ON A THEOREM OF GIFFEN

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0. Introduction. This paper is based on some ideas of Giffen concerning the Karoubi conjecture. Let R be a ring with unit and an involution. Let $\mathrm{GL}(R) = \varinjlim \mathrm{GL}_n(R)$ be the infinite general linear group and ${}_\epsilon O(R) = \varinjlim {}_\epsilon O_{n,n}(R)$ be the infinite orthogonal group where $\epsilon = \pm 1$ and ${}_\epsilon O_{n,n}(R)$ is the group of automorphisms of R^{2n} preserving the ϵ-hermitian pairing

$$\langle x, y \rangle = x \left[\begin{array}{c|c} 0 & I_n \\ \hline \epsilon I_n & 0 \end{array}\right] \bar{y}^t.$$

Applying the plus-construction to the classifying spaces $B\mathrm{GL}(R)$ and $B{}_\epsilon O(R)$ of these groups we obtain the classifying spaces of algebraic K-theory and hermitian K-theory, respectively.

During the early development of K-theory, Karoubi studied the relation between these two theories. The natural inclusion ${}_\epsilon O_{n,n}(R) \to \mathrm{GL}_{2n}(R)$ and the hyperbolic map $\mathrm{GL}_n(R) \to {}_\epsilon O_{n,n}(R)$ which takes

$$g \mapsto \left[\begin{array}{c|c} g & 0 \\ \hline 0 & (\bar{g}^t)^{-1} \end{array}\right],$$

induce mappings $B{}_\epsilon O(R)^+ \to B\mathrm{GL}(R)^+$ and $B\mathrm{GL}(R)^+ \to B{}_\epsilon O(R)^+$. Motivated by periodicity in topological K-theory, Karoubi [7] conjectured that the homotopy fibers ${}_\epsilon\mathcal{V}(R)$ and ${}_\epsilon\mathcal{U}(R)$ of these maps are related by a homotopy equivalence, $\Omega\, {}_\epsilon\mathcal{U}(R) \simeq {}_{-\epsilon}\mathcal{V}(R)$. This conjecture was proven to be true by Karoubi and Loday ([9], [10]) under the assumption that 2 is invertible in R.

More recently, Giffen attempted to reinterpret this conjecture in a categorical framework. For this, he introduced a category ${}_\epsilon W(R)$ intended as a model for a delooping of ${}_\epsilon\mathcal{U}(R)$. These ideas have never appeared in print, but have significant implications in light of recent work by the current authors. The main purpose of this paper, therefore, is to describe the category ${}_\epsilon W(R)$ and to prove that there exists a homotopy fibration of infinite loop spaces

$$(0.1) \qquad K_0(R) \times B\mathrm{GL}(R)^+ \to K_0^H(R) \times B{}_\epsilon O(R)^+ \to |{}_\epsilon W(R)|$$

as predicted by Giffen.

Our own interest in this problem stems from the study of compactifications of moduli spaces. In [5], we construct maps from the Satake compactification of Siegel space, $\mathfrak{S}_n^*$, to certain subspaces $|{}_{-1}W_n(\mathbf{Z})|$ of $|{}_{-1}W(\mathbf{Z})|$, and prove that these maps induce isomorphisms on rational cohomology. Using a special case of the fibration (0.1) (proved in the appendix of [5]) and computations of Borel, we are able to determine this cohomology in degrees $< n$. Recent work of the authors

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suggests that this relation between ${}_\epsilon W(R)$ categories and Satake compactifications can be greatly generalized and, in some cases, sharpened to a correspondence of integral cohomologies. This is our motivation for the current work.

1. The category ${}_\epsilon W(R)$. Let R be a ring with unit and an involution $r \mapsto \bar{r}$ (i.e., $\bar{\bar{r}} = r$ and $(\overline{rs}) = \bar{s}\bar{r}$). Fix $\epsilon = \pm 1$. For any right R-module M, the dual module $M^* = \mathrm{Hom}_R(M, R)$ will be viewed as a right R-module via $(f \cdot r)(m) = \bar{r} f(m)$. There is a natural pairing

$$\mu_M: (M \oplus M^*) \times (M \oplus M^*) \to R$$
$$(m, f) \times (n, g) \mapsto g(m) + \epsilon \overline{f(n)}.$$

In particular, if M is a free R-module with basis $e_1, \ldots, e_n$, and $e_1^*, \ldots, e_n^*$ is the dual basis for M^*, then with respect to this basis, the pairing takes the form

$$\mu_M(x, y) = x \left[\begin{array}{c|c} 0 & I_n \\ \hline \epsilon I_n & 0 \end{array} \right] \bar{y}^t.$$

We denote by $H(M)$ the module $M \oplus M^*$ together with the pairing μ_M. In general, for any right R-module P and pairing $\lambda: P \times P \to R$ we say that (P, λ) is a *hyperbolic R-module* if there exists an isometry $(P, \lambda) \cong H(M)$ for some finitely generated projective R-module M. The isometry classes of hyperbolic R-modules form a commutative monoid under direct sum, $H(M) \oplus H(N) \cong H(M \oplus N)$. The Grothendieck group of this monoid will be denoted $K_0^H(R)$.

The set of hyperbolic R-modules (P, λ) constitute the objects of the category ${}_\epsilon W(R)$. To define morphisms in ${}_\epsilon W(R)$, recall that a submodule $L \subset P$ is *isotropic* (with respect to λ) if $L \subseteq L^\perp$, where

$$L^\perp = \{x \in P \mid \lambda(x, y) = 0 \ \forall y \in L\}.$$

For such an L, λ induces a pairing

$$\lambda_L: L^\perp/L \times L^\perp/L \to R.$$

A morphism $(P', \lambda') \to (P, \lambda)$ in ${}_\epsilon W(R)$ consists of a pair (L, φ), where L is an isotropic direct summand of P and $\varphi: L^\perp/L \to P'$ is an isometry of $(L^\perp/L, \lambda_L)$ to (P', λ'). (It follows that $(L^\perp/L, \lambda_L)$ must be hyperbolic.) Note that (L, φ) is an isomorphism in ${}_\epsilon W(R)$ if and only if $L = 0$ and $\varphi: P \to P'$ is an isometry. In particular, if $(P, \lambda) \cong H(M)$ with M a free module of rank n, then $\mathrm{Aut}(P, \lambda) \cong {}_\epsilon O_{n,n}(R)$.

For $L \subset P$ an isotropic direct summand, we will call a summand $L_D \subset P$ a *hyperbolic dual* for L if $L \cap L_D = \{0\}$ and the map

$$L \oplus L_D \to L \oplus L^*$$
$$(x, y) \mapsto (x, \lambda_y) \quad \lambda_y(x) = \lambda(x, y)$$

is an isometry of $(L \oplus L_D, \lambda \mid_{L \oplus L_D})$ to $H(L)$. Such hyperbolic duals always exist [3, §2] and give rise to isometries $P = (L^\perp \cap L_D^\perp) \oplus L \oplus L_D \cong L^\perp/L \oplus L \oplus L^* \cong P' \oplus H(L)$.

The direct sum operation $(P,\lambda)\oplus(P',\lambda')=(P\oplus P',\lambda\oplus\lambda')$ gives rise to an associative H-space structure on ${}_\epsilon W(R)$ with the identity element given by the zero module. The requirement that an object (P,λ) in ${}_\epsilon W(R)$ be hyperbolic insures that there exists a direct summant $M\subset P$ with $M^\perp=M$, and hence there exists a morphism $(M,\mathrm{id}_0)\colon 0\to(P,\lambda)$. It follows that ${}_\epsilon W(R)$ is connected and so $|{}_\epsilon W(R)|$ is an H-group, that is, an associative H-space with homotopy inverses (cf. [6, p. 227]).

