ON THE ACCESSIBILITY OF THE BOUNDARY OF A SIMPLY CONNECTED DOMAIN

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Let D be a simply connected plane domain, not the whole plane, and let $w=f(z)$ map $|z|<1$ one-to-one and conformally onto D. As is well known, for almost every θ $(0 \le \theta \le 2\pi)$, $f(z)$ has a finite radial limit $f(e^{i\theta})$ at $e^{i\theta}$. Consequently, the image under f of the radius at such an $e^{i\theta}$ determines an (ideal) accessible boundary point of D whose complex coordinate is $f(e^{i\theta})$ [2, pp. 357–363]. We will denote both the (ideal) accessible boundary point and its complex coordinate by $f(e^{i\theta})$; no confusion will arise provided that we treat $f(e^{i\theta_1})$ and $f(e^{i\theta_2})$ to be distinct whenever $\theta_1 \neq \theta_2$ (even though the complex coordinates may be equal).

We introduce the following metric on D: the *arc-length distance* $l_D(w_1, w_2)$ between two points of D is defined to be the infimum of the Euclidean lengths of the rectifiable arcs lying in D and joining w_1 to w_2. This arc-length metric is seen to agree locally with the Euclidean metric. Let R be the set of rectifiably accessible points of ∂D. For $w \in D$ and $w_0 \in R$ we let $l_D(w, w_0)$ be the infimum of the Euclidean lengths of rectifiable curves lying in D and joining w to w_0. The arc-length distance between two points of R is defined similarly. It is easily shown that l_D is a metric for $D \cup R$. The distance between two subsets S_1 and S_2 of $D \cup R$ will be denoted by $l_D(S_1, S_2)$ and is defined in the usual manner. Any limits involving elements of R will be taken using the arc-length metric.

We will let $\Delta(w,r)$ denote the open disc which is centered at w of radius r. Let $w_0 \in R$. Corresponding to each positive number r small enough so that the domain D contains a disc of radius r, let

$$\delta(r, w_0) = \inf\{l_D(w, w_0) : \Delta(w, r) \subseteq D\}.$$

We say that w_0 is *broadly accessible* if $\liminf_{r \to 0} \delta(r, w_0)/r = 1$. In more picturesque language, $w_0 \in R$ is broadly accessible if we can find discs in D close to w_0 such that the center of each disc can be joined to w_0 by an arc whose length is only slightly larger than the radius of the disc. We will use $\delta(r,\theta)$ to abbreviate $\delta(r, f(e^{i\theta}))$. Concerning the broad accessibility criterion, we will prove the following theorem.

THEOREM. *Let D be a simply connected plane domain, not the whole plane. Let f map $|z|<1$ one-to-one and conformally onto D. Then for almost every θ,*

$$\lim_{r \to 1} f(re^{i\theta}) = f(e^{i\theta})$$

is a broadly accessible point of ∂D.

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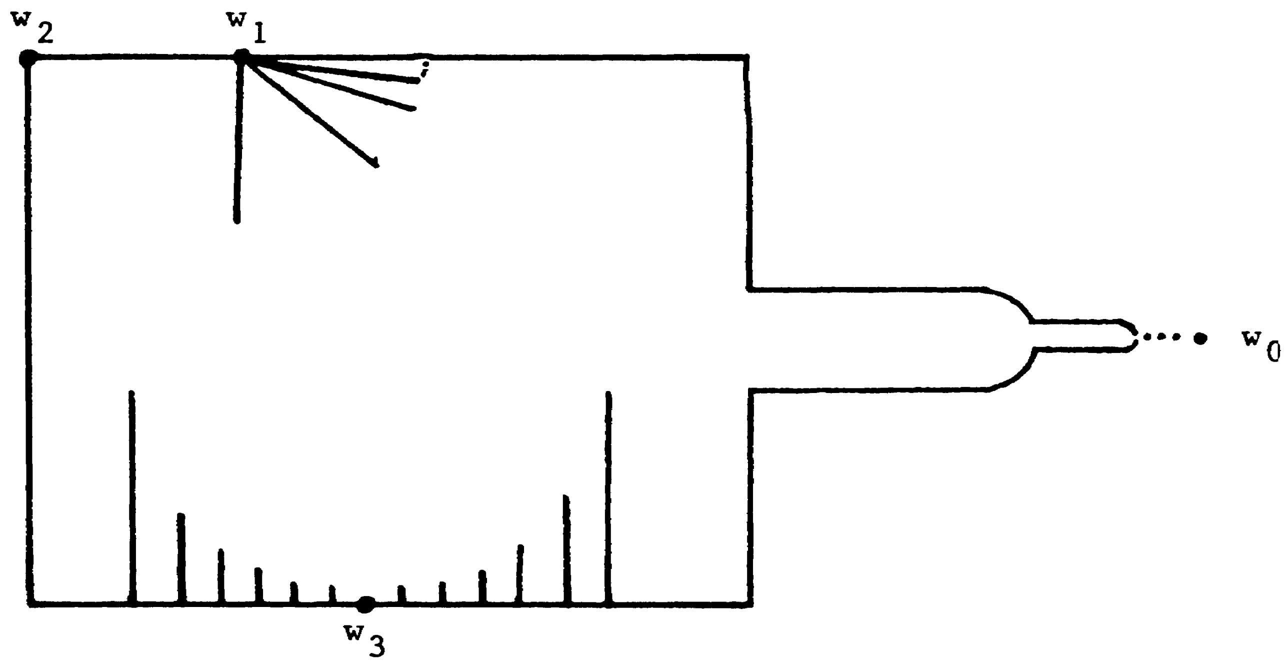

