, and $\lambda-\lambda'+s(t-t')=p+q'\theta$.

This completes the proof of the theorem. □

REMARK 1. If s is an integer which is relatively prime to n, then the irreducibility and cohomology relations for $R^{(n,\lambda,s,t)}$ hold even when θ has unbounded partial quotients. When $t=0$, this is precisely the cocycles studied in [3] where no continued fraction distinctions were made.

REMARK 2. We know of no irreducible n-dimensional cocycle which we can show is not cohomologous to one in our list. This is not, however, a strong statement, since there are many simple cocycles whose relationship to the $R^{(n,\lambda,s,t)}$'s is undetermined. We make no conjecture about the completeness of our list.

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