As noted in the Introduction, there is a map $B\mathrm{GL}(R)^+\to B\,{}_\epsilon O(R)^+$ induced by the group homomorphism

$$g\mapsto\left[\begin{array}{c|c} g & 0 \\ \hline 0 & (\bar g^t)^{-1}\end{array}\right].$$

Combining this with the homomorphism $K_0(R)\to K_0^H(R)$ induced by $P\mapsto H(P)$, we get a map

$$h\colon K_0(R)\times B\mathrm{GL}(R)^+\to K_0^H(R)\times B\,{}_\epsilon O(R)^+.$$

Our main theorem is as follows.

THEOREM. *There is a homotopy fibration of infinite loop spaces*

$$K_0(R)\times B\mathrm{GL}(R)^+\xrightarrow{h}K_0^H(R)\times B\,{}_\epsilon O(R)^+\to|{}_\epsilon W(R)|.$$

The proof is found in Section 3.

2. Preliminaries.

2.1. The proof of the main theorem uses category-theoretic techniques developed by Quillen in [6] and [12]. We first recall some terminology and results from [12]. For a functor $\theta\colon A\to B$ and an object b in B, there are three "fiber categories" as follows.

(i) $\theta^{-1}(b)$: This is the category whose objects are $\{a\in\operatorname{obj}A\mid\theta(a)=b\}$, and whose morphisms $a\xrightarrow{\alpha}a'$ are morphisms in A such that $\theta(\alpha)=\mathrm{id}_b$.

(ii) θ/b: The set of objects of this category is $\{(a,\theta(a)\xrightarrow{\beta}b)\mid a\in\operatorname{obj}A,\ \beta$ is a B-morphism$\}$, and morphisms $(a,\theta(a)\xrightarrow{\beta}b)\xrightarrow{\alpha}(a',\theta(a')\xrightarrow{\beta'}b)$ are A-morphisms $a\xrightarrow{\alpha}a'$ such that the diagram

$$\begin{array}{ccc} \theta(a) & \xrightarrow{\theta(\alpha)} & \theta(a') \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\beta'} & \\ & b & \end{array}$$

commutes in B.

(iii) $b\backslash\theta$: Here we take the objects to be $\{(a,\theta(a)\xleftarrow{\beta}b)\mid a\in\operatorname{obj}A,\ \beta$ is a B-morphism$\}$, and morphisms $(a,\theta(a)\xleftarrow{\beta}b)\xrightarrow{\alpha}(a',\theta(a')\xleftarrow{\beta'}b)$ to be A-morphisms $a\xrightarrow{\alpha}a'$ such that the diagram

$$\begin{array}{ccc} \theta(a) & \xrightarrow{\theta(\alpha)} & \theta(a') \\ & {\scriptstyle\beta}\nwarrow \quad \nearrow{\scriptstyle\beta'} & \\ & b & \end{array}$$

commutes in B.

For each b, there are natural inclusions

$$\theta^{-1}(b) \to b\backslash\theta \qquad \theta^{-1}(b) \to \theta/b$$

$$a \mapsto (a, \theta(a) \xrightarrow{\text{id}} b) \qquad a \mapsto (a, \theta(a) \xleftarrow{\text{id}} b)$$

and projections

$$b\backslash\theta \xrightarrow{{}_b\pi} A \qquad \theta/b \xrightarrow{\pi_b} A$$

$$(a, \theta(a) \to b) \mapsto a \qquad (a, \theta(a) \leftarrow b) \mapsto a.$$

Associated to a morphism $\beta: b \to b'$ in B is a *base-change* functor

$$\beta^*: b'\backslash\theta \to b\backslash\theta$$

$$(a, \theta(a) \xleftarrow{\gamma} b') \mapsto (a, \theta(a) \xleftarrow{\gamma\circ\beta} b).$$

and a *cobase-change* functor

$$\beta_*: \theta/b \to \theta/b'$$

$$(a, \theta(a) \xrightarrow{\gamma} b) \mapsto (a, \theta(a) \xrightarrow{\beta\circ\gamma} b').$$

In Theorem B of [12], Quillen proves that if every base-change functor is a homotopy equivalence then the diagram

$$\begin{array}{ccc} b\backslash\theta & \xrightarrow{{}_b\pi} & A \\ \downarrow & & \downarrow \theta \\ b & \longrightarrow & B \end{array}$$

is homotopy cartesian, or in other words the natural map from $|b\backslash\theta|$ into the homotopy fiber F_b of $|\theta|$ over b is a homotopy equivalence. For the proof of our main theorem we will need a homological version of Quillen's theorem.

2.2. PROPOSITION. *Suppose $\theta: A \to B$ is a functor such that every base-change (resp. cobase-change) induces isomorphisms on homology. Then for every object $b \in B$, the natural map from $|b\backslash\theta|$ (resp. $|\theta/b|$) into F_b induces isomorphisms on homology.*

Proof. The proof follows Quillen's [12] proof of Theorem B, replacing the notion of quasi-fibration by that of homology-fibration (McDuff and Segal [11]). In particular, Proposition 4 of [11] substitutes for Quillen's main lemma. We leave the details to the reader. □

2.3. REMARK. Let $\theta: A \to B$ be as in the proposition. If A and B are monoidal categories and θ is a monoidal functor, then $0\backslash\theta$ inherits a monoidal structure and the natural map $|0\backslash\theta| \to F_0$ is an H-space map. If, in addition, $\pi_0(0\backslash\theta)$ is a group under the monoid law, then $|0\backslash\theta|$ and F_0 are H-groups and the proposition implies that $|0\backslash\theta| \to F_0$ is actually a homotopy equivalence.

2.4. We next recall the "localization" construction from [6]. Suppose S is a monoidal category all of whose morphisms are isomorphisms, and suppose there is a left action of S on a category X, $S\times X\xrightarrow{+}X$. We can form the "localized" category $S^{-1}X$ whose objects are pairs (A,B) with $A\in\operatorname{obj} S$, $B\in\operatorname{obj} X$, and whose morphisms $(A,b)\to(A',B')$ are equivalence classes of triples (C,α,β), with $C\in\operatorname{obj} S$ and $\alpha\colon C+A\to A'$, $\beta\colon C+B\to B'$ morphisms in S and X, respectively. The equivalence relation is given by $(C,\alpha,\beta)\sim(C',\alpha',\beta')$, if there exists an isomorphism $\gamma\colon C\cong C'$ in S such that the diagrams

$$\begin{array}{ccc} C+A & \xrightarrow{\gamma+\mathrm{id}} & C'+A \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\alpha'} & \\ & A' & \end{array} \qquad\qquad \begin{array}{ccc} C+B & \xrightarrow{\gamma+\mathrm{id}} & C'+B \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\beta'} & \\ & B' & \end{array}$$

commute. The homology groups of X and $S^{-1}X$ are related by the equation

$$H_*(S^{-1}X)=\pi_0(S)^{-1}H_*(X).$$

If S acts *invertibly* on X (i.e., if $C+\colon X\to X$ is a homotopy equivalence for every C), then the inclusion $X\to S^{-1}X$, $A\mapsto(0,A)$, is a homotopy equivalence. If T is another monoidal category and $f\colon T\to S$ a monoidal functor, then f induces an action of T on X and a functor $f^{-1}X\colon T^{-1}X\to S^{-1}X$. If f is *cofinal* (i.e., if for any $A\in\operatorname{obj} S$, $\exists B\in\operatorname{obj} S$ and $C\in\operatorname{obj} T$ such that $A+B\cong f(T)$), then $f^{-1}X$ is a homotopy equivalence.