Figure 1

Before proceeding with the proof of the theorem we refer to Figure 1 for several examples. A rectifiably accessible boundary point need not be broadly accessible, as is seen by considering w_1 and w_2 in Figure 1. w_1 is the complex coordinate of countably many rectifiably accessible boundary points, none of which are broadly accessible. w_3 will be broadly accessible provided that the spikes are arranged so that w_3 is a point of inner tangency for ∂D, that is, so that D contains Stolz angles at w_3 of angular opening arbitrarily near to, but not greater than, π. In the construction suggested on the right side of the figure, each passage is of length $3r_i$, where r_i is the radius of the disc at the end of the passage. We let w_0 denote the point of ∂D which is accessible via the passages. If $r_i=(1/2)^i$ for each i, then w_0 will be rectifiably, but not broadly, accessible. However, if $r_1=1/2$ and if for $i\geq 2$, $r_i=(1/2)^i r_{i-1}$, then w_0 will be broadly accessible. We also note that in either construction w_0 will not be a point of inner tangency for ∂D.

While every point of inner tangency of ∂D is a broadly accessible point, the last example given shows that the converse of this is not true. It should be noted that it is possible for the points of inner tangency of ∂D to correspond under f to a set of measure zero on $|z|=1$ [3, pp. 65–66].

We now turn to the proof of the theorem. The proof makes use of the geometric quantity called extremal length [1, pp. 10–16], and the following lemma (which is concerned with an extremal length estimate) is essential to the argument. We are indebted to McMillan for an earlier version of this lemma, in which the setting was the upper half-plane [3, pp. 56–57].

LEMMA. *Let* $0<\delta<\pi/2$ *and let* $A\subseteq(0,\delta)$ *with outer measure* $m^*(A)$. *For each* θ *such that* $0<\theta<\delta$ *let* γ_θ *denote the arc which is contained in the unit disc* *of the circle orthogonal to* $|z|=1$ *at* $e^{i\theta}$ *and at* $e^{-i\theta}$. *Set* $\Gamma=\{\gamma_\theta:\theta\in A\}$. *Then the extremal length* $\lambda(\Gamma)$ *of the family of curves satisfies*

$$\lambda(\Gamma)\le\pi/\log k,$$

where $k=\sin\delta/\sin[\delta-m^*(A)]$.

Proof of Lemma. Suppose $\rho(z)$ is an arbitrary admissible function for extremal length, that is, $\rho(z)$ is a non-negative, measurable function defined in the whole plane such that the integral

$$A(\rho)=\iint\rho^2\,dx\,dy,$$

taken over the whole plane, is finite and non-zero. Set

$$L(\Gamma,\rho)=\inf_{\gamma_\theta\in\Gamma}L(\gamma_\theta,\rho),\quad\text{where } L(\gamma_\theta,\rho)=\int_{\gamma_\theta}\rho\,|dz|,$$

and where the integral is taken to be infinite if ρ is not measurable on γ_θ and may be infinite in any case. For almost every $\theta\in A$ both of the following integrals are finite, and by Schwarz's inequality

$$L(\gamma_\theta,\rho)^2=\left(\int_{\gamma_\theta}\rho\,|dz|\right)^2\le(\pi-2\theta)\tan\theta\int_{\gamma_\theta}\rho^2\,|dz|,$$