Of particular interest is the case where $X=S$ with the action given by the monoid structure of S. If S is a *symmetric* monoidal category, (i.e., if the monoid operation satisfies certain commutativity and associativity relations; see [1]), then $|S^{-1}S|$ is an infinite loop space. For example, the categories

$$S=\text{finitely generated projective } R\text{-modules and isomorphisms},$$
$$S_H=\operatorname{Iso}{}_\epsilon W(R)=\text{hyperbolic } R\text{-modules and isometries}$$

are symmetric monoidal and give rise to the infinite loop spaces

$$|S^{-1}S|=K_0(R)\times B\mathrm{GL}(R)^+,$$
$$|S_H^{-1}S_H|=K_0^H(R)\times B_\epsilon O(R)^+.$$

The functor $H\colon S\to S_H$ which takes $P\mapsto H(P)$, $\alpha\mapsto\alpha\oplus\alpha^{*-1}$, induces an infinite loop space map $|S^{-1}S|\to|S_H^{-1}S_H|$ which clearly corresponds to the map h defined in Section 1.

The category $W={}_\epsilon W(R)$ is also a symmetric monoidal category under $\oplus$ and, as observed in Section 2 above, the action of W on itself is invertible. It follows that $|W|\simeq|W^{-1}W|$ is an infinite loop space. Letting $\tau\colon S_H\to W$ be the inclusion functor, the functors

(2.4.1) $$S^{-1}S\xrightarrow{H^{-1}H}S_H^{-1}S_H\xrightarrow{\tau^{-1}\tau}W^{-1}W$$

induce infinite loop-space maps

$$K_0(R)\times BGL(R)^+ \xrightarrow{h} K_0^H(R)\times B_\epsilon O(R)^+ \to |W|.$$

The proof of the main theorem consists in showing that this diagram represents a homotopy fibration.

3. The fibration sequence. We now restate and prove the main theorem.

3.1. THEOREM. *There is a homotopy fibration of infinite loop spaces*

$$K_0(R)\times BGL(R)^+ \xrightarrow{h} K_0^H \times B_\epsilon O(R)^+ \to |_\epsilon W(R)|.$$

Proof. The hyperbolic functor $H: S\to S_H$ and the inclusion functor $\tau: S_H\to W$ are both cofinal, so it follows from the discussion in 2.4 that the diagram (2.4.1) is homotopy equivalent to the diagram

$$S^{-1}S \xrightarrow{S^{-1}H} S^{-1}S_H \xrightarrow{S^{-1}\tau} S^{-1}W.$$

Now the action of S on W,

$$A+(P,\lambda) = H(A)\oplus(P,\lambda) = (A\oplus A^*\oplus P, \mu_A\oplus\lambda)$$

is not only invertible, it is actually trivial in the sense that for any A, the translation $A+: W\to W$ is homotopic to the identity. The homotopy is induced by the natural transformation $\eta_A: \mathrm{id}_W\to(A+)$,

$$\eta_A(P) = (A, \mathrm{id}_P): (P,\lambda)\to(A\oplus A^*\oplus P, \mu_A\oplus\lambda).$$

This enables us to define a homotopy inverse ψ to the inclusion $W\to S^{-1}W$,

$$\begin{array}{ccc} & \psi: S^{-1}W\to W & \\ (B,P,\lambda) & \mapsto & (P,\lambda) \\ {\scriptstyle (A,\alpha,\beta)}\downarrow & & \downarrow{\scriptstyle \beta\circ\eta_A(P)} \\ (B',P',\lambda') & \mapsto & (P',\lambda'). \end{array}$$

Letting $\theta: S^{-1}S_H\to W$ be the composite functor $\theta=\psi\circ(S^{-1}\tau)$, it follows that the homotopy fiber F_θ of $|\theta|$ is the same as that of $|S^{-1}\tau|$. To identify this fiber, we will apply Proposition 2.2.

3.2. LEMMA. $\theta: S^{-1}S_H\to W$ *satisfies the hypotheses of Proposition* 2.2.

3.3. REMARK. As noted below, the fiber categories $(P,\lambda)\backslash\theta$ are all of the form $S^{-1}((P,\lambda)\backslash\tau)$ where $(P,\lambda)\backslash\tau$ is a groupoid (i.e., every morphism is an isomorphism). Up to natural isomorphism, a groupoid $\underline{\underline{C}}$ is simply a disjoint union of classifying categories of groups

$$\underline{\underline{C}} \cong \coprod_{\{x_\alpha\}} \underline{\underline{\mathrm{Aut}}}(x_\alpha),$$

where the union runs over a set of representatives $\{x_\alpha\}$ for the isomorphism classes of objects of $\underline{\underline{C}}$. Suppose now that R is a ring such that every finitely generated projective module is free. Then the groups $\mathrm{Aut}(x)$ for an object x in $(P,\lambda)\backslash\tau$ are simply matrix groups of the form

$$G_{n,k}=\left\{g\in {}_{\epsilon}O_{n,n}(R)\;\middle|\; g=\left[\begin{array}{cc|cc} X & * & * & * \\ 0 & I_k & * & 0 \\ \hline 0 & 0 & \tilde{X} & 0 \\ 0 & 0 & * & I_k \end{array}\right],\ \tilde{X}=(\bar{X}')^{-1}\right\}.$$

In this case, Lemma 3.2 reduces to the claim that for $k'<k$, the maps

$$\varinjlim_{n} G_{n,k'}\to\varinjlim_{n} G_{n,k},$$

defined by the inclusions $G_{n,k'}\subset G_{n,k}$ (as subgroups of ${}_{\epsilon}O_{n,n}(R)$), induce isomorphisms on homology. The reader who is not completely comfortable with category theory may wish to consider this special case and translate the arguments below into group-theoretic arguments.

Proof of Lemma 3.2. For any object (P,λ) in W, the requirement that (P,λ) be hyperbolic means that P contains an isotropic subspace M such that $M=M^{\perp}$. Hence (M,id_0) is a morphism in W from the 0-object to (P,λ). It follows that any morphism $\beta:(P,\lambda)\to(P',\lambda')$ fits into a commutative diagram of the form

$$\begin{array}{ccc} (P,\lambda) & \xrightarrow{\ \beta\ } & (P',\lambda') \\ {\scriptstyle (M,\mathrm{id})}\nwarrow & & \nearrow{\scriptstyle (M',\mathrm{id})} \\ & 0. & \end{array}$$

It therefore suffices to show that all the base-change maps

$$(M,\mathrm{id})^{*}:(P,\lambda)\backslash\theta\to 0\backslash\theta$$

induce isomorphisms on homology.

Let $\tau: S_H\to W$ be the inclusion as above. It is straightforward to verify that for any (P,λ), $(P,\lambda)\backslash\theta=S^{-1}((P,\lambda)\backslash\tau)$, where S acts on $(P,\lambda)\backslash\tau$ by

$$T+[(P,\lambda)\xrightarrow{\alpha}(P',\lambda')]=[(P,\lambda)\xrightarrow{\alpha\circ\eta_T(P')}H(T)\oplus(P',\lambda')].$$

Moreover, for any morphism β in W, the induced base change β^{*} on $-\backslash\tau$ commutes with the action of S and the corresponding map $S^{-1}\beta^{*}$ is precisely the base change on $-\backslash\theta$.

We remark that there are also *right* actions of S on S_H and W. On W, this action is invertible (in fact trivial) via the natural transformation $\rho_T:\mathrm{id}_W\to(+T)$,

$$\rho_T(P)=(0\oplus T,\mathrm{id}_P):(P,\lambda)\to(P,\lambda)\oplus H(T),$$

so we have an induced right action on $(P,\lambda)\backslash\tau$:

$$[(P,\lambda)\xrightarrow{\alpha}(P',\lambda')]+T=[(P,\lambda)\xrightarrow{\alpha\circ\rho_T(P')}(P',\lambda')\oplus H(T)].$$

Equivalently, this action may be viewed as the composite $\rho_T(P)^{*}\circ R_T$ of the functor

$$(P,\lambda)\backslash\tau\xrightarrow{R_T}(P,\lambda)\oplus H(T)\backslash\tau,$$

which adds $H(T)$ (on the right) to *both* the target and source of an arrow, and the base-change functor

$$H(T)\oplus(P,\lambda)\backslash\tau \xrightarrow{\rho_T(P)^*} (P,\lambda)\backslash\tau$$

induced by $\rho_T(P)$. It follows from the fact that ρ_T is a natural transformation that these two definitions are equivalent. Clearly, the right action of T on $(P,\lambda)\backslash\tau$ is naturally isomorphic to the left action, hence inverting the left S action gives rise to a homotopy equivalence

$$(+T): (P,\lambda)\backslash\theta \to (P,\lambda)\backslash\theta.$$

Now consider an arbitrary morphism $\beta=(M,\gamma): (P',\lambda')\to(P,\lambda)$ in W. In addition to the base change functor, $\beta^*: (P,\lambda)\backslash\tau\to(P',\lambda')\backslash\tau$, we can define a functor $\beta_*^{\hat M}: (P',\lambda')\backslash\tau\to(P,\lambda)\backslash\tau$ as follows. Using the right action of M on *both* W and S_H we get a functor

$$R_M: (P',\lambda')\backslash\tau\to(P',\lambda')\oplus H(M)\backslash\tau$$

$$[(P',\lambda')\xrightarrow{(L,\varphi)}(P'',\lambda'')]\mapsto[(P',\lambda')\oplus H(M)\xrightarrow{(L,\varphi)\oplus \mathrm{id}_{H(M)}}(P'',\lambda'')\oplus H(M)].$$

On the other hand, $(P',\lambda')\oplus H(M)$ is isomorphic (in W) to (P,λ). Choosing an isomorphism $\delta: (P,\lambda)\xrightarrow{\cong}(P',\lambda')\oplus H(M)$, we thus get a functor

$$\delta^*\circ R_M: (P',\lambda')\backslash\tau\to(P,\lambda)\backslash\tau.$$

In particular, we can obtain such an isomorphism δ by choosing a hyperbolic dual space $\hat M\subset P$ for M (cf. §1). Then

$$P=(M^\perp\cap\hat M^\perp)\oplus M\oplus\hat M=M^\perp/M\oplus M\oplus M^*$$

(where the second equal sign is actually a canonical isomorphism), and we set $\delta=\gamma\oplus\mathrm{id}_{M\oplus M^*}$. (Recall that $\beta=(M,\gamma)$ so $\gamma: M^\perp/M\to P'$ is an isometry.) The functor $\delta^*\circ R_M$ arising from this choice of δ will be denoted $\beta_*^{\hat M}$. The significance of this choice of δ is that the composite map $\delta\circ\beta: (P',\lambda')\to(P',\lambda')\oplus H(M)$ is simply the morphism $\rho_M=(0\oplus M,\mathrm{id}_{P'})$. Thus the composite

$$(P',\lambda')\backslash\tau\xrightarrow{\beta_*^{\hat M}}(P,\lambda)\backslash\tau\xrightarrow{\beta^*}(P',\lambda')\backslash\tau$$

satisfies

$$\beta^*\circ\beta_*^{\hat M}=\beta^*\circ\delta^*\circ R_M=\rho_M^*\circ R_M=(+M),$$

where $(+M)$ is the right action of M. It follows from the discussion above that the induced composite $S^{-1}\beta^*\circ S^{-1}\beta_*^{\hat M}$ is a homotopy equivalence.

The situation with the reverse composite $S^{-1}\beta_*^{\hat M}\circ S^{-1}\beta^*$ is slightly more complicated. For this we restrict our attention to the case

$$(P',\lambda')=0, \qquad \beta=(M,\mathrm{id}): 0\to(P,\lambda).$$

As noted at the beginning of this proof, it suffices to consider this case. Let $\hat\beta=(M\oplus 0,\mathrm{id}): (P,\lambda)\to(P\oplus P,\lambda\oplus\lambda)$. Then $0\oplus\hat M$ is dual to $0\oplus M$ in $P\oplus P$.

Consider the diagram

$$\begin{array}{ccc} (P,\lambda)\backslash\tau & \xrightarrow{\beta^*} & 0\backslash\tau \\ \hat{\beta}_*^{0\oplus\hat{M}} \downarrow & & \downarrow \beta_*^{\hat{M}} \\ (P\oplus P,\lambda\oplus\lambda)\backslash\tau & \xrightarrow{\hat{\beta}^*} & (P,\lambda)\backslash\tau. \end{array}$$

By the previous paragraph, the composite $\hat{\beta}^*\circ\hat{\beta}_*^{0\oplus\hat{M}}$ becomes a homotopy equivalence after inverting S. Unfortunately, the diagram does not commute. (For example, the object $(P,\lambda)\xrightarrow{\text{id}}(P,\lambda)$ in $(P,\lambda)\backslash\tau$ goes to $(P,\lambda)\xrightarrow{(M\oplus 0,\delta)}(P,\lambda)\oplus H(M)$ under one composite and to $(P,\lambda)\xrightarrow{(0\oplus M,\text{id})}(P,\lambda)\oplus H(M)$ under the other.) However, if we let $\sigma: P\oplus P\to P\oplus P$ be the isometry which switches the factors, then we obtain a commutative diagram

$$\begin{array}{ccc} (P,\lambda)\backslash\tau & \xrightarrow{\beta^*} & 0\backslash\tau \\ \hat{\beta}_*^{0\oplus\hat{M}} \downarrow & & \downarrow \beta_*^{\hat{M}} \\ (P\oplus P,\lambda\oplus\lambda)\backslash\tau & \xrightarrow{(\hat{\beta}\circ\sigma)^*} & (P,\lambda)\backslash\tau. \end{array}$$

On the other hand, since σ is an isomorphism in W, the induced base change σ^* induces the identity map on homology. This follows from the fact that for $y\in\text{obj}\, W$, the category $y\backslash\tau$ is a groupoid (every morphism is an isomorphism), and σ^* acts trivially on $\pi_0(y\backslash\tau)$. In other words, restricting to a skeletal subcategory of $y\backslash\tau$, we get a disjoint union of classifying categories of groups with σ^* acting on each component as an inner automorphism.

Thus we conclude that the map induced on homology by $\beta_*^{\hat{M}}\circ\beta^*=\hat{\beta}^*\circ\sigma^*\circ\hat{\beta}_*^{0\oplus\hat{M}}$ is the same as the map induced by $\hat{\beta}^*\circ\hat{\beta}_*^{0\oplus\hat{M}}$. Inverting S, therefore, we see that $S^{-1}\beta_*^{\hat{M}}\circ S^{-1}\beta^*$ induces an isomorphism on homology. Combining this with the fact that $S^{-1}\beta^*\circ S^{-1}\beta_*^{\hat{M}}$ is a homotopy equivalence, we see that both maps, $S^{-1}\beta^*$ and $S^{-1}\beta_*^{\hat{M}}$, induce isomorphisms on homology. This completes the proof of Lemma 3.2. □

It now follows from Proposition 2.2 and Remark 2.3 that $|0\backslash\theta|$ is the homotopy fiber of $|\theta|$. It remains to compare $0\backslash\theta=S^{-1}(0\backslash\tau)$ with $S^{-1}S$. For this, note that the objects of $0\backslash\tau$ may be viewed as triples

$$(M,P,\lambda):=(0\xrightarrow{(M,\text{id})}(P,\lambda))$$

such that $M\subset P$ is a summand with $M^\perp=M$. In this notation, a morphism $(M,P,\lambda)\to(M',P',\lambda')$ in $0\backslash\tau$ is simply an isometry $\alpha: P\cong P'$ such that $\alpha(M)=M'$, and the action of S on $0\backslash\tau$ is given by

$$T+(M,P,\lambda)=(T\oplus M, T\oplus T^*\oplus P, \mu_T\oplus\lambda).$$

Clearly, then, $M\mapsto(M,H(M))$, $\alpha\mapsto H(\alpha)$, defines a functor $\Delta: S\to 0\backslash\tau$ and a factorization of H,

$$\begin{array}{ccc} S & \xrightarrow{\Delta} & 0\backslash\tau \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle 0^\pi} & \\ & S_H, & \end{array}$$

where ${}_0\pi$ is the projection $(M, P, \lambda) \to (P, \lambda)$ as in 2.1. Inverting the action of S, we get a commutative diagram

$$\begin{array}{ccc} S^{-1}S & \xrightarrow{S^{-1}\Delta} & S^{-1}(0\backslash\tau) = 0\backslash\theta \\ & {\scriptstyle S^{-1}H}\searrow \quad \swarrow{\scriptstyle S^{-1}({}_0\pi)} & \\ & S^{-1}S_H & \end{array}$$

The following lemma will complete the proof of the main theorem.