where it is seen that $(\pi-2\theta)\tan\theta$ is the length of γ_θ. Hence the inequality

$$L(\Gamma,\rho)^2[(\pi-2\theta)\tan\theta]^{-1}\le\int_{\gamma_\theta}\rho^2\,|dz|$$

holds for each θ in a measurable subset A_0 of $(0,\delta)$ that contains A. It is readily seen that

$$\int_{A_0}[\pi-2\theta]^{-1}\cot\theta\,d\theta\ge\int_{\delta-m(A_0)}^{\delta}\frac{1}{\pi}\cot\theta\,d\theta$$
$$\ge\int_{\delta-m^*(A)}^{\delta}\frac{1}{\pi}\cot\theta\,d\theta,$$

where $m(A_0)$ denotes the measure of A_0. This last integral is equal to $(1/\pi)\log k$, with $k=\sin\delta/\sin[\delta-m^*(A)]$. Since

$$\int_{A_0}\left(\int_{\gamma_\theta}\rho^2\,|dz|\right)d\theta\le A(\rho),$$

we have that

$$L(\Gamma,\rho)^2(1/\pi)\log k\le A(\rho).$$

Since ρ is an arbitrary admissible function,

$$\lambda(\Gamma)\equiv\sup_\rho L(\Gamma,\rho)^2/A(\rho)\le\pi/\log k,$$

as claimed. □

Proof of Theorem. Let ϵ_n and μ_k be sequences of positive numbers monotonically decreasing to zero. Let A be the set of points of $|z|=1$ for which $\lim_{r\to 1} f(re^{i\theta})$ exists and is rectifiably accessible. Then $m(A)=2\pi$ [4, pp. 311–312]. Since the set of $e^{i\theta}\in A$ for which $f(e^{i\theta})$ is not broadly accessible is the countable union

$$\bigcup_n \bigcup_k \{e^{i\theta}\in A: \delta(r,\theta)/r > (1+\epsilon_n) \text{ if } r<\mu_k\},$$

it suffices to prove for each $\epsilon>0$ and each $\mu>0$ that $m(E)=0$, where

$$E=\{e^{i\theta}\in A: \delta(r,\theta)/r>(1+\epsilon) \text{ if } r<\mu\}.$$

Let $w\in D$, $\Delta(w,r)\subseteq D$ and $r<\mu$. Then for any $e^{i\theta}\in E$,

$$l_D(w,f(e^{i\theta}))>(1+\epsilon)r,$$

and as a consequence

$$l_D(\Delta(w,r),f(e^{i\theta}))>\epsilon r. \tag{1}$$

Fix a point $e^{i\theta_0}\in E$ and let z_n be any sequence radially approaching $e^{i\theta_0}$. Let $w_n=f(z_n)$ and let $\Delta(w_n,r_n)$ be the largest disc centered at w_n which is contained in D. Since $\lim f(z_n)\neq\infty$, we see that $r_n\to 0$. By choosing a subsequence and relabeling we may assume that $r_n<\mu$ for all n. Inequality (1) holds for each $\Delta(w_n,r_n)$, using the fixed $e^{i\theta_0}$. For each n let α_n be a radius of $\Delta(w_n,r_n)$ whose closure contains a point of the boundary of D. Each α_n determines a point $f(\xi_n)\in\partial D$ which is seen to be broadly accessible, and from the definition of E we see that $\xi_n\in A-E$.

In what follows we will denote the length of a curve γ by $l(\gamma)$. For each n we define the family $\Gamma(\alpha_n)$ to consist of all curves γ lying in D which join a point of α_n to a point $f(e^{i\theta})$, $e^{i\theta}\in E$. From the definitions of $\Gamma(\alpha_n)$ and E and from (1) we see that for each n,

$$l(\gamma)\geq\epsilon r_n=\epsilon l(\alpha_n) \quad \text{for each } \gamma\in\Gamma(\alpha_n). \tag{2}$$

We now prove that $\lambda(\Gamma(\alpha_n))\geq\epsilon/(2+\pi\epsilon)$, a quantity independent of n, where $\lambda(\Gamma(\alpha_n))$ denotes the extremal length of the family $\Gamma(\alpha_n)$.