3.4. LEMMA. *$S^{-1}\Delta$: $S^{-1}S \to 0\backslash\theta$ is a homotopy equivalence.*

Proof. Since $|S^{-1}\Delta|$ is an H-space map, it suffices to show that it induces isomorphisms on homology. Since S is a groupoid, restricting to a skeletal subcategory gives $S \cong \coprod \underline{\underline{\mathrm{Aut}}}(P)$, where the disjoint union runs over a set of representatives P for $\pi_0(S)$ and $\underline{\underline{\mathrm{Aut}}}(P)$ denotes the classifying category of the group of automorphisms of P. Since the free R-modules form a cofinal set in $\pi_0(S)$, it follows that

$$\begin{aligned} H_*(S^{-1}S) &= \pi_0(S)^{-1}H_*(S) \\ &= \varinjlim H_*(\mathrm{GL}_n(R)) \\ &= H_*(\mathrm{GL}(R)), \end{aligned}$$

where $\mathrm{GL}(R) = \varinjlim \mathrm{GL}_n(R)$ and the limit is taken with respect to the lower right inclusion maps $\mathrm{GL}_n \to \mathrm{GL}_{n+1}$ (cf. [6]). Similarly, $0\backslash\tau$ is a groupoid with the triples

$$(R^n, R^{2n}, \mu_n), \quad \mu_n(x, y) = x\left[\begin{array}{c|c} 0 & I \\ \hline \epsilon I & 0 \end{array}\right]\bar{y}^t,$$

representing a cofinal set in $\pi_0(0\backslash\tau)$; hence

$$\begin{aligned} H_*(S^{-1}(0\backslash\tau)) &= \pi_0(S)^{-1}H_*(0\backslash\tau) \\ &= \varinjlim H_*(\mathrm{Aut}(R^n, R^{2n}, \mu_n)). \end{aligned}$$

More explicitly, letting

$$\begin{aligned} G_n(R) &= \mathrm{Aut}(R^n, R^{2n}, \mu_n) \\ &= \left\{ g \in {}_\epsilon O_{n,n}(R) \,\middle|\, g = \left[\begin{array}{c|c} X & * \\ \hline 0 & \tilde{X} \end{array}\right], \ \tilde{X} = (\bar{X}^t)^{-1} \right\}, \end{aligned}$$

there is a canonical isomorphism

$$\begin{aligned} R + (R^n, R^{2n}, \mu_n) &= (R \oplus R^n, R \oplus R^* \oplus R^{2n}, \mu_R \oplus \mu_n) \\ &\cong (R^{n+1}, R^{2(n+1)}, \mu_{n+1}) \end{aligned}$$

which induces

$$G_n(R) \xrightarrow{i_n} G_{n+1}(R),$$

$$\left[\begin{array}{c|c} X & * \\ \hline 0 & \tilde{X} \end{array}\right] \mapsto \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & X & 0 & * \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \tilde{X} \end{array}\right].$$

Let $G(R) = \varinjlim G_n(R)$ defined by the maps l_n. Then

$$H_*(S^{-1}(0\backslash\tau)) = \varinjlim H_*(G_n(R)) = H_*(G(R)).$$

The functor $\Delta\colon S \to 0\backslash\tau$ takes R^n to (R^n, R^{2n}, μ_n) and $\mathrm{GL}_n(R)$ to $G_n(R)$ by the hyperbolic map

$$\mathrm{GL}_n(R) \xrightarrow{h_n} G_n(R),$$

$$X \mapsto \left[\begin{array}{c|c} X & 0 \\ \hline 0 & \tilde{X} \end{array}\right].$$

Hence the lemma reduces to proving that, in the limit, the map $h = \varinjlim h_n$ induces isomorphisms on homology, or (equivalently) that h induces isomorphisms on $H_*(\ ;k)$ for every algebraically closed field k. The maps h_n are split injective with splittings

$$G_n(R) \xrightarrow{r_n} \mathrm{GL}_n(R),$$

$$\left[\begin{array}{c|c} X & * \\ \hline 0 & \tilde{X} \end{array}\right] \mapsto X.$$

These splittings commute with the inclusions $\mathrm{GL}_n \subset \mathrm{GL}_{n+1}$, $G_n \subset G_{n+1}$, so the limit map h is likewise split injective with splitting $r = \varinjlim r_n$. It follows that the induced map h_* on $H_*(\ ;k)$ is injective. It remains to prove that h_* is surjective, or (equivalently) that r_* is injective.

We now fix an algebraically closed field k and assume all homology to have k-coefficients, that is, $H_*(\ ) \equiv H_*(\ ;k)$. We begin by showing that the lemma holds for certain finite extensions of R. By a lemma of Quillen [13, p. 208], for any integer $d > 0$ there exists an order $\mathcal{O}$ in a number field of degree d over $\mathbf{Q}$, such that if N is an $\mathcal{O}$-module then (for $0 < i < d$) $H_i(N, k)$ decomposes as a direct sum of non-trivial 1-dimensional representations of $\mathcal{O}^*$ over k. A careful reading of Quillen's proof shows that if d is assumed to be odd, then the characters $\sigma\colon \mathcal{O}^* \to k^*$ of these representations cannot be of order 2. (The crucial observation here is that for d odd, $q = (p^d - 1)/(p-1)$ is also odd and hence the congruence on page 215 of Quillen's proof, $\sum_{0 \le a < d}(m_a + n_a)p^a \equiv 0 \bmod q$, holds in the case of order-2 characters as well as trivial characters.) Assume, therefore, that d is an odd integer >0 and $\mathcal{O}$ is as in Quillen's lemma. Let A be the R-algebra, $A = R \otimes_{\mathbf{Z}} \mathcal{O}$. Then the involution on R induces an involution $\overline{s \otimes z} = \bar{s} \otimes z$ on A. Let $i\colon \mathcal{O} \to A$ be the ring homomorphism $i(z) = 1 \otimes z$. Note that $i(\mathcal{O})$ is contained in the center of A and is fixed by the involution. Consider the exact sequence

$$1 \to N \to G_n(A) \xrightarrow{r^A} \mathrm{GL}_n(A) \to 1.$$

The kernel N can be identified with the additive group of $n\times n$ matrices over A satisfying $M^t=-\epsilon\bar{M}$. Letting $z\in\mathcal{O}$ act on $M\in N$ via left multiplication by

$$D_z=\begin{bmatrix} i(z) & & 0 \\ & \ddots & \\ 0 & & i(z)\end{bmatrix}$$

gives N the structure of an $\mathcal{O}$-module. The restriction of this action to the group of units $\mathcal{O}^*$ in $\mathcal{O}$ extends to an action $\mathcal{O}^*$ on $G_n(A)$:

$$z\cdot\left[\begin{array}{c|c} X & M \\ \hline 0 & \tilde{X}\end{array}\right]=\left[\begin{array}{c|c} X & z\cdot M \\ \hline 0 & \tilde{X}\end{array}\right].$$

The induced action on $\mathrm{GL}_n(A)$ is trivial. It follows that $\mathcal{O}^*$ acts on the spectral sequence

$$E^2_{p,q}=H_p(\mathrm{GL}_n(A);H_q(N))\Rightarrow H_{p+q}(G_n(A))$$

and that the entire spectral sequence decomposes into a direct sum of spectral sequences, ${}_\sigma E^*_{**}$, on which $\mathcal{O}^*$ acts by the character $\sigma: D^*\to k^*$. By Quillen's lemma, if σ is trivial or of order 2, then ${}_\sigma E^2_{p,q}={}_\sigma E^\infty_{p,q}=0$ for $0<q<d$. On the other hand, from the equation

$$z^2\cdot\left[\begin{array}{c|c} X & M \\ \hline 0 & \tilde{X}\end{array}\right]=\left[\begin{array}{c|c} D_z & 0 \\ \hline 0 & D_z^{-1}\end{array}\right]\left[\begin{array}{c|c} X & M \\ \hline 0 & \tilde{X}\end{array}\right]\left[\begin{array}{c|c} D_z^{-1} & 0 \\ \hline 0 & D_z\end{array}\right]$$

we see that z^2 acts on $G_n(A)$ as an inner automorphism. (Recall that $\overline{i(z)}=i(z)$ so $D_z^{-1}=\tilde{D}_z$.) Hence it acts trivially on homology. It follows that if σ is *not* of order 1 or 2, then ${}_\sigma E^\infty_{p,q}=0$ for all p,q. Combining these we see that

$$E^\infty_{p,0}=H_p(\mathrm{GL}_n(A))$$
$$E^\infty_{p,q}=0 \quad \text{for } 0<q<d.$$

We conclude that $r^A_*: H_i(G_n(A))\cong H_i(\mathrm{GL}_n(A))$ for all $i<d$.

Next note that since $\mathcal{O}$ is a free $\mathbf{Z}$-module of rank d, $A=R\otimes_{\mathbf{Z}}\mathcal{O}$ is a free R-module of rank d. Thus, choosing a basis we can identify $A\cong R^d$. (For convenience, choose a basis of the form $\{1\otimes z_i\}$.) This gives rise to inclusion maps $j_n:\mathrm{GL}_n(A)\hookrightarrow\mathrm{GL}_n(R)$ and $\hat{j}_n: G_n(A)\hookrightarrow G_n(R)$. Passing to the limit as $n\to\infty$, we get inclusions $j:\mathrm{GL}(A)\hookrightarrow\mathrm{GL}(R)$ and $\hat{j}: G(A)\hookrightarrow G(R)$. Consider the commutative diagram

$$\begin{array}{ccccc} G(R) & \xrightarrow{\hat{t}} & G(A) & \xrightarrow{\hat{j}} & G(R) \\ r\downarrow & & r^A\downarrow & & r\downarrow \\ \mathrm{GL}(R) & \xrightarrow{t} & \mathrm{GL}(A) & \xrightarrow{j} & \mathrm{GL}(R), \end{array}$$

where $t,\hat{t}$ are induced by the inclusion $r\mapsto r\otimes 1$ of R into A. The composite $\hat{j}\circ\hat{t}$ takes each entry r of a matrix in $G(R)$ to a $d\times d$ diagonal matrix

$$\begin{pmatrix} r\otimes 1 & & 0 \\ & \ddots & \\ 0 & & r\otimes 1\end{pmatrix}.$$

On the other hand, this is precisely the map which defines multiplication by d on $H_*(G(R))$. In particular, the kernel of $\hat{j}_* \circ \hat{i}_*$ is the d-torsion subgroup ${}_dH_*(G(R))$. If we assume that $d > i$, then from the commutative diagram

$$\begin{array}{ccccc} H_i(G(R)) & \xrightarrow{\hat{i}_*} & H_i(G(A)) & \xrightarrow{\hat{j}_*} & H_i(G(R)) \\ r_* \downarrow & & \downarrow \cong & & \downarrow r_* \\ H_i(\mathrm{GL}(R)) & \xrightarrow{i_*} & H_i(\mathrm{GL}(A)) & \xrightarrow{j_*} & H_i(\mathrm{GL}(R)), \end{array}$$

we see that $\ker r_* \subseteq \ker \hat{i}_* \subseteq \ker \hat{j}_* \hat{i}_* \subseteq {}_dH_i(G(R))$. But this holds for *every* odd integer $d > i$. This is clearly impossible unless $\ker r_* = 0$. This completes the proof of Lemma 3.4 and Theorem 3.1. □

3.4. We conclude this section with a discussion of some alternate versions of $W = {}_\epsilon W(R)$. We consider first a "split" version of $W = {}_\epsilon W(R)$. By this we mean the category W^{sp} whose objects are the same as the objects of W, but whose morphisms $(P, \lambda) \to (P', \lambda')$ consist of a W-morphism (L, φ) together with a choice of hyperbolic dual space L_D for L in P'. Recall that such a choice determines a decomposition $P' = (L^\perp \cap L_D^\perp) \oplus L \oplus L_D = L^\perp/L \oplus L \oplus L^*$ and hence determines an isometry $(P, \lambda) \oplus H(L) \cong (P', \lambda')$. Conversely, given $M \in \mathrm{obj}\, S$ and an isometry $\varphi: (P, \lambda) \oplus H(M) \cong (P', \lambda')$, the triple $(\varphi(M), \varphi(M^*), \varphi \mid \bar{P}^1)$ represents a morphism in W^{sp}. It follows that W^{sp} is precisely the category $\langle S, S_H \rangle$ described in [6] and hence (by [6, p. 223]) there exists a homotopy fibration

$$S^{-1}S \xrightarrow{S^{-1}H} S^{-1}S_H \xrightarrow{\pi_2} W^{\mathrm{sp}},$$

where π_2 is projection on the second factor. Comparing this fibration with the homotopy fibration

$$S^{-1}S \xrightarrow{S^{-1}H} S^{-1}S_H \xrightarrow{\theta} W$$

constructed above, we have a commutative diagram

$$\begin{array}{ccccc} S^{-1}S & \longrightarrow & S^{-1}S_H & \xrightarrow{\pi_2} & W^{\mathrm{sp}} \\ \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow F \\ S^{-1}S & \longrightarrow & S^{-1}S_H & \xrightarrow{\theta} & W \end{array}$$

where F is the forgetful functor. We conclude with the following.

3.5. COROLLARY. *The forgetful functor $F: W^{\mathrm{sp}} \to W$ is a homotopy equivalence.*

3.6. It is also interesting to consider what happens if we do not require the objects (P, λ) of W to be hyperbolic, but insist only that P be a finitely generated projective module and that λ be bi-linear, ϵ-hermitian (i.e., $\lambda(x, y) = \epsilon\lambda(\overline{y, x})$), and nonsingular (i.e., $\mathrm{ad}\,\lambda: P \to P^*$ is an isomorphism). Let $\hat{W} = {}_\epsilon\hat{W}(R)$ denote the category of such pairs (P, λ) with morphisms defined as for W. In particular, a morphism $(L, \varphi): (P, \lambda) \to (P', \lambda')$ exists if and only if there exists an isometry $(P', \lambda') \cong (P, \lambda) \oplus H(L)$ (though the isometry is not specified by the morphism). It is clear that in general this category is not connected but rather that $\pi_0(\hat{W})$ is the set of *stable* isometry classes of objects (P, λ) of $\hat{W}$. If, however, we assume that 2 is invertible in R, then for every object (P, λ) in $\hat{W}$, $(P, \lambda) \oplus (P, -\lambda)$ is hyperbolic and so $\pi_0(\hat{W})$ is a group. It follows that $\hat{W}$ is an H-group and that all of its connected components are homotopy equivalent. The component of the 0-object, $\hat{W}^0$, consists of those objects (P, λ) which are stably hyperbolic, that is, $(P, \lambda) \oplus H(M)$ is hyperbolic for some M. (We may, of course, choose M to be a free module.) Filtering $\hat{W}^0$ by the full subcategories F^i,

$$\operatorname{obj} F^i = \{(P, \lambda) \mid (P, \lambda) \oplus H(R^i) \text{ is hyperbolic}\},$$

we obtain a filtration

$$W = F^0 \subset F^1 \subset \cdots \subset \bigcup F^i = \hat{W}^0$$

such that each inclusion $F^i \subset F^{i+1}$ has a homotopy inverse (defined by adding $H(R)$). It follows that the inclusion of W into $\hat{W}^0$ is a homotopy equivalence, and we conclude that $|\hat{W}| \simeq \pi_0(\hat{W}) \times |W|$. Equivalently, we have the following.