For each n let $V_n=\{w\in D: \operatorname{dist}(w,\alpha_n)<\epsilon l(\alpha_n)\}$, where $\operatorname{dist}(w,\alpha_n)$ is the Euclidean distance from the point w to α_n. Let $\gamma\in\Gamma(\alpha_n)$. If γ contains points of $D-V_n$, then there is a component γ_1 of $\gamma\cap V_n$ whose length is at least equal to $\epsilon l(\alpha_n)$, and thus

$$l(\gamma\cap V_n)\geq l(\gamma_1)\geq\epsilon l(\alpha_n).$$

If γ is contained in V_n, then by (2) we have the same inequality. Hence for any $\gamma\in\Gamma(\alpha_n)$,

$$l(\gamma\cap V_n)\geq\epsilon l(\alpha_n).$$

Define the function ρ_n by setting $\rho_n(w)$ equal to 1 if $w\in V_n$ and equal to 0 elsewhere. Since V_n is open, ρ_n is a measurable function. We see that for any $\gamma\in\Gamma(\alpha_n)$,

$$\int_\gamma \rho_n|dw|\geq\epsilon l(\alpha_n),$$

and consequently,

$$L(\Gamma(\alpha_n), \rho_n) = \inf \int_\gamma \rho_n |dw| \geq \epsilon l(\alpha_n),$$

where the infimum is taken over all $\gamma \in \Gamma(\alpha_n)$. Since $A(\rho_n)$ is the area integral for V_n, we see that

$$A(\rho_n) \leq [2\epsilon l(\alpha_n)] l(\alpha_n) + \pi[\epsilon l(\alpha_n)]^2.$$

Thus, for each n,

$$\begin{aligned} \lambda(\Gamma(\alpha_n)) &\equiv \sup_\rho \frac{L(\Gamma(\alpha_n, \rho)^2}{A(\rho)} \\ &\geq \frac{L(\Gamma(\alpha_n), \rho_n)}{A(\rho_n)} \\ &\geq \frac{\epsilon^2}{2\epsilon + \pi\epsilon^2}. \end{aligned} \tag{3}$$

Since a set of positive measure has points of outer density, we will prove $m(E)=0$ by proving that each point $e^{i\theta} \in E$ is not a point of outer density for E. We continue to consider the same fixed $e^{i\theta_0} \in E$. For each n define α_n^z by requiring that $f(\alpha_n^z) = \alpha_n$. Since the mapping function f is a normal function, it has no Koebe arcs [4, pp. 262–267]; consequently, each α_n^z has an endpoint $e^{i\theta_n}$ on $|z|=1$ and $e^{i\theta_n} \to e^{i\theta_0}$. For each n, $e^{i\theta_n} \notin E$, and in particular we have $e^{i\theta_n} \neq e^{i\theta_0}$. By replacing $\{\theta_n\}$ by a certain subsequence we can suppose, without loss of generality, that $\theta_0 < \theta_n < (\theta_0 + \pi/2)$ for all n; the other case is analogous.

For each n, let Γ_n' be the family of curves in $|z|<1$ which corresponds under f to the family $\Gamma(\alpha_n)$, that is,

$$\Gamma_n' = \{\gamma' : \gamma' \subseteq \{|z|<1\}, f(\gamma') \in \Gamma(\alpha_n)\}.$$

Since extremal length is a conformal invariant [1, p. 14], $\lambda(\Gamma_n') = \lambda(\Gamma(\alpha_n))$ for each n, and this together with (3) implies that

$$\lambda(\Gamma_n') \geq c > 0 \quad \text{for each } n, \tag{4}$$

where c is a constant independent of n.

For each n let

$$E_n = \{e^{i(\theta_n - \xi)} \in E : 0 < \xi < \theta_n - \theta_0\},$$

and let Γ_n'' be the family of circular arcs contained in $|z|<1$ which are orthogonal to $|z|=1$ at $e^{i(\theta_n+\xi)}$ and $e^{i(\theta_n-\xi)}$ for some $e^{i(\theta_n-\xi)} \in E_n$ (see Figure 2).