3.7. COROLLARY. *If $\frac{1}{2} \in R$, then there is a homotopy fibration*

$$K_0(R) \times B\mathrm{GL}(R)^+ \to {}_\epsilon L_0(R) \times B_\epsilon O(R)^+ \to |{}_\epsilon\hat{W}(R)|.$$

If 2 is not invertible in R, the situation is, of course, more complicated. In this case, $\hat{W}$ need not be an H-group. We could "group complete" $\hat{W}$ by passing to $\hat{W}^{-1}\hat{W}$, but the relation between W and $\hat{W}^{-1}\hat{W}$ is not clear. As our main theorem demonstrates, W is the appropriate category to work with in this context since it gives a delooping of Karoubi's ${}_\epsilon\mathfrak{U}(R)$.

4. The homotopy and homology of ${}_\epsilon W(R)$.

4.1 Suppose X is a commutative H-space and $\sigma: X \to X$ is an involution on X which respects the H-space structure, that is, $\sigma^2 = \mathrm{id}_X$ and $\sigma(x+y) = \sigma(x) + \sigma(y)$. Suppose also that multiplication by 2 is invertible in X, that is, that the map

$$(\times 2): X \xrightarrow{\Delta} X \times X \xrightarrow{+} X$$

is a homotopy equivalence. Then X is homotopy equivalent to a product of H-spaces, $X \simeq X^\sigma \times X^{-\sigma}$, where

$$X^\sigma = \operatorname{image}\{(\mathrm{id} + \sigma): X \to X\}$$
$$X^{-\sigma} = \operatorname{image}\{(\mathrm{id} - \sigma): X \to X\}.$$

If multiplication by 2 is not invertible, we can "localize at 2" by taking the homotopy limit

$$X_{(2)} = \operatorname{holim}(X \xrightarrow{\times 2} X \xrightarrow{\times 2} X \xrightarrow{\times 2} \cdots).$$

The resulting space $X_{(2)}$ is a commutative H-space with 2 invertible and so, as above, $X_{(2)} \simeq X^\sigma_{(2)} \times X^{-\sigma}_{(2)}$.

Consider the involutions on the categories S and S_H defined by

$$\tau: S \to S \quad \text{by } P \mapsto P^*, \ \alpha \mapsto (\alpha^*)^{-1},$$
$$\sigma: S_H \to S_H \quad \text{by } (P, \lambda) \mapsto (P, -\lambda), \ \alpha \mapsto \alpha.$$

These induce H-space involutions (which we again denote τ and σ) on $X = |S^{-1}S|$ and $Y = |S_H^{-1}S_H|$, from which we obtain decompositions

$$X_{(2)} \simeq X_{(2)}^{\tau} \times X_{(2)}^{-\tau}, \qquad Y_{(2)} \simeq Y_{(2)}^{\sigma} \times Y_{(2)}^{-\sigma}.$$

Karoubi [8] proves that the hyperbolic map (localized at 2), $h_{(2)}: X_{(2)} \to Y_{(2)}$, is homotopic to a map

$$X_{(2)}^{\tau} \times X_{(2)}^{-\tau} \xrightarrow{\text{proj}} X_{(2)}^{\tau} \simeq Y_{(2)}^{\sigma} \xrightarrow{\text{inc}} Y_{(2)}^{\sigma} \times Y_{(2)}^{-\sigma}.$$

It follows from Theorem 3.1 that $|W|_{(2)}$ is homotopic to $Y_{(2)}^{-\sigma} \times BX_{(2)}^{-\tau}$, where $BX_{(2)}^{-\tau}$ is a connected delooping of $X_{(2)}^{-\tau}$. (As a model for $BX_{(2)}^{-\tau}$ we can take Quillen's Q-category on finitely generated projective R-modules [11] and define an involution $\rho: Q \to Q$, $P \mapsto P^*$, $(P \overset{\alpha}{\twoheadleftarrow} V \overset{\beta}{\rightarrowtail} P') \mapsto (P^* \overset{\alpha^*}{\rightarrowtail} V^* \overset{\beta^*}{\twoheadleftarrow} P'^*)$. Then it is easily verified that $BX_{(2)}^{-\tau} \simeq |Q|_{(2)}^{-\rho}$.) The involutions τ and σ induce corresponding involutions on the K-groups, $K_i(R) = \pi_i(X)$, and on the L-groups, ${}_{\epsilon}L_i(R) = \pi_i(Y)$ $(i \geq 1)$. Letting $\mathbf{Z}' = \mathbf{Z}[\frac{1}{2}]$, $i \geq 1$, we have

$$\pi_i(BX_{(2)}^{-\tau}) = K_{i-1}^{-\tau}(R) \otimes \mathbf{Z}',$$
$$\pi_i(Y_{(2)}^{-\sigma}) = {}_{\epsilon}L_i^{-\sigma}(R) \otimes \mathbf{Z}'.$$

The latter are also known as the Witt groups of R.

4.2. THEOREM. *For* $\mathbf{Z}' = \mathbf{Z}[\frac{1}{2}]$, $i \geq 1$,

$$\pi_i({}_{\epsilon}W(R)) \otimes \mathbf{Z}' \cong ({}_{\epsilon}L_i^{-\sigma}(R) \otimes \mathbf{Z}') \oplus (K_{i-1}^{-\tau}(R) \otimes \mathbf{Z}').$$

4.3. Consider, in particular, the case where R is a ring of integers in a number field with the trivial involution. Let r_1 and r_2 denote (respectively) the number of real and complex places of R. In this case, the Witt groups of R are known to be

$${}_1L_i^{-\sigma}(R) \otimes \mathbf{Z}' = \begin{cases} (\mathbf{Z}')^{r_1} & i \equiv 0 \ (4) \\ 0 & \text{otherwise}, \end{cases}$$

$${}_{-1}L_i^{-\sigma}(R) \otimes \mathbf{Z}' = \begin{cases} (\mathbf{Z}')^{r_1} & i \equiv 2 \ (4) \\ 0 & \text{otherwise}. \end{cases}$$

The K-groups of R are unfortunately not so well understood. However, by computations of Borel [4], we do know the *ranks* of the K-groups of R:

$$\operatorname{rank} K_i(R) = \dim(K_i(R) \otimes \mathbf{Q}) = \begin{cases} 0 & i \equiv 0 \ (4) \ \ i > 0 \\ r_1 + r_2 & i \equiv 1 \ (4) \ \ i > 1 \\ 0 & i \equiv 2 \ (4) \\ r_2 & i \equiv 3 \ (4), \end{cases}$$

and by Dirichlet, $\operatorname{rank} K_1(R) = r_1 + r_2 - 1$. Borel also computes the ranks of the L-groups of R,

$$\operatorname{rank}{}_1L_i(R)=\dim({}_1L_i(R)\otimes\mathbf{Q})=\begin{cases} r_1 & i\equiv 0\ (4)\ i>0\\ 0 & i\equiv 1\ (4)\\ 0 & i\equiv 2\ (4)\\ r_2 & i\equiv 3\ (4),\end{cases}$$

$$\operatorname{rank}{}_{-1}L_i(R)=\dim({}_{-1}L_i(R)\otimes\mathbf{Q})=\begin{cases} 0 & i\equiv 0\ (4)\ i>0\\ 0 & i\equiv 1\ (4)\\ r_1 & i\equiv 2\ (4)\\ r_2 & i\equiv 3\ (4).\end{cases}$$