If $\Gamma_n'' = \phi$ for any n, then $m(E_n)=0$ and the outer density of E for linear measure on $|z|=1$ is obviously zero at $e^{i\theta_0}$. We proceed with the case that $\Gamma_n'' \neq \phi$ for any n. Each $\gamma'' \in \Gamma_n''$ contains some curve $\gamma' \in \Gamma_n'$. By the comparison principle for extremal length [1, p. 10],

$$\lambda(\Gamma_n'') \geq \lambda(\Gamma_n'). \tag{5}$$

By the lemma,

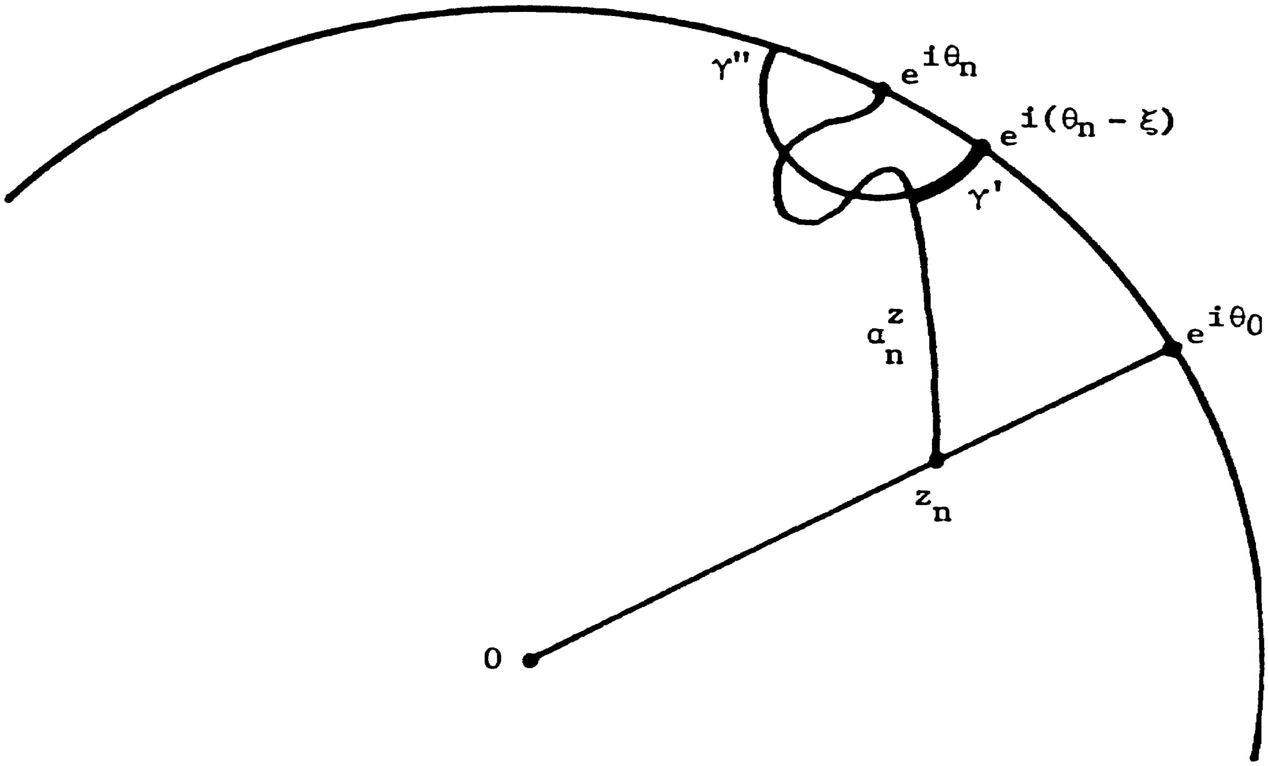

Figure 2

$$\lambda(\Gamma_n'') \leq \pi/\log k_n,$$

where $k_n = \sin(\theta_n - \theta_0)/\sin[(\theta_n - \theta_0) - m^*(E_n)]$. This inequality together with (4) and (5) yields that

$$\begin{aligned}\sin(\theta_n - \theta_0) &\leq e^{\pi/c} \sin[(\theta_n - \theta_0) - m^*(E_n)] \\ &\leq e^{\pi/c}[(\theta_n - \theta_0) - m^*(E_n)],\end{aligned}$$

the last inequality following from $0 \leq (\theta_n - \theta_0) - m^*(E_n)] < \pi/2$. Consequently,

$$m^*(E_n)/(\theta_n - \theta_0) \leq 1 - e^{-\pi/c}[\sin(\theta_n - \theta_0)/(\theta_n - \theta_0)].$$

Hence, $\limsup_n m^*(E_n)/(\theta_n - \theta_0) < 1$, and thus $e^{i\theta_0}$ is not a point of outer density for E on $|z| = 1$. Since $e^{i\theta_0}$ was an arbitrary point of E, no point of E is a point of outer density for E, and consequently E has measure zero. The proof of the theorem is now complete. □

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