(For $i=1$; this follows from the fact that ${}_\epsilon L_1(R)\otimes\mathbf{Q}\cong H_1({}_\epsilon O_{n,n}(R);\mathbf{Q})$. The latter is zero by Borel's computations.) Since the Witt groups are the cokernels of the hyperbolic map, we conclude that for $i\geq 1$,

$$h_{\mathbf{Q}}: K_i(R)\otimes\mathbf{Q}\to {}_\epsilon L_i(R)\otimes\mathbf{Q}$$

is an isomorphism when $i\equiv 3 \bmod 4$ and is the zero-map otherwise. The theorem below follows immediately.

4.4. THEOREM. *Let R be a ring of integers in a number field with the trivial involution. Then for $j\geq 1$,*

$$\operatorname{rank}\pi_i({}_1W(R))=\begin{cases} r_1+r_2-1 & i=2\\ r_1 & i=4j\\ r_1+r_2 & i=4j+2\\ 0 & \textit{otherwise},\end{cases}$$

$$\operatorname{rank}\pi_i({}_{-1}W(R))=\begin{cases} 2r_1+r_2-1 & i=2\\ 2r_1+r_2 & i=4j+2\\ 0 & \textit{otherwise}.\end{cases}$$

The rational cohomology ring of ${}_\epsilon W(R)$ is a polynomial algebra

$$H^*({}_1W(R);\mathbf{Q})=\Big(\bigotimes^{r_1+r_2-1}\mathbf{Q}[x_2]\Big)\otimes\Big(\bigotimes^{r_1}\mathbf{Q}[x_{4j}]\Big)\otimes\Big(\bigotimes^{r_1+r_2}\mathbf{Q}[x_{4j+2}]\Big),\quad j\geq 1,$$

$$H^*({}_{-1}W(R);\mathbf{Q})=\Big(\bigotimes^{2r_1+r_2-1}\mathbf{Q}[x_2]\Big)\otimes\Big(\bigotimes^{2r_1+r_2}\mathbf{Q}[x_{4j+2}]\Big),\quad j\geq 1.$$

4.5. We continue to assume that R is a ring of integers in a number field with the trivial involution. In this final section we consider a filtration of W which is used in [5] to establish a connection between K-theory and the theory of moduli spaces. Let W_n be the full subcategory of W whose objects are pairs (P,λ) with rank $P\leq 2n$, and let $I_n: W_n\to W_{n+1}$ be the inclusion functor. We claim that I_n induces isomorphisms on homology in dimensions $i<n$. (Our argument follows that of Quillen [14] for the Q_n categories.) For an object (P,λ) in W_{n+1}, consider the category $I_n/(P,\lambda)$ defined as in 2.1. If rank $P\leq 2n$, then $I_n/(P,\lambda)$ has a final object $((P,\lambda),\mathrm{id}_{(P,\lambda)})$; hence $I_n/(P,\lambda)$ is contractible. If rank $P=2(n+1)$, then (up to isomorphism) an object in $I_n/(P,\lambda)$, $((P',\lambda'),(P',\lambda')\xrightarrow{(l,\varphi)}(P,\lambda))$, is

completely determined by the (non-zero) isotropic summand $L \subset P$; a morphism corresponds to an inclusion $L' \subseteq L$ of such summands. In other words, $I_n/(P, \lambda)$ is naturally isomorphic to the partially ordered set (or "poset") of non-zero isotropic direct summands of (P, λ). On the other hand, tensoring with the field of fractions F gives an isomorphism of this poset with the poset of non-zero isotropic subspaces of the $2(n+1)$-dimensional vector space $P \otimes_R F$. By a theorem of Vogtmann [15], the latter has the homotopy type of a wedge of n-spheres. It follows that for $n \geq 1$, the spectral sequence associated to I_n,

$$E^2_{p,q} = H_p(W_{n+1}; \mathcal{H}_q) \Rightarrow H_{p+q}(W_n)$$

where $\mathcal{H}_q$ is the functor $(P, \lambda) \mapsto H_q(I_n/(P, \lambda))$, has E^2-terms

$$E^2_{p,0} = H_p(W_{n+1})$$

$$E^2_{p,q} = 0 \quad (0 < q < n-1).$$

As a consequence, we obtain the following.

4.6. THEOREM. *For R a ring of integers in a number field with the trivial involution, the inclusion functor $I_n: {}_\epsilon W_n(R) \to {}_\epsilon W_{n+1}(R)$ induces isomorphisms $H_i({}_\epsilon W_n(R)) \cong H_i({}_\epsilon W_{n+1}(R))$ for $i < n$.*

REFERENCES

1. J. F. Adams, *Infinite loop spaces,* Ann. Math. Studies 90, Princeton Univ. Press, Princeton, N.J., 1978.
2. A. Bak, *On modules with quadratic forms.* Algebraic K-theory and its geometric applications (Conf., Hull, 1969), 55–66, Lecture Notes in Math., 108, Springer, Berlin, 1969.
3. ———, *K-theory of forms,* Ann. of Math. Studies, 98, Princeton Univ. Press, Princeton, N.J., 1981.
4. A. Borel, *Stable real cohomology of arithmetic groups,* Ann. Sci. École. Norm. Sup. (4) 7 (1974), 235–272.
5. R. Charney and R. Lee, *Cohomology of the Satake compactification,* Topology (22) 4 (1983), 389–423.
6. D. Grayson (after D. Quillen), *Higher algebraic K-theory* II. Algebraic K-theory (Evanston, Ill., 1976), 217–240, Lecture Notes in Math., 551, Springer, Berlin, 1976.
7. M. Karoubi, *Périodicité de la K-théorie hermitienne.* Algebraic K-theory, III: Hermitian K-theory and geometric applications (Seattle, Wash., 1972), 301–411, Lecture Notes in Math., 343, Springer, Berlin, 1973.
8. ———, *Théorie de Quillen et homologie du groupe orthogonal,* Ann. of Math. (2) 112 (1980), 207–257.
9. ———, *Le théorème fondamental de la K-théorie hermitienne,* Ann. of Math. (2) 112 (1980), 259–282.
10. J.-L. Loday, *K-théorie algèbrique et représentations de groupes,* Ann. Sci. École Norm. Sup. (4) 9 (1976), 309–377.
11. D. McDuff and G. Segal, *Homology fibrations and the "group completion" theorem,* Invent. Math. 31 (1976), 279–284.

12. D. Quillen, *Higher algebraic K-theory* I. Algebraic K-theory, I: Higher K-theories (Seattle, Wash., 1972), 85–147, Lecture Notes in Math., 341, Springer, Berlin, 1973.
13. ———, *Characteristic classes of representations.* Algebraic K-theory (Evanston, Ill., 1976), 189–216, Lecture Notes in Math., 551, Springer, Berlin, 1976.
14. ———, *Finite generation of the groups K_i of rings of algebraic integers.* Algebraic K-theory, I: Higher K-theories (Seattle, Wash., 1972), 179–198, Lecture Notes in Math., 341, Springer, Berlin, 1973.
15. K. Vogtmann, *Homology stability for $O_{n,n}$.* Comm. Algebra 7 (1979), 9–38.